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Impact of Interactions between Collinearity, Leverage Points and Outliers on Ridge, Robust, and Ridge-type Robust Estimators

O. Ufuk Ekiz

Department of Statistics, Faculty of Science, Gazi University, Ankara, Turkey

Abstract This study proposes a framework to compare performances of various ridge, robust and ridge-type robust estimators when a data set is contaminated by collinearity, collinearity-influential observations, as well as outliers. This is achieved by first generating fifteen different synthetic data sets with known level of contamination. These data sets are then used to evaluate performances of twelve different estimators based on the Monte-Carlo estimates of total mean square, total variance and total bias. It has shown that these results can be used as lookup tables to select the best estimator for various cases of contamination. The results reveal that the interactions between leverage points and collinearity can be misleading for estimation selection problem. It is also shown that the notion of directionality of outliers and the strength of collinearity can also drastically impact estimator performance. Finally, an example application is presented to validate the results.

Keywords Collinearity-influential, Leverage, Outliers, Ridge, Robust, Ridge-type robust

1. Introduction

There is a growing interest in the literature for understanding the performance of ridge, robust, and ridge-type robust estimators that are less prone to the contaminations caused by outliers and collinearity in data [1-7]. However, the degree to which these contaminations effect estimator performance is not yet well understood. Hence, for practical applications this lack of insight makes it challenging to decide the best and the most efficient estimator. Another challenge is to construct proper platforms for comparing and validating estimator performances when data is subject to various levels of contaminations due to collinearity, outliers, leverage points and their interactions. In the rest of the study, the term contamination will be used to describe the negative effects of various levels of collinearity, type of outliers, leverage points and their interactions on the estimation performance. This study proposes a framework to address some of these challenges.

To begin our discussion, we consider the linear regression model [8],

$$Y = X\beta + \varepsilon \tag{1}$$

* Corresponding author:
ufukekiz@gazi.edu.tr (O. Ufuk Ekiz)
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where Y (dependent variable) is a vector of size $n \times 1$, X is an $n \times p$ full rank design matrix, n is the sample size, and p is the number of explanatory variables. In Equation (1), ε is an $n \times 1$ error vector that satisfies the expected value $E(\varepsilon) = 0$ and covariance $Cov(\varepsilon) = \sigma^2 I$ where I is the $n \times n$ identity matrix. Moreover, β is the $p \times 1$ unknown parameter vector, and σ^2 is the variance. In this setup, it is well known that ordinary least squares (*OLS*)

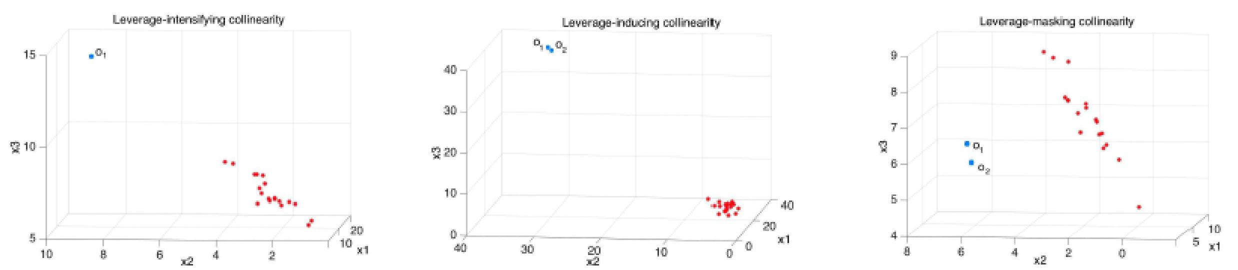
$$\hat{\beta}_{OLS} = \left(X'X\right)^{-1} X'Y \tag{2}$$

is the best linear unbiased estimator [8] where X' is the transpose of matrix X. However, it is also known that collinearity in data introduces sign switches for the *OLS* estimator and inflates its variance [1,3,4]. There is a vast amount of work in literature [4,9,10,11] that have proposed and studied ridge regression which is one of the most commonly used methods to overcome the challenges introduced by collinearity [3]. Furthermore, these studies have revealed that presence of high-leverage points (observations) in data is also critical since they can drastically change the effect of collinearity in estimation.

Not only collinearity but outliers may also effect the performance of *OLS* and ridge estimators. In their studies [7,12,13], authors have proposed various robust estimators that target to eliminate the impact of outliers in estimation. Properties of these estimators vary depending upon the type of outliers present in the data; i.e. if the observations are classified as outliers based on their X (Type 1) or Y (Type 2) distances then estimator performance may vary when data contains only Type 1, only Type 2 or both Type 1 and Type 2 outliers. Finally, ridge-type and Liu-type robust estimators have been proposed in studies [1,2,5,6,14] in order to overcome the effects caused by both collinearity and outliers. However, the interactions between collinearity, outliers and leverage points are not yet well understood and it is not trivial to conclude which estimator performs better when data is subject to various levels of contamination.

In [15] and [16], the impact of outliers on the performance of various robust estimators when the sample size is small is studied. In this study, the details of a simulation study in which the synthetic data sets are generated so as to involve predefined levels of contamination caused by outliers, collinearity and leverage points are presented. Then, performances of a subset of well-known ridge, robust and ridge-type robust estimators are investigated. This is achieved by first classifying collinearity-influential observations (high-leverage points) in data into three subgroups based on their type of influence-leverage-masking collinearity, leverage-inducing collinearity, and leverage-intensifying collinearity (similar classifications were presented in studies [17] and [18]).

We simulate and present example data sets in Figure 1 to illustrate masking, inducing and intensifying effects of high-leverage observations on the estimate of variance inflation factor $\hat{VIF} = 1/(1-\hat{R}^2)$, where $\hat{R}^2$ is the estimate of maximum determination coefficient computed for all explanatory variables such that $\hat{VIF}$ is the maximum value of the diagonal elements of inverse sample correlation matrix of the explanatory variables [19,20]. Moreover, $\hat{VIF}$ and $\hat{R}^2$ are the estimates of VIF and R^2. Thus, when R^2 is the maximum determination coefficient among the explanatory variables $VIF = 1/(1-R^2)$. In the figure o_i is an observation, where $i = 1,\ldots,n$. Let us consider the observation o_1 (shown in blue) in Figure 1a. It is obvious that even if we exclude o_1 there exists collinearity between the rest of the observations. However, it is also clear that including o_1 in the computations will intensify the strength of collinearity. Hence, observation o_1 is classified as leverage-intensifying collinearity. Similarly in Figure 1b, it is shown that including high-leverage observations o_1 and o_2 in computations will induce collinearity (leverage-inducing collinearity) that would not be as prominent without o_1 and o_2. Finally, Figure 1c illustrates an example in which o_1 and o_2 mask the effect of collinearity (leverage-masking collinearity), that is, including o_1 and o_2 in estimation drastically decreases level of collinearity already present for the rest of the data.



(a) An example data set involving leverage-intensifying collinearity (b) An example data set involving leverage-inducing collinearity (c) An example data set involving leverage-masking collinearity

Figure 1. Example data sets involving three types of collinearity-influential observations

Then the goal becomes introducing a framework to compare estimators when data involves various levels of structured contamination as listed in Table 1. In the table, first column represents enumerated cases of distinct contaminations, second column is the collinearity level computed by using the data set that excludes leverage observations, third column is the type(s) of outlier(s) in data, and fourth column represents the influence of leverage observations on the collinearity (as shown in Figure 1).

To measure the performance of an estimator for each case presented in Table 1, Monte-Carlo estimates of total mean square error (*MSE*), total variance (*VAR*) and total bias (*BIAS*), [21-23], given by

$$MSE\left(\hat{\beta}\right) = E\left(\left\|\hat{\beta}-\beta\right\|^2\right) = Tr\left(\sigma^2_{\hat{\beta}}\right) + \left\|Bias\left(\hat{\beta}\right)\right\|^2 \quad (3)$$

$$E\left(\left\|\hat{\beta}-\beta\right\|^2\right) = E\left(\sum_{j=1}^{p}\left(\hat{\beta}_j-\beta_j\right)^2\right) \quad (4)$$

$$VAR = Tr\left(\sigma^2_{\hat{\beta}}\right) = \sum_{j=1}^{p}\sigma^2_{\hat{\beta}_j} \quad (5)$$

$$BIAS = \left\|Bias\left(\hat{\beta}\right)\right\|^2 = \sum_{j=1}^{p}\left(E(\hat{\beta}_j)-\beta_j\right)^2 \quad (6)$$

where $\hat{\beta}$ is the estimate of β and $\sigma^2_{\hat{\beta}}$ is the variance of $\hat{\beta}$ is used. *Tr* is the trace operator and *E* is the expectation. Finding the most efficient estimator (by means of *MSE*, *VAR* and *BIAS*) for each case in Table 1 requires the knowledge of the following properties of the data: (1) ratio of outliers in data, (2) ratio of high-leverage points in data, (3) type of leverage points (masking, inducing or intensifying), (4) level of collinearity in data with and without the leverage points, (5) type of outlier(s) in data (Type 1, Type 2 or both). Thus, in Section 3.1, we will summarize our methodology by first describing the steps for generating synthetic data with known properties. Furthermore, the synthetic data will then be used to construct a lookup table that shows the estimator performance for each case in Table 1.

Table 1. 15 different cases of data contamination investigated in this study

Case no:	Collinearity level	Outlier type	Leverage points
1	high	-	leverage-masking
2	high	Type 1	leverage-masking
3	high	Type 2	leverage-masking
4	high	Type 1 and 2	leverage-masking
5	low	-	leverage-masking
6	low	Type 1	leverage-masking
7	low	Type 2	leverage-masking
8	low	Type 1 and 2	leverage-masking
9	moderate	-	leverage-intensifying
10	moderate	Type 1	leverage-intensifying
11	moderate	Type 2	leverage-intensifying
12	moderate	Type 1 and 2	leverage-intensifying
13	high	Type 2	-
14	low	Type 2	-
15	moderate	Type 2	-

In order to conduct a formal study, we first introduce a subset of ridge, robust and ridge-type robust estimators in Section 2. In Section 3, we present the details for generating synthetic data and a simulation study for comparing the estimator performances. Moreover, we discuss an application of the proposed framework to real data. Finally, Section 4 concludes the study and discusses possible future work directions.

2. Estimators

In this section we present the details of commonly used estimators compared in our simulation studies. First, we will describe the details of ridge estimators and summarize three of them using different ridge parameters. Second we will discuss five frequently used robust estimators - *Least Median Square, Re-weighted Least Square, Least Trimmed Squares, M-estimator and S-estimator*. Finally, we will present three ridge-type robust estimators.

2.1. Ridge Estimators

The ridge regression estimator is introduced in [3] as,

$$\hat{\beta}_{RR} = \left(X'X + kI\right)^{-1} X'Y \tag{7}$$

where I is the identity matrix and k is the ridge parameter that controls the bias of the regression. Many different methods are given in the literature to determine the value of the parameter k. In [4], [10], and [9] the ridge parameter is given by

$$\hat{k}_1 = \frac{p\hat{\sigma}^2}{\hat{\beta}'_{OLS}\hat{\beta}_{OLS}}, \tag{8}$$

$$\hat{k}_2 = \frac{p\hat{\sigma}^2}{\sum_{j=1}^{p} \lambda_j \hat{\beta}^2_{OLS_j}}, \tag{9}$$

and

$$\hat{k}_3 = \frac{\hat{\sigma}^2}{\min\left(\frac{\hat{\sigma}^2}{\hat{\beta}^2_{OLS_j}} + \frac{1}{\lambda_j}\right)} \tag{10}$$

respectively. Here, λ_j is the j^{th} eigenvalue of the matrix $X'X$ where $j = 1,\dots,p$ and $\hat{\sigma}^2$ is given as follows

$$\hat{\sigma}^2 = \left(Y - X\hat{\beta}_{OLS}\right)'\left(Y - X\hat{\beta}_{OLS}\right)/\left(n-p\right) \tag{11}$$

In Section 1, we have introduced $\hat{\beta}_{OLS}$ in Equation (2). Henceforth, $\hat{\beta}_{RR1}$, $\hat{\beta}_{RR2}$ and $\hat{\beta}_{RR3}$ denote the ridge estimators computed by the ridge parameters $\hat{k}_1$, $\hat{k}_2$ and $\hat{k}_3$, respectively.

2.2. Robust Estimators

In what follows, we describe frequently used robust estimators in literature to estimate parameter β in Equation (1), [7,13,24].

- Least median square (*LMS*)

One of the most commonly used robust estimator is the *LMS* estimator, [13]. For our purposes, we summarize the steps of *LMS* algorithm as follows.

Step 1: Generate all possible subsamples of observations with size p from n, and randomly select m subsamples out of all generated subsamples.

Step 2: Perform regression analysis for m distinct subsamples (with size p).

Step 3: For each regression compute the residual r_i of i^{th} observation where $i = 1,\dots,n$.

Step 4: Solve the objective function $min_{\hat{\beta}^t_{OLS}}\left\{median_i\left(r_i^2\right)\right\}$ where $t = 1,\dots,m$ and $\hat{\beta}^t_{OLS}$ is the ordinary least square estimate $\hat{\beta}_{OLS}$ for the t^{th} subsample.

Step 5: Finally, $\hat{\beta}^t_{OLS}$ that minimizes the objective function is referred to as $\hat{\beta}^t_{LMS}$.

Furthermore, *LMS* estimate of the variance is calculated by

$$\hat{\sigma}_{LMS}^2 = \frac{\sum_{i=1}^{n} w_i r_i^2}{\sum_{i=1}^{n} w_i - p} \tag{12}$$

[7,24] where the weights are given as

$$w_i = \begin{cases} 1, & \left| r_i / \hat{\sigma}_{LMS} \right| \le 2.5 \\ 0, & \text{otherwise} \end{cases}. \tag{13}$$

Note that only for the first iteration we compute S_0,

$$S_0 = 1.4826(1 + \frac{5}{n-p})\sqrt{median(r_i^2)}, \tag{14}$$

and replace it by $\hat{\sigma}_{LMS}$ in Equation (12).

• Re-weighted least square (*RLS*)

In order to compute *RLS* estimator [13], the goal becomes solving another objective function

$$\min_{\hat{\beta}_\ell} \sum_{i=1}^{n} w_i r_i^2, \quad i = 1,...,n. \tag{15}$$

In Equation (15), $\hat{\beta}_\ell$ is the well-known *weighted least square* estimator calculated from the ℓ^{th} iteration where w_i and r_i are the weight and residual for each observation, respectively. The $\hat{\beta}_\ell$ that minimizes Equation (15) is defined as *RLS* estimator $\hat{\beta}_{RLS}$. The *RLS* estimator of variance is computed by

$$\hat{\sigma}_{RLS}^2 = \frac{\sum_{i=1}^{n} w_i r_i^2}{\sum_{i=1}^{n} w_i - p} \tag{16}$$

[7,24], and the weights are given as

$$w_i = \begin{cases} 1, & \left| r_i / \hat{\sigma}_{RLS} \right| \le 2.5 \\ 0, & \text{otherwise} \end{cases}. \tag{17}$$

Only for the initial iteration $\ell = 1$, $\hat{\beta}_{LMS}$ and associated residuals are used as initial conditions and $\hat{\sigma}_{RLS}^2$ is replaced by $\hat{\sigma}_{LMS}^2$ in Equation (17).

• Least trimmed squares (*LTS*)

For the LTS estimator, [13], we generate subsamples of data with size $h = \lfloor n/2 \rfloor + \lfloor (p+1)/2 \rfloor$ from *n* observations, where total number of subsamples is *m* and each subsample is enumerated as $t = 1,...,m$. For a subsample, we perform OLS regression analysis and compute residuals for *n* observations. Moreover, square of the residuals are ordered such that $r_{1:n}^2 \le r_{2:n}^2 \le ... \le r_{n:n}^2$ and $\hat{\beta}_t$ that minimizes

$$\min_{\hat{\beta}_k} \sum_{i=1}^{h} r_{i:n}^2 \tag{18}$$

is the *LTS* estimator $\hat{\beta}_{LTS}$. Note that when *m* is too large, a fixed number of randomly selected subsamples are used as an approximation to the optimal solution of $\hat{\beta}_t$ for computational reasons. In [13], the details for finding the number of randomly selected subsamples are discussed for a desired probability of distance to the optimal value.

• *M*-estimator

This type of robust estimator is obtained from the solution of

$$\min_{\hat{\beta}_l} \sum_{i=1}^{n} \rho(r_i), \tag{19}$$

such that $\hat{\beta}_\ell$ computed at the ℓ^{th} iteration is defined as *M*-estimator, $\hat{\beta}_M$ [24]. The details of the stopping rule that determines ℓ could be found in [24]. In the first iteration $\hat{\beta}_{LMS}$ is used as the initial point and for the rest of the iterations weighted least squares is estimated by using

$$\hat{\beta}_\ell = \left(X'W^{\ell-1}X \right)^{-1} X'W^{\ell-1}Y \tag{20}$$

where $W^{\ell-1}$ is the current diagonal weight matrix with diagonal elements $w(r_i)$, [13]. In this study Tukey's bi-weight function

$$\psi(r_i) = \begin{cases} r_i(1 - r_i/c)^2, & |r_i| \le c \\ 0, & |r_i| > c \end{cases} \tag{21}$$

where the derivative $\rho'(r_i) = \psi(r_i)$ and the weights $w(r_i) = \psi(r_i)/r_i$ is used, [13]. Finally, in simulations the constant *c* is selected as 1.547 (the reason for this selection will be discussed in the following section)

• *S*-estimator

S-estimator is proposed in [25] and it is computed by solving

$$\min_{\hat{\beta}_l} s(r_1(\beta), r_2(\beta), ..., r_n(\beta)) \tag{22}$$

where $s(r_1(\beta), r_2(\beta), ..., r_n(\beta))$ is the estimate of the variance of the residuals. Here, in the iterative solution $\hat{\beta}_{LMS}$ is also used as an initial point and $\hat{\beta}_\ell$ obtained from the ℓ^{th} iteration is defined as *S*-estimator, $\hat{\beta}_S$.

To compute *s*, in each iteration, the equation

$$K = \frac{1}{n} \sum_{i=1}^{n} \rho\left(\frac{r_i}{s(r_1(\hat{\beta}), r_2(\hat{\beta}), ..., r_n(\hat{\beta}))}\right) \tag{23}$$

is solved. Then the new weighted least squares is estimated by Equation (20). Here K is set to $\int \rho(x) d_{\varphi(x)}$ such that $\hat{\beta}_S$ and $s(r_1(\beta_S), r_2(\beta_S), ..., r_n(\beta_S))$ are asymptotically consistent estimates of β and σ^2 for the Gaussian regression model and is usually taken as $E_\phi(\rho)$. $\phi(x)$ is the standard normal distribution. In this study, Tukey's biweight function (Equation (21)) is used with $c = 1.547$. This value is selected such that $K/\rho(c)$ becomes 0:5 (breakdown point of S estimator), [24,25]. Same value of c is used in both S and M-estimator in order to be able to compare their performances.

- Ridge-type robust estimators

Ridge-type robust estimator is proposed in [26] and it is computed by

$$\hat{\beta}_{RTR} = \left(X'WX + \hat{k}_R \right)^{-1} X'WX \hat{\beta}_R \tag{24}$$

Where $\hat{k}_R$ is obtained by

$$\hat{k}_R = \frac{p\hat{\sigma}_R^2}{\left\| \hat{\beta}_R \right\|^2} \tag{25}$$

Here, $\hat{\beta}_R$ and $\hat{\sigma}_R^2$ can be selected as any type of robust estimator. In this study we use ridge-type *RLS* (*RTRLS*), ridge-type *S* (*RTS*), and ridge-type *M* (*RTM*) to compare their performances with the rest of the estimators presented above.

3. Simulations and an Example

Here, we generate fifteen different synthetic data sets with varying levels of contamination. These data sets are then used to evaluate performances of twelve different estimators based on the Monte-Carlo estimates of *total mean square*, *total variance* and *total bias*. We show that these results can be used as lookup tables to select the best estimator for various cases of contamination. Moreover, we present an example to demonstrate an application of using these tables.

3.1. Simulations

In this study, we generate contaminated normal distributed data with three explanatory variables $p = 3$ that has the joint probability density function F for (Y, X_1, X_2, X_3) (in Equation (1)), where $F = (1-\eta)G + \eta H$. Here, $G \sim N_{p+1}(\mu, \Sigma)$, $H \sim N_{p+1}(\theta, \Sigma)$ and $\eta \in [0,1]$ is the mixture parameter that satisfies $\eta << 1$, [24]. Here, the location parameters μ and θ are used as design specifications. We parse them into elements such that $\mu = (\mu_Y, \mu_X)'$ and $\theta = (\theta_Y, \theta_X)'$ where $\mu_X = (\mu_{X_1}, \mu_{X_2}, \mu_{X_3})$ and $\theta_X = (\theta_{X_1}, \theta_{X_2}, \theta_{X_3})$, respectively. μ_Y (or θ_Y) is the mean of Y for distribution G (or H). Thus this parsing will allow us to use the set of design parameters $\mu_Y, \mu_{X_1}, \mu_{X_2}, \mu_{X_3}, \theta_Y, \theta_{X_1}, \theta_{X_2}, \theta_{X_3}$ to manipulate the level and the type of contamination corresponding to the cases presented in Table 1.

In the simulations $\mu = (10, 5, 2, 7)$. To generate data with *leverage-masking*, *leverage-inducing*, and *leverage-intensifying collinearity* θ and Σ are selected as

- $\theta = (\theta_Y, \theta_{X_1}, \theta_{X_2}, \theta_{X_3}) = (10, 5, 7, 7)$

$$\Sigma = \begin{bmatrix} \sigma_Y^2 & \Sigma_{Y,X} \\ \Sigma_{X,Y} & \Sigma_{X,X} \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.9000 & 0.9000 & 0.9000 \\ 0.9000 & 1.0000 & 0.9990 & 0.9000 \\ 0.9000 & 0.9990 & 1.0000 & 0.9000 \\ 0.9000 & 0.9000 & 0.9000 & 1.0000 \end{bmatrix},$$

- $\theta = (\theta_Y, \theta_{X_1}, \theta_{X_2}, \theta_{X_3}) = (10, 35, 32, 37)$

$$\Sigma = \begin{bmatrix} \sigma_Y^2 & \Sigma_{Y,X} \\ \Sigma_{X,Y} & \Sigma_{X,X} \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.9000 & 0.7100 & 0.4100 \\ 0.9000 & 1.0000 & 0.4000 & 0.2000 \\ 0.7100 & 0.4000 & 1.0000 & 0.2000 \\ 0.4100 & 0.2000 & 0.2000 & 1.0000 \end{bmatrix},$$

- $\theta = (\theta_Y, \theta_{X_1}, \theta_{X_2}, \theta_{X_3}) = (10, 10, 7, 7)$

$$\Sigma = \begin{bmatrix} \sigma_Y^2 & \Sigma_{Y,X} \\ \Sigma_{X,Y} & \Sigma_{X,X} \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.9107 & 0.9351 & 0.7223 \\ 0.9107 & 1.0000 & 0.9706 & 0.7753 \\ 0.9351 & 0.9706 & 1.0000 & 0.7824 \\ 0.7223 & 0.7753 & 0.7824 & 1.0000 \end{bmatrix},$$

respectively. In order to generate data including leverage-masking collinearity, we use a covariance matrix (Σ) with diagonal elements 1. Non-diagonal elements are close to 1 which guarantees strong collinearity between explanatory variables. Moreover, high-leverage observations drawn from the distribution $H \sim N_{p+1}(\theta, \Sigma)$ with $\theta = (10, 5, 7, 7)$ are added to data with ratio η. $\hat{VIF}_G$ and $\hat{VIF}_F$ are estimates of *VIF* computed from the observations generated from G and F, respectively. For instance, one may observe from Table 3 (Appendices) that even when η is small $\hat{VIF}_F$ is much smaller than $\hat{VIF}_G$. Hence, a small number of high-leverage observations can mask the underlined strong collinearity associated with the majority of the data. In leverage inducing collinearity case, the covariance matrix has smaller non-diagonal elements and the corresponding *VIF* (value computed from Σ) is smaller which implies no collinearity between explanatory variables. Similarly, high-leverage observations drawn from distribution H and with a different $\theta = (10, 35, 32, 37)$ added to data with ratio η. In this case, we observe that $\hat{VIF}_F$ value is much higher than $\hat{VIF}_G$ computed without leverage points. This reveals that a small set of high-leverage observations can induce collinearity. A similar approach is taken for the third case of manipulation in which the high-leverage points intensify the strength of the collinearity.

In what follows, we will describe the steps for generating data involving all combinations of contamination consisting of outliers and high-leverage points. Below, we list five manipulation methods used in our simulations.

(1) The data (with size n) is generated by having only high-leverage points (masking, inducing or intensifying) with ratio η. To achieve this, leverage points are drawn from the distribution $N_p\left(\theta_X, \Sigma_{XX}\right)$ and non-leverage points from $N_p\left(\mu_X, \Sigma_{XX}\right)$. Then the observations of dependent variable for the leverage points are drawn from $N\left(\mu_{Y\backslash X}, \sigma^2_{Y\backslash X}\right)$ where $\mu_{Y\backslash X} = \mu_Y - \Sigma_{YX}\Sigma_{XX}^{-1}(X - \theta_X)$ and $\sigma^2_{Y\backslash X} = \sigma^2_Y - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}$. Non-leverage points are also drawn from the same distribution $N\left(\mu_{Y\backslash X}, \sigma^2_{Y\backslash X}\right)$ by just replacing θ_X with μ_X.

(2) The data is generated by having only Type 1 outliers, which are also high-leverage points, with ratio η. The observations of dependent variable associated with these leverage points are drawn from $N\left(10\mu_{Y\backslash X}, \sigma^2_{Y\backslash X}\right)$. Here, $\mu_{Y\backslash X} = \mu_{YX} - \Sigma_{XX}^{-1}(X - \theta_X)$ and $\sigma^2_{Y\backslash X} = \sigma^2_Y - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}$. Rest of the observations are drawn from $N\left(\mu_{Y\backslash X}, \sigma^2_{Y\backslash X}\right)$ where $\mu_{Y\backslash X} = \mu_Y - \Sigma_{YX} - \Sigma_{XX}^{-1}(X - \mu_X)$ and same variance $\sigma^2_{Y\backslash X}$.

(3) The data is generated by having only Type 2 outliers with ratio η that are drawn from $N\left(10\mu_{Y\backslash X}, \sigma^2_{Y\backslash X}\right)$ where $\mu_{Y\backslash X} = \mu_Y - \Sigma_{YX} - \Sigma_{XX}^{-1}(X - \mu_X)$ and $\sigma^2_{Y\backslash X} = \sigma^2_Y - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}$. Rest of the observations are generated from $N\left(\mu_{Y\backslash X}, \sigma^2_{Y\backslash X}\right)$ where $\mu_{Y\backslash X} = \mu_Y - \Sigma_{YX} - \Sigma_{XX}^{-1}(X - \mu_X)$ and same variance $\sigma^2_{Y\backslash X}$.

(4) The data is generated by having both high-leverage points and Type 2 outliers with ratio $\eta/2$. To achieve this, a combination of manipulations (1) for high-leverage points and (3) for Type 2 outliers is used. That is, leverage points are drawn from the distribution $N_p\left(\theta_X, \Sigma_{XX}\right)$. Type 2 outliers with ratio $\eta/2$ are drawn from $N\left(10\mu_{Y\backslash X}, \sigma^2_{Y\backslash X}\right)$ where $\mu_{Y\backslash X} = \mu_Y - \Sigma_{YX} - \Sigma_{XX}^{-1}(X - \mu_X)$ and $\sigma^2_{Y\backslash X} = \sigma^2_Y - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}$. Rest of the observations with ratio $1-\eta$ are generated from $N\left(\mu_{Y\backslash X}, \sigma^2_{Y\backslash X}\right)$ where $\mu_{Y\backslash X} = \mu_Y - \Sigma_{YX} - \Sigma_{XX}^{-1}(X - \mu_X)$ and same variance $\sigma^2_{Y\backslash X}$.

(5) The data is generated by having both Type 1 and Type 2 outliers each with ratio high-leverage points and Type 2 outliers with ratio $\eta/2$. To achieve this a combination of manipulations (2) and (3) is used. For Type 1 outliers, we generate high-leverage points from $N_p\left(\theta_X, \Sigma_{XX}\right)$ with also ratio $\eta/2$. The observations of dependent variable associated with these leverage points are drawn from $N\left(10\mu_{Y\backslash X}, \sigma^2_{Y\backslash X}\right)$. Here, $\mu_{Y\backslash X} = \mu_Y - \Sigma_{YX} - \Sigma_{XX}^{-1}(X - \theta_X)$ and $\sigma^2_{Y\backslash X} = \sigma^2_Y - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}$. Moreover, Type 2 outliers are generated from $N\left(10\mu_{Y\backslash X}, \sigma^2_{Y\backslash X}\right)$ with ratio $\eta/2$ where $\mu_{Y\backslash X} = \mu_Y - \Sigma_{YX} - \Sigma_{XX}^{-1}(X - \mu_X)$ and $\sigma^2_{Y\backslash X} = \sigma^2_Y - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}$. Rest of the observations with ratio $1-\eta$ are generated from $N\left(\mu_{Y\backslash X}, \sigma^2_{Y\backslash X}\right)$ where $\mu_{Y\backslash X} = \mu_Y - \Sigma_{YX} - \Sigma_{XX}^{-1}(X - \mu_X)$ and same variance $\sigma^2_{Y\backslash X}$.

Each manipulation is performed for leverage-masking, inducing and intensifying observations. Hence, in total there are 15 cases to be investigated as presented in Table 1 for which number of iterations is fixed to 10000 in order to compute the Monte-Carlo estimations of *MSE*, *VAR*, and *BIAS* given in Equation (4)-(6). Moreover, each case is investigated for various values of mixture parameter η that contributes to the level of the manipulation. We present simulation results for the case when $n = 50$ and $p = 3$ since we observe that the results were similar for various (n, p) combinations.

Structurally, we first generate data with only leverage-masking collinearity associated with the cases 1 to 4 in Table 1 and present our results in Tables 3-6. (*Appendices*). Similarly, same steps are followed for leverage-inducing and leverage intensifying collinearity associated with the cases 5 to 8 and 9 to 12 in Table 1, and the results are shown in Tables 7-10 (*Appendices*) and 11-14 (*Appendices*), respectively. Finally, for the cases 13-15 in which estimator performances are compared when data has only Type 2 outliers with no leverage observations but relatively high, low and moderate level of collinearity the results are presented in Tables 15-17 (*Appendices*). In all of the Tables 2-17, *OLS*, *RR1-RR2-RR3-RLS-LMS-LTS-M-S*, and *RTRLS-RTM-RTS* denote the ordinary least squares, ridge, robust, and ridge-type robust estimators, respectively, presented in Section 2. Furthermore, in the top row of each table simulation parameters (η, β, R^2) as well as the known *VIF* value calculated from the covariance matrix (Σ) are shown. In the second column of Tables 3-17, $\hat{VIF}_G$ and $\hat{VIF}_F$ are computed from the first iteration of Monte-Carlo simulation in which the data is generated from G and F distributions, respectively. Each table presents the results based on *MSE*, *VAR* and *BIAS* computed by Equation (4)-(6).

Let us first consider the cases when data only involves leverage observations. Figure 2a illustrates that the estimator RR3 (shown with red) performs the best based on variance. However, in a practical application, since $\hat{VIF}_F$ value (computed from all of the observations, Table 3) is relatively small due the masking effect of leverage masking observations and there are no outliers, *OLS* (shown in blue) is expected to outperform the rest of the estimators, [6,8]. In contrast, in Figure 2c we observe that *OLS* performs the best even though ridge estimators (shown in red) are expected to perform the best for high $\hat{VIF}_F$ value (Table 7) and no outliers [3]. Hence, we conclude that a small ratio of leverage-inducing observations in data causes a drastic increase in $\hat{VIF}_F$ that is used to select estimators in practical applications. In Figure 2e there are only leverage-intensifying observations that strengthens an existing high level of collinearity (as one may observe from $\hat{VIF}_F$ in Table 11) and we conclude that ridge estimator performs the best.

If we observe Figure 2b and 2d a similar argument can be made for data that involves both high-leverage observations as well as outliers in X and Y directions. For instance, based on low $\hat{VIF}_F$ value (Table 6) due to the masking effect of leverage masking collinearity, one may expect robust estimators to be the best performing estimator, [13]. However, Figure 2b illustrates that for varying range of mixture parameter ridge-type *S* estimator (*RTS*) outperforms the rest. Similarly, when the leverage observations induce collinearity (high $\hat{VIF}_F$ value in Table 10), Figure 2d shows that robust estimator (*RLS*) performs better for $\eta <$ 0:30. This validates that observations classified as leverage-inducing collinearity are misleading since one expects ridge-type robust estimators to be more efficient for data sets with high collinearity and outliers [5,6].

Hence, we conclude that for practical applications in order to select the best estimator with respect to variance, $\hat{VIF}$ value should be computed by extracting the leverage points from the data for the best results. This suggests the use of robust $\hat{VIF}$ calculations as we will discuss in Subsection 3.2. However, the precise contribution of robust $\hat{VIF}$ computations to solve estimator selection problem is yet to be understood and this will be the subject of our future work.

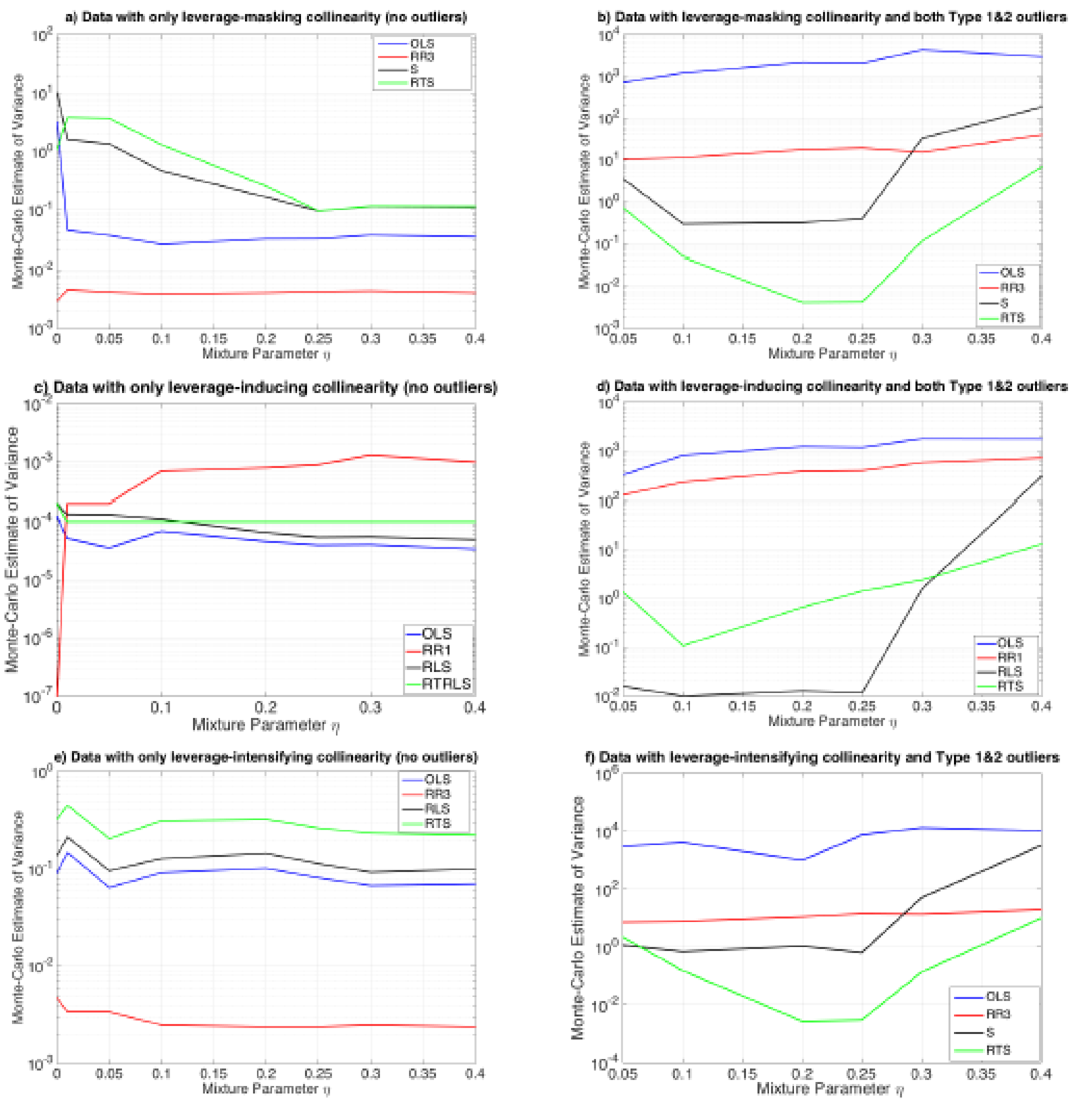


Figure 2. 6 plots illustrating the logarithm of the Monte-Carlo estimates of variance as a function of mixture parameter η . Color coding in the figures is arranged such that *OLS*, ridge, robust, and ridge-type robust estimators are represented by blue, red, black, and green, respectively. The set of parameters used in each subplot (a)-(f) are presented in Tables 3, 6, 7, 10, 11, and 14, respectively

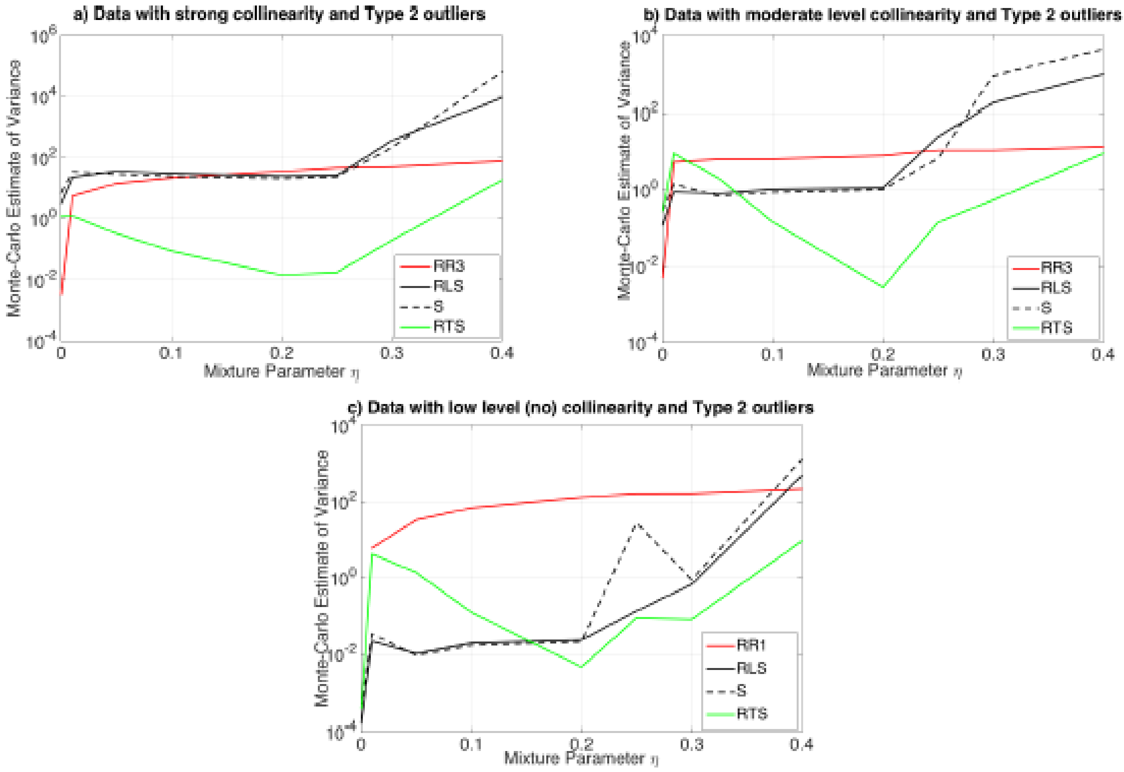


Figure 3. 3 plots illustrating the logarithm of the Monte-Carlo estimates of variance as a function of mixture parameter η . Color coding in the figures is arranged such that ridge, robust, ridge-type robust estimators are represented by red, black and blue, green, respectively. The set of parameters used in each subplot (a), (b), (c) are presented in Tables 15, 17, 16, respectively

In Figure 3a-3c, we present results gathered from Tables 15-17, respectively. In Figure 3a, it is observed that ridge-type *S* estimator performs the best based on variance (*RTS* shown in green). Moreover, Figure 3b illustrates that for small values of $\eta < 0.07$ robust estimators (*RLS* and *S* shown in black solid and dashed lines) but for higher values of $\eta > 0.07$ ridge-type *S* estimator (*RTS*) has the lowest variance. Finally, Figure 3c, shows a similar trend around $\eta = 0.16$.

These results validate that when there are outliers only in *Y* direction, as the level of collinearity increases ridge-type *S* estimator (*RTS*) performs better with lower variance than robust estimators (*S* and *RLS*). Hence, the strength of collinearity is also critical for selecting estimators.

Remark: In this section, we have compared estimator performances only with respect to their variances. However, in some practical applications bias-variance trade-off should be taken into account for selecting the appropriate estimator. Hence, we presented *BIAS* and *MSE* values computed from Monte-Carlo estimates as well as *VAR* in the Tables 3-17.

3.2. Example

In this section, we compare estimator performances (presented in Section 2) when they are applied to the data set introduced in [27]. This synthetic regression data is constructed such that there are three explanatory variables and in total $n = 75$ observations that are enumerated from 1-75 such that the observations 1-10 are outliers in Type 1 direction ($\eta = 0.13$). Moreover, data is constructed such that the following 4 observations between 11 and 14 are leverage observations $\hat{VIF}$ computed from 75 observations is 23.6842. This reveals that collinearity exists in data. $\hat{VIF}$ obtained by excluding both outliers and leverage observations is found as 1.0163. Hence, we classify these (in total 15) observations as leverage-inducing collinearity. If we exclude only leverage observations and outliers, $\hat{VIF}$ values are 34.1178 and 13.0166, respectively. These results indicate that Type 1 outliers induce collinearity more prominently than the leverage observations.

Using this knowledge, one may use Table 8 (for data involving Type 1 outliers as well as leverage-inducing collinearity) which concludes that *RLS* estimator performs the best based on variance when mixture parameter is in the range $0.01 < \eta < 0.25$. Moreover, for the same range of η we observe from Table 8 that *RLS* and *RTRLS* have approximately the same value of *VAR* but *RLS* has lower *BIAS* and lower *MSE* values.

In Table 2 (Appendices), we present results gathered from distinct methods to estimate the regression parameters for $Y = \beta_0 + \beta_1 X_2 + \beta_3 X_3 + \varepsilon$ as well as $\hat{VIF}_E$ values. $\hat{VIF}_E$ is the estimate of VIF . Moreover, it is computed by the set of observations used for calculating $\hat{\beta}$ for each estimation method. In Table 2, $\hat{VIF}_E$ value computed from *RLS* is equal to 13.0166. This is the same value computed by excluding only Type 1 outliers from data. Hence, this suggests that *RLS* estimator uses the subsample excluding all Type 1 observations which maximize its efficiency and this is in line with our results presented in Table 8.

However, it is non-trivial to make the same conclusion for the rest of the robust estimators. Note that, since ridge estimators (*RR1*, *RR2*, *RR3*) use all the data their $\hat{VIF}_E$ values (in Table 2) are found as 23.6842 which is estimated from the total data set with 75 observations. Ridge-type robust estimators *RTRLS*, *RTM* and *RTS* have the same $\hat{VIF}_E$ values as *RLS*, *M* and *S* in that they use the same subsample.

4. Conclusions

This study aims to construct a framework in order to compare estimator performances for various levels of contamination caused by collinearity, collinearity-influential observations and outliers in data. Furthermore, it targets to investigate the influence of their interactions by examining fifteen different data sets. These synthetically generated data sets involve distinct combinations of outliers classified as Type 1 or 2 based on their distance, and three distinct collinearity-influential observations, namely, leverage masking, leverage-inducing and leverage-intensifying collinearity. For the analysis, we use ridge, robust and ridge-type robust estimators from the literature and compute the Monte-Carlo estimates of total mean square, total variance and total bias for evaluating their estimation performances.

We compare estimators based on variance and observe that when data involves high collinearity and only leverage-masking observations (no outliers), *RR3* (ridge estimator) has the smallest variance. However, in a practical application due to the masking effect of the leverage observations, $\hat{VIF}$ would be small and *OLS* estimator would be expected to perform the best for no collinearity and no outliers case, [6, 8]. Similar effects are observed when data involves leverage-inducing observations. Hence, we conclude that the interactions between collinearity and collinearity-influential observations can be misleading when selecting an estimator. These results reveal that $\hat{VIF}$, which is used for estimator selection, should be computed robustly by excluding the leverage points from the data set.

We also observe that directionality in outliers and the strength of collinearity in data are also relevant for estimator selection. We show that the results presented in this study can be used as lookup tables to decide on the best estimator based on total variance, total mean square error and total bias values.

Appendices

Table 2. Parameter estimations and $\hat{VIF}_E$ values for the data

	OLS	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
$\hat{\beta}_0$	−0.3866	0.2893	1.2756	1.2746	−0.1798	−0.5224	−0.5896	−0.5404	−0.1800	−0.1592	−0.4583	−0.0159
$\hat{\beta}_1$	0.2459	0.4116	0.9759	0.9952	0.0833	0.2882	0.2097	0.1952	0.0834	0.0727	0.1538	0.0046
$\hat{\beta}_2$	−0.3347	0.3745	−2.4001	−2.3013	0.0398	0.0457	0.0791	0.0502	0.0398	0.0378	0.0439	0.0105
$\hat{\beta}_3$	0.3811	0.4210	4.0419	3.9238	−0.0523	−0.1126	−0.1194	− 0.0867	−0.0523	−0.0480	−0.0713	−0.0101
$\hat{VIF}_E$	23.6842	23.6842	23.6842	23.6842	13.0166	100.9109	72.2954	14.4149	13.0194	13.0166	14.4149	13.0194

Table 3. Case 1: *MSE*, *VAR*, *BIAS* and $\hat{VIF}_F$ values when data involves only leverage-masking collinearity (no outliers)

			$n=50, R^2=0.8527, VIF=501.3193, \beta=[5.0298 \quad 0.2375 \quad 0.2375 \quad 0.4726]$											
η	$\hat{VIF}_G$ $\hat{VIF}_F$		*OLS*	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
0	496.3414	*MSE*	3.1753	23.1763	23.2682	20.6044	4.4854	16.2551	18.7782	14.6596	10.1293	7.0128	11.4274	12.6297
	496.3414	*VAR*	3.1738	0.3389	0.4226	0.0030	4.4819	16.2439	18.7963	14.6510	10.1192	2.1351	2.5846	1.1034
		BIAS	0.0014	4.7788	4.7797	4.5388	0.0591	0.1058	0.0274	0.0927	0.0151	2.2085	2.9477	3.3950
0.01	571.9829	*MSE*	0.0451	26.0160	26.0232	23.6590	2.4058	8.8678	11.5210	7.9443	1.6027	2.7540	4.3902	6.0464
	5.0794	*VAR*	0.0451	0.0064	0.0064	0.0046	2.3962	8.8723	11.5043	7.9376	1.6025	2.1990	3.1204	3.8046
		BIAS	0.0020	5.0999	5.1006	4.8635	0.0979	0.0622	0.1292	0.0818	0.0141	0.7449	1.1268	1.4972
0.05	522.0611	*MSE*	0.0369	26.5338	26.5410	24.1553	1.9752	8.8006	9.1045	8.3059	1.3470	2.3198	4.4066	5.5537
	4.6591	*VAR*	0.0368	0.0057	0.0058	0.0042	1.9733	8.7927	9.1014	8.2976	1.3459	1.8892	3.1454	3.6806
		BIAS	0.0100	5.1505	5.1512	4.9143	0.0435	0.0888	0.0556	0.0911	0.0331	0.6562	1.1230	1.3686
0.10	543.2944	*MSE*	0.0262	26.6435	26.6508	24.2517	0.6390	1.6505	2.5166	2.1801	0.4641	0.5565	1.0352	1.4895
	4.7935	*VAR*	0.0261	0.0054	0.0054	0.0039	0.6384	1.6494	2.5145	2.1783	0.4637	0.5530	0.9544	1.3018
		BIAS	0.0100	5.1612	5.1619	4.9242	0.0244	0.0331	0.0458	0.0424	0.0200	0.0591	0.2842	0.4332
0.20	516.9158	*MSE*	0.0322	23.6475	23.6542	21.3373	0.1887	0.3915	0.5909	0.4234	0.1654	0.1771	0.3285	0.2901
	5.6986	*VAR*	0.0322	0.0063	0.0063	0.0041	0.1886	0.3911	0.5905	0.4229	0.1653	0.1563	0.3069	0.2550
		BIAS	0.0000	4.8625	4.8629	4.6187	0.1000	0.0200	0.0200	0.0223	0.0100	0.1442	0.1469	0.1873
0.25	542.7095	*MSE*	0.0324	0.0062	0.0063	0.0043	0.0437	0.1610	0.1709	0.1424	0.0960	0.0451	0.1421	0.0963
	5.7860	*VAR*	0.0324	0.0062	0.0063	0.0043	0.0437	0.1610	0.1709	0.1424	0.0960	0.0451	0.1421	0.0963
		BIAS	0.0100	5.8671	5.5165	5.2818	0.0100	0.0141	0.0141	0.0100	0.0000	0.1545	0.1095	0.1466
0.30	503.0554	*MSE*	0.0373	28.0853	28.0929	25.6103	0.0517	0.1897	0.2083	0.1675	0.1148	0.0768	0.1816	0.1467
	6.0808	*VAR*	0.0373	0.0667	0.0067	0.0044	0.0517	0.1895	0.2082	0.1673	0.1147	0.0541	0.1671	0.1154
		BIAS	0.0000	5.2989	5.2996	5.0602	0.0000	0.0141	0.0100	0.0141	0.0100	0.1506	0.1204	0.1769
0.40	546.9026	*MSE*	0.0349	29.3827	29.3906	26.8942	0.0486	0.1869	0.2385	0.1719	0.1123	0.0734	0.1984	0.1581
	4.4796	*VAR*	0.0349	0.0055	0.0055	0.0041	0.0485	0.1866	0.2381	0.1713	0.1120	0.0526	0.1719	0.1143
		BIAS	0.0000	5.4200	5.4208	5.1853	0.0100	0.0173	0.0141	0.0244	0.0173	0.1442	0.1627	0.2092

Table 4. Case 2: *MSE*, *VAR*, *BIAS* and $\hat{VIF}_F$ values when data involves leverage-masking collinearity and Type 1 outliers

							$n = 50, R^2 = 0.8527, VIF = 501.3193, \beta = [5.0298 \quad 0.2375 \quad 0.2375 \quad 0.4726]$							
η	$\hat{VIF}_G$ $\hat{VIF}_F$		*OLS*	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
0	529.7096	*MSE*	2.7029	25.5297	25.6452	22.7997	3.6606	13.7355	15.0799	11.5488	8.1338	5.6640	9.6678	11.0165
	529.7046	*VAR*	2.7004	0.4305	0.5381	0.0044	3.6579	13.7302	15.0656	11.5363	8.1301	2.1890	2.8968	1.2274
		BIAS	0.0500	5.0099	5.0106	4.7744	0.0519	0.0728	0.1195	0.1118	0.0608	1.8641	2.6021	3.1287
0.01	506.8597	*MSE*	234.7457	52.0732	71.6673	15.6463	18.8920	76.3627	83.2257	67.9060	8.0675	8.5941	11.7337	16.8300
	7.3117	*VAR*	234.6389	23.4638	36.7325	5.8190	18.8741	76.2910	83.1782	67.8431	8.0607	4.4120	4.5776	2.9981
		BIAS	0.3268	5.4414	5.9105	3.1348	0.0179	0.2677	0.2179	0.2507	0.0824	2.0450	2.6750	3.7191
0.05	507.2957	*MSE*	774.6900	146.6279	223.7402	9.3643	18.4033	74.2908	84.0283	63.4357	2.3853	8.0086	10.3919	23.1763
	5.7908	*VAR*	772.5825	90.7528	134.4433	9.2803	18.3478	74.2887	83.8854	63.3891	2.3718	4.1583	4.0285	0.6855
		BIAS	1.4517	7.4749	9.4497	0.2898	0.2355	0.0458	0.3780	0.2158	0.1161	1.9622	2.5225	4.8096
0.10	532.7264	*MSE*	1164.60000	277.5993	458.7673	13.8984	8.6148	34.2484	32.3658	28.4982	0.3566	3.2838	4.8290	24.2297
	6.5089	*VAR*	1148.8000	151.3497	256.9425	13.1450	8.6063	34.2141	32.3422	28.4795	0.3563	2.4646	2.5934	0.0434
		BIAS	3.9749	11.2360	14.2065	0.8679	0.0921	0.1852	0.1536	0.1367	0.0173	0.9050	1.4951	4.9179
0.20	553.9037	*MSE*	2043.7000	581.8891	891.2881	19.2273	8.4847	26.3266	28.2697	18.8594	0.4469	3.2554	5.0730	24.4059
	4.3107	*VAR*	1960.3000	233.3516	362.1131	18.5762	8.4638	26.3109	28.2437	18.8409	0.4467	2.3912	2.0658	0.0044
		BIAS	9.1323	18.6691	23.0038	0.8069	0.1445	0.1252	0.1612	0.1360	0.0141	0.9296	1.7341	4.9397
0.25	553.3740	*MSE*	2817.2000	808.5669	1203.1000	24.7868	0.4291	1.2420	1.3164	0.8208	0.3841	0.7088	3.7290	24.4790
	5.5146	*VAR*	2711.8000	260.9063	399.4000	24.2745	0.4287	1.2417	1.3156	0.8206	0.3838	0.4422	0.8055	0.0041
		BIAS	10.2664	23.4021	28.3496	0.7157	0.0200	0.0173	0.0282	0.0141	0.0173	0.5163	1.7098	4.9472
0.30	517.1702	*MSE*	3324.9000	1064.0000	1530.0000	23.7944	2.9897	4.0234	10.3315	3.3049	3.4036	1.7266	5.0248	24.5726
	5.1232	*VAR*	3072.0000	290.3000	428.3000	23.2684	2.9895	4.0219	10.3296	3.3045	3.4026	1.4652	1.2421	0.0629
		BIAS	17.9694	27.8154	38.7517	0.7252	0.0141	0.0387	0.0435	0.0200	0.0316	0.5112	1.9449	4.9507
0.40	494.8730	*MSE*	4382.3000	1713.8000	454.2000	32.5802	201.4496	214.9626	504.5490	208.0729	687.1087	160.2099	97.4845	31.9000
	4.0389	*VAR*	4020.8000	362.5000	344.0000	32.5375	187.6392	200.8633	479.1744	193.8089	674.8161	154.0125	96.0372	7.5042
		BIAS	19.0131	36.7600	10.4976	0.2066	3.7162	3.7549	5.0373	3.7767	3.5060	2.4894	1.2030	4.9392

Table 5. Case 3: *MSE*, *VAR*, *BIAS* and $\hat{VIF}_F$ values when data involves leverage-masking collinearity and Type 2 outliers

							$n = 50, R^2 = 0.8527, VIF = 501.3193, \beta = [5.0298 \quad 0.2375 \quad 0.2375 \quad 0.4726]$							
η	$\hat{VIF}_G$ $\hat{VIF}_F$		*OLS*	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
0.01	272.8787	*MSE*	0.0503	25.2357	25.2427	22.9021	2.1206	8.1764	8.9112	6.8895	1.4957	2.8207	5.6036	7.7735
	4.0340	*VAR*	0.0503	0.0056	0.0056	0.0043	2.1185	8.1683	8.9056	6.8827	1.4942	2.0458	3.3809	3.7270
		BIAS	0.0000	5.0229	5.0236	4.7851	0.0458	0.0900	0.0748	0.0824	0.0387	0.8802	1.4908	2.0115
0.05	559.9026	*MSE*	118.7196	64.6932	121.4574	1.7749	1.7444	7.6584	8.8648	6.6441	0.0333	2.5933	5.1324	24.1069
	5.2819	*VAR*	0.0330	0.0043	0.0082	0.0020	1.7428	7.6469	8.8588	6.6258	0.0333	1.8359	3.1307	0.0000
		BIAS	10.8943	8.0429	11.0205	1.3315	0.0400	0.1072	0.0774	0.1352	0.0000	0.8702	1.4148	4.9098
0.10	483.4270	*MSE*	237.0975	121.6504	409.2771	1.6402	1.3128	4.2851	4.6516	3.9097	0.0380	1.3577	1.9408	24.3180
	3.7752	*VAR*	0.0329	0.0334	0.0044	0.0006	1.3116	4.2847	4.6778	3.9067	0.0380	1.2225	1.6196	0.0000
		BIAS	15.3969	11.0293	20.2304	1.2804	0.0346	0.0200	0.0616	0.0547	0.0000	0.3676	0.5667	4.9313
0.20	353.9559	*MSE*	2810.3000	758.0234	1197.4000	7.9827	0.1051	0.3190	0.3159	0.3073	0.0526	0.1018	0.2769	24.4402
	6.1584	*VAR*	0.0000	0.0084	0.0000	0.0000	0.1050	0.3188	0.3157	0.3071	0.0526	0.0958	0.1989	0.0001
		BIAS	53.0122	27.5320	34.6034	2.8253	0.0100	0.0141	0.0141	0.0141	0.0000	0.0774	0.2792	4.9436
0.25	505.3948	*MSE*	45.5900	129.1147	763.6897	37.2091	0.0584	0.1678	0.1853	0.1150	0.0520	0.0645	0.2266	24.4660
	3.5610	*VAR*	0.0375	0.0560	0.0041	0.0563	0.0583	0.1676	0.1851	0.1149	0.0520	0.0587	0.1256	0.0001
		BIAS	7.0393	11.3553	27.6348	6.0953	0.0100	0.0141	0.0141	0.0100	0.0000	0.0761	0.3178	4.9463
0.30	576.6891	*MSE*	362.8712	535.9134	1414.1000	0.8485	0.0690	0.1888	0.2084	0.1157	0.0641	0.0788	0.3280	24.5368
	6.5954	*VAR*	0.0296	0.0035	0.0000	0.0005	0.0689	0.1887	0.2083	0.1156	0.0640	0.0689	0.1258	0.0001
		BIAS	19.0484	23.1497	37.6045	0.9208	0.0100	0.0100	0.0100	0.0100	0.0100	0.0994	0.4496	4.9534
0.40	523.3333	*MSE*	4158.2000	2105.0000	2425.3000	10.8696	0.0588	0.1398	0.1307	0.0653	0.0562	0.0674	0.5352	24.6301
	4.5606	*VAR*	0.0000	0.0000	0.0000	0.0000	0.0588	0.1398	0.1305	0.0653	0.0562	0.0586	0.0903	0.0001
		BIAS	64.4841	45.8802	49.2473	3.2969	0.0000	0.0000	0.0141	0.0000	0.0000	0.0938	0.6670	4.9628

Table 6. Case 4: *MSE*, *VAR*, *BIAS* and $\hat{VIF}_F$ values when data involves leverage-masking collinearity and Type 1 and 2 outliers

							$n = 50, R^2 = 0.8527, VIF = 501.3193, \beta = [5.0298 \quad 0.2375 \quad 0.2375 \quad 0.4726]$							
η	$\hat{VIF}_G$ $\hat{VIF}_F$		*OLS*	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
0.05	450.8200	*MSE*	724.2683	134.0927	206.0272	10.3703	23.3461	92.5618	102.3391	77.5495	3.4574	8.8865	10.8898	23.1750
	4.9234	*VAR*	715.8429	79.7951	118.0946	10.3230	23.3459	92.4717	102.2656	77.4722	3.4540	4.5442	4.2093	0.6992
		BIAS	2.9026	7.3686	9.3772	0.2174	0.0141	0.3001	0.2711	0.2780	0.0583	2.0303	2.5846	4.7408
0.10	482.9687	*MSE*	1222.5000	237.4938	369.5466	12.3218	8.8520	34.9045	37.5003	32.8024	0.2941	3.1836	5.3833	24.2505
	3.8655	*VAR*	1205.2000	111.1412	175.3777	11.4861	8.8513	34.8917	37.4699	32.7732	0.2940	2.4032	2.8963	0.0498
		BIAS	4.1593	11.2406	13.9344	0.9141	0.0264	0.1131	0.1743	0.1708	0.0100	0.8834	1.5770	4.9194
0.20	463.0453	*MSE*	2254.5000	572.3891	840.7474	18.6280	6.4794	12.2573	17.0612	9.7103	0.3153	1.7591	3.7149	24.4456
	3.7138	*VAR*	2170.9000	199.1137	294.0424	17.5879	6.4781	12.2507	16.9889	9.6886	0.3151	1.3523	1.3695	0.0042
		BIAS	9.1433	19.3203	23.3817	1.0198	0.0360	0.0812	0.2688	0.1473	0.0141	0.6378	1.5314	4.9438
0.25	464.0975	*MSE*	2151.2000	737.2531	1105.1000	20.0360	4.7305	10.0741	10.7348	8.5565	0.3879	1.3398	3.6747	24.4650
	3.5378	*VAR*	2070.3000	221.9921	334.0000	19.2329	4.7262	10.0713	10.7324	8.5555	0.3878	1.0076	1.1203	0.0043
		BIAS	8.9944	22.6993	27.7686	0.8961	0.0655	0.0529	0.0489	0.0316	0.0100	0.5763	1.5982	4.9457
0.30	492.5841	*MSE*	4503.2000	1131.5000	1507.6000	16.0786	19.3466	39.0542	64.8170	29.5726	32.6410	7.4794	9.8347	24.6329
	3.5481	*VAR*	4306.7000	311.8000	410.7000	15.3436	19.3286	39.0188	64.7843	29.5513	32.6221	6.0876	2.4851	0.1137
		BIAS	14.0178	28.6304	33.1194	0.8573	0.1341	0.1881	0.1808	0.1459	0.1374	1.1797	2.7110	4.9516
0.40	455.5961	*MSE*	3406.3000	1610.0000	2295.7000	39.4425	195.0126	191.7856	523.9091	189.1410	193.6565	178.1621	106.7396	31.1167
	3.5675	*VAR*	2993.4000	311.7000	420.3000	39.4364	182.2653	179.5783	503.1533	176.7983	181.4105	169.7089	105.9778	6.6797
		BIAS	20.3199	36.0319	43.3058	0.0781	2.9575	3.4938	4.5558	3.5132	3.4994	2.9074	0.8728	4.9433

Table 7. Case 5: *MSE*, *VAR*, *BIAS* and $\hat{VIF}_F$ values when data involves only leverage-inducing collinearity in data (no outliers)

$n = 50, R^2 = 0.9902, VIF = 1.2121, \beta = [4.3545 \quad 0.7061 \quad 0.3894 \quad 0.1909]$

η	$\hat{VIF}_G$ / $\hat{VIF}_F$		*OLS*	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
0	1.3297	*MSE*	0.1225[a]	35.0931	35.0931	34.3358	0.0002	0.0006	0.7079 [a]	0.0005	0.3595[a]	0.0198	0.0005	0.3593[a]
	1.3297	*VAR*	0.1225[a]	0.0000	0.0000	0.0001	0.0002	0.0006	0.7077[a]	0.0005	0.3593	0.0002	0.0005	0.3590[a]
		BIAS	0.0223[a]	5.9239	5.9239	5.8596	0.0000	0.0000	0.0141[a]	0.0000	0.0141[a]	0.1400	0.0000	0.0173[a]
0.01	1.0964	*MSE*	0.5186[b]	68.0867	68.0869	64.6400	0.1339[a]	0.4997[a]	0.5630[a]	0.4596[a]	0.1845[a]	0.0198	0.4595[a]	0.1856
	29.2360	*VAR*	0.5182[b]	0.0002	0.0002	0.0019	0.1337	0.4997	0.5624	0.4594	0.1843[a]	0.0001	0.4590	0.1854
		BIAS	0.0200[b]	8.2514	8.2514	8.0397	0.0141	0.0000	0.0244	0.0141	0.0141	0.1403	0.0223	0.0141
0.05	1.1341	*MSE*	0.0358[a]	66.4768	66.4770	62.7613	0.1316[a]	0.5346[a]	0.6028[a]	0.4740[a]	0.1487[a]	0.0198	0.4737	0.1466[a]
	31.4088	*VAR*	0.0358[a]	0.0002	0.0002	0.0026	0.1311[a]	0.5341	0.6026[a]	0.4735[a]	0.1486	0.001	0.4733	0.1463
		BIAS	0.0000[a]	8.1533	8.1533	7.9220	0.0223[a]	0.0223[a]	0.0141[a]	0.0223[a]	0.0100[a]	0.1403	0.0141[a]	0.0173[a]
0.10	1.1486	*MSE*	0.6867[b]	156.7150	156.7162	144.2453	0.1109[a]	0.4602[a]	0.0005	0.4031[a]	0.2490[a]	0.0197	0.4038[a]	0.2508[a]
	109.2607	*VAR*	0.6861[b]	0.0007	0.0007	0.0021	0.1108[a]	0.4599	0.0005	0.4028[a]	0.2488[a]	0.0001	0.4037[a]	0.2507[a]
		BIAS	0.0244[b]	12.5185	12.5186	12.0101	0.0100[a]	0.0173[a]	0.0000	0.0173[a]	0.0141[a]	0.1400	0.0100[a]	0.0100[a]
0.20	1.1430	*MSE*	0.0460[a]	232.1828	232.1852	213.6815	0.0649[a]	0.2557[a]	0.2819[a]	0.2359[a]	0.1462[a]	0.0001	0.2358[a]	0.1572[a]
	137.8677	*VAR*	0.0459[a]	0.0008	0.0008	0.0014	0.0648[a]	0.2557[a]	0.2813[a]	0.2358[a]	0.1461[a]	0.0001	0.2358[a]	0.1572[a]
		BIAS	0.0100[a]	15.2375	15.2375	14.6178	0.0100[a]	0.0000[a]	0.0244[a]	0.0100[a]	0.0100[a]	0.1400	0.0141[a]	0.0000[a]
0.25	1.2587	*MSE*	0.0399[a]	312.9033	312.9072	287.3791	0.0541[a]	0.2237[a]	0.2578[a]	0.1943[a]	0.1400[a]	0.0001	0.1942[a]	0.1401[a]
	154.3004	*VAR*	0.0399[a]	0.0009	0.0009	0.0010	0.0541[a]	0.2235[a]	0.2576[a]	0.1942[a]	0.1399[a]	0.0001	0.1942[a]	0.1400[a]
		BIAS	0.0000[a]	17.6891	17.6891	16.9522	0.0000[a]	0.0141[a]	0.0141[a]	0.0100[a]	0.0100[a]	0.1400	0.0100[a]	0.0100[a]
0.30	1.2195	*MSE*	0.4065[b]	476.9743	476.9820	442.2079	0.0550[a]	0.1899[a]	0.2054[a]	0.1675[a]	0.1150[a]	0.0197	0.1678[a]	0.1149[a]
	179.9356	*VAR*	0.4061[b]	0.0013	0.0013	0.0004	0.0550[a]	0.1899[a]	0.2053[a]	0.1673[a]	0.1149[a]	0.0001	0.1674[a]	0.1149[a]
		BIAS	0.0200[b]	21.8397	21.8398	21.0287	0.0000[a]	0.0000[a]	0.0100[a]	0.0141[a]	0.0100[a]	0.1400	0.0200[a]	0.0000[a]
0.40	1.6609	*MSE*	0.0339[a]	563.0693	563.0748	525.1727	0.0496[a]	0.1960[a]	0.2201[a]	0.1817[a]	0.1122[a]	0.0197	0.1818[a]	0.1123[a]
	156.1624	*VAR*	0.0339[a]	0.0010	0.0010	0.0005	0.0495[a]	0.1958[a]	0.2201[a]	0.1816[a]	0.1121[a]	0.0001	0.1817[a]	0.1122[a]
		BIAS	0.0000[a]	23.7290	23.7291	22.9166	0.0100[a]	0.0141[a]	0.0000[a]	0.0100[a]	0.0100[a]	0.1400	0.0100[a]	0.0100[a]

[a] 1.0e-003
[b] 1.0e-004

Table 8. Case 6: *MSE*, *VAR*, *BIAS* and $\hat{VIF}_F$ values when data involves leverage-inducing collinearity and Type 1 outliers

$n = 50, R^2 = 0.9902, VIF = 1.2121, \beta = [4.3545 \quad 0.7061 \quad 0.3894 \quad 0.1909]$

η	$\hat{VIF}_G$ / $\hat{VIF}_F$		*OLS*	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
0	1.5443	*MSE*	0.1473[a]	32.4473	32.4473	31.7776	0.1926[a]	0.7745[a]	0.0008	0.6765[a]	0.4377[a]	0.0198	0.6759[a]	0.4370[a]
	1.5443	*VAR*	0.1472[a]	0.0000	0.0000	0.0001	0.1924[a]	0.7744[a]	0.0008	0.6763[a]	0.4373[a]	0.0002	0.6754[a]	0.4366[a]
		BIAS	0.0100[a]	5.6962	5.6962	5.6371	0.0141[a]	0.0100[a]	0.0000	0.0141[a]	0.0200[a]	0.1400	0.0223[a]	0.0200[a]
0.01	1.1994	*MSE*	87.0330	109.0760	685.8694	66.5989	0.0139	0.0478	0.0556	0.0438	0.0135	0.0335	0.0450	6.8259
	65.4066	*VAR*	86.9539	23.1091	572.4999	15.4946	0.0139	0.0478	0.0556	0.0438	0.0135	0.0141	0.0446	4.0448
		BIAS	0.2812	9.2718	10.6475	7.1487	0.0000	0.0000	0.0000	0.0000	0.0000	0.1392	0.0141	1.6676
0.05	1.4831	*MSE*	789.2460	328.2240	2668.0000	38.2615	0.0182	0.0748	0.0857	0.0652	2.4541	0.0379	0.0723	17.0467
	60.4842	*VAR*	782.3389	187.5411	2423.1000	27.8880	0.0182	0.0747	0.0856	0.0651	2.4542	0.0189	0.0677	0.9704
		BIAS	2.6281	11.8609	15.6492	3.2207	0.0000	0.0100	0.0100	0.0100	0.0100	0.1378	0.0678	4.0095
0.10	1.1915	*MSE*	1255.8000	619.3360	8812.5000	50.8950	0.0082	0.0297	0.0357	0.0267	58.4559	0.0277	0.0295	18.3724
	65.71050	*VAR*	1237.4000	261.1380	8115.2000	42.2497	0.0082	0.0297	0.0357	0.0267	55.6374	0.0082	0.0276	0.0728
		BIAS	4.2895	18.9261	26.4064	2.9402	0.0000	0.0000	0.0000	0.0000	1.6788	0.1396	0.0435	4.2778
0.20	1.4365	*MSE*	2174.6000	1803.7000	48311.0000	102.7260	0.0052	0.0177	0.0192	0.0129	640.5338	0.0248	0.0143	18.1470
	159.3482	*VAR*	2109.1000	319.0000	44969.0000	87.8773	0.0052	0.0177	0.0192	0.0129	400.3337	0.0053	0.0132	0.4668
		BIAS	8.0932	37.5859	57.8100	3.8534	0.0000	0.0000	0.0000	0.0000	15.4983	0.1396	0.0331	4.2047
0.25	1.0534	*MSE*	4862.5000	3341.1000	53790.0000	116.0437	0.0063	0.0196	0.0205	0.0129	1216.6000	0.0259	0.0150	18.5455
	143.4449	*VAR*	4735.1000	1016.6000	48516.0000	100.0015	0.0063	0.0196	0.0205	0.0129	632.0000	0.0064	0.0134	0.7321
		BIAS	11.2871	48.2130	72.6223	4.0052	0.0000	0.0000	0.0000	0.0000	24.1785	0.1396	0.0400	4.2205
0.30	1.2016	*MSE*	6408.6000	4962.4000	88194.0000	129.6032	13.6780	13.6130	38.4597	13.6909	1590.1000	4.7099	2.0940	18.1419
	187.6202	*VAR*	6321.7000	1220.2000	80165.0000	115.3542	13.6710	13.6080	38.3686	13.6843	565.2000	4.6961	2.0936	0.6786
		BIAS	9.3220	61.1735	89.6046	3.7747	0.0836	0.0707	0.3018	0.0812	32.0140	0.1174	0.0200	4.1789
0.40	1.4662	*MSE*	7258.8000	8332.8000	151670.0000	159.6657	439.8639	462.6390	1109.3000	446.7136	2859.2000	207.3655	106.2435	21.6628
	210.7239	*VAR*	7030.1000	1475.4000	137260.0000	142.3450	430.8633	453.3259	1096.9000	438.0765	1182.4000	199.4386	101.7644	3.5534
		BIAS	15.1228	82.8094	120.0416	4.1618	3.0000	3.0517	3.5213	2.9388	40.9487	2.8154	2.1163	4.2555

[a] 1.0e-003

Table 9. Case 7: *MSE*, *VAR*, *BIAS* and $\hat{VIF}$ values when data involves leverage-inducing collinearity and Type 2 outliers

$n = 50, R^2 = 0.9902, VIF = 1.2121, \beta = [4.3545 \quad 0.7061 \quad 0.3894 \quad 0.1909]$

η	$\hat{VIF}_G$ / $\hat{VIF}_F$		*OLS*	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
0.01	1.2186	*MSE*	0.5160[b]	70.0554	70.0556	66.5532	0.1917[a]	0.6866[a]	0.0008	0.6114[a]	0.2158[a]	0.1917[a]	0.6116[a]	0.0002
	33.6208	*VAR*	0.5156[b]	0.0002	0.0002	0.0019	0.1916[a]	0.6860[a]	0.0008	0.6109[a]	0.2156[a]	0.1916[a]	0.6110[a]	0.0002
		BIAS	0.0200[b]	8.3698	8.3699	8.1578	0.0100[a]	0.0244[a]	0.0000	0.0223[a]	0.0141[a]	0.0100[a]	0.0244[a]	0.0000
0.05	1.1167	*MSE*	221.8993	85.6393	1.1664[a]	4.6518	0.1067[a]	0.3951[a]	0.4249[a]	0.3388[a]	0.0402[a]	0.1069[a]	0.3394[a]	17.9387
	30.8481	*VAR*	0.0000	0.0000	0.0000[a]	0.0000	0.1067[a]	0.3948[a]	0.4246[a]	0.3387[a]	0.0402[a]	0.1068[a]	0.3391[a]	0.0000
		BIAS	14.8962	9.2541	1.0800	2.1568	0.0000[a]	0.0173[a]	0.0173[a]	0.0100[a]	0.0000[a]	0.0100[a]	0.0173[a]	4.2354
0.10	1.1753	*MSE*	356.8385	345.3085	819.3369	3.6846	0.0532[a]	0.1830[a]	0.1992[a]	0.1589[a]	0.3745[b]	0.0532[a]	0.1591[a]	18.3426
	82.5808	*VAR*	0.0000	0.0000	0.0003	0.0000	0.0531[a]	0.1829[a]	0.1990[a]	0.1588[a]	0.3742[b]	0.0531[a]	0.1590[a]	0.0000
		BIAS	18.8901	18.5824	28.6240	1.9195	0.0100[a]	0.0100[a]	0.0141[a]	0.0100[a]	0.0173[b]	0.0100[a]	0.0100[a]	4.2828
0.20	1.0911	*MSE*	94.3430	26.3706	4589.3000	2600.1000	0.0586[a]	0.1925[a]	0.2169[a]	0.1422[a]	22.1783	0.0587[a]	0.1425[a]	18.4820
	95.0118	*VAR*	0.0000	0.0001	0.0000	0.0000	0.0586[a]	0.1924[a]	0.2168[a]	0.1421[a]	0.0000	0.0586[a]	0.1424[a]	0.0000
		BIAS	9.71301	5.1352	67.7443	50.9911	0.0100[a]	0.0100[a]	0.0100[a]	0.0100[a]	4.7093	0.0100[a]	0.0100[a]	4.2990
0.25	1.2042	*MSE*	636.8290	878.6124	26034.0000	6.7473	0.0582[a]	0.1825[a]	0.1982[a]	0.1235[a]	0.0002	0.0583[a]	0.1237[a]	18.6131
	136.1976	*VAR*	0.0000	0.0000	0.0000	0.0000	0.0581[a]	0.1824[a]	0.1980[a]	0.1234[a]	0.0002	0.0581[a]	0.1235[a]	0.0000
		BIAS	25.2354	29.6413	161.3505	2.5975	0.0100[a]	0.0100[a]	0.0141[a]	0.0100[a]	20.6919	0.0141[a]	0.0141[a]	4.3142
0.30	1.2800	*MSE*	20012.0000	975.8000	208610.0000	16.2083	0.0620[a]	0.1798[a]	0.1895[a]	0.1123[a]	339.4904	0.0620[a]	0.1123[a]	18.1431
	151.4842	*VAR*	0.0000	0.0000	0.0000	0.0000	0.0620[a]	0.1796[a]	0.1894[a]	0.1123[a]	0.0000	0.0620[a]	0.1122[a]	0.0000
		BIAS	141.4637	97.8560	456.7384	4.0259	0.0000[a]	0.0141[a]	0.0100[a]	0.0000[a]	18.4252	0.0000[a]	0.0100[a]	4.2594
0.40	1.1366	*MSE*	13486.0000	10692.0000	69950.0000	14.8033	0.1398[a]	0.3604[a]	0.3680[a]	0.1629[a]	6143.2000	0.1399[a]	0.0002	19.5331
	190.6226	*VAR*	0.0000	0.0000	0.0000	0.0000	0.1397[a]	0.3602[a]	0.3677[a]	0.1628[a]	0.0000	0.1398[a]	0.0002	0.0000
		BIAS	116.1292	103.4021	264.4806	3.8475	0.0100[a]	0.0141[a]	0.0173[a]	0.0100[a]	78.3785	0.0100[a]	0.0000	4.4196

[a] 1.0e-003
[b] 1.0e-004

Table 10. Case 8: *MSE*, *VAR*, *BIAS* and $\hat{VIF}$ values when data involves leverage-inducing collinearity and Type 1 and 2 outliers

$n = 50, R^2 = 0.9902, VIF = 1.2121, \beta = [4.3545\ \ 0.7061\ \ 0.3894\ \ 0.1909]$														
η	$\hat{VIF}_G$ $\hat{VIF}_F$		*OLS*	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
0.05	1.2556	*MSE*	330.6950	223.1207	1057.0000	38.1527	0.0158	0.0589	0.0673	0.0524	2.6999	0.0163	0.0569	16.4480
	36.8277	*VAR*	328.2389	127.2435	866.0000	27.0700	0.0158	0.0589	0.0671	0.0524	2.6979	0.0160	0.0541	1.3614
		BIAS	1.5671	9.7916	13.8202	3.3290	0.0000	0.0000	0.0141	0.0000	0.0447	0.0173	0.0529	3.8841
0.10	1.4250	*MSE*	826.7091	478.5488	3835.1000	34.8870	0.0101	0.0349	0.0383	0.0297	31.4060	0.0104	0.0320	18.2540
	47.3145	*VAR*	809.8106	229.5509	3392.7000	29.9037	0.0101	0.0349	0.0382	0.0297	31.0912	0.0102	0.0304	0.1089
		BIAS	4.1107	15.7796	21.0333	2.2323	0.0000	0.0000	0.0100	0.0000	0.5610	0.0141	0.0400	4.2597
0.20	1.1146	*MSE*	1279.7000	976.8048	7461.9000	76.3914	0.0128	0.0418	0.0445	0.0295	467.7813	0.0131	0.0339	18.5530
	46.4068	*VAR*	1229.9000	387.1526	6332.9000	64.6805	0.0128	0.0418	0.0444	0.0295	419.0799	0.0129	0.0306	0.6672
		BIAS	7.0569	24.2827	33.6005	3.4221	0.0000	0.0000	0.0100	0.0000	6.9786	0.0141	0.0574	4.2291
0.25	1.1207	*MSE*	1293.5000	1092.7000	7805.7000	183.9447	0.0118	0.0354	0.0367	0.0239	762.9913	0.0120	0.0295	19.1237
	51.3307	*VAR*	1185.4000	398.5000	6023.7000	144.8279	0.0118	0.0354	0.0367	0.0239	637.9481	0.0118	0.0254	1.4456
		BIAS	10.3971	26.3476	42.2137	6.2543	0.0000	0.0000	0.0000	0.0000	11.1822	0.0141	0.0640	4.2045
0.30	1.3245	*MSE*	1903.7000	1818.6000	14426.0000	70.0651	1.6031	1.6200	29.1744	1.6136	1212.9000	1.6033	1.5889	20.5080
	43.1085	*VAR*	1776.6000	574.3000	12248.0000	57.9722	1.6029	1.6198	29.1460	1.6133	941.1000	1.6025	1.5873	2.4349
		BIAS	11.2738	35.2746	46.6690	3.7474	0.0141	0.0141	0.1685	0.0173	16.4863	0.0282	0.0400	4.2512
0.40	1.1369	*MSE*	2047.7000	2484.7000	13691.0000	170.3236	311.2770	515.1878	897.1322	452.6443	2311.3000	146.3168	113.0968	33.4976
	42.6806	*VAR*	1762.7000	718.5000	10212.0000	135.9125	301.3969	502.0352	893.9175	440.6842	1696.7000	139.3986	107.7842	13.2200
		BIAS	16.8819	42.0261	58.9830	5.8660	3.1432	3.6266	1.7929	3.4583	24.7911	2.6302	2.3050	4.5030

Table 11. Case 9: *MSE*, *VAR*, *BIAS* and $\hat{VIF}$ values when data involves only leverage-intensifying collinearity (no outliers)

$n = 50, R^2 = 0.8748, VIF = 17.4806, \beta = [8.0858\ \ 0.0612\ \ 0.8962\ \ -0.0263]$														
η	$\hat{VIF}_G$ $\hat{VIF}_F$		*OLS*	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
0	20.6421	*MSE*	0.0919	3.4480	3.4488	2.6934	0.1389	0.4916	0.5638	0.4452	0.3088	0.2033	0.5716	0.5525
	20.6421	*VAR*	0.0918	0.0149	0.0149	0.0047	0.1385	0.4914	0.5635	0.4449	0.3084	0.1520	0.4632	0.3275
		BIAS	0.0100	1.8528	1.8530	1.6397	0.0141	0.0141	0.0173	0.0173	0.0200	0.0513	0.3292	0.4743
0.01	27.6444	*MSE*	0.1476	4.5513	4.5525	3.6778	0.2132	0.7545	0.8679	0.6301	0.4383	0.2781	0.9193	1.0307
	45.3646	*VAR*	0.1475	0.0296	0.0296	0.0034	0.2131	0.7542	0.8670	0.6294	0.4381	0.2597	0.6359	0.4507
		BIAS	0.0100	2.1264	2.1267	1.9168	0.0100	0.0173	0.0300	0.0264	0.0141	0.5096	0.5323	0.7615
0.05	17.2903	*MSE*	0.0646	5.1146	5.1169	3.9859	0.0960	0.3655	0.3911	0.3108	0.2067	0.1609	0.3683	0.3313
	44.3722	*VAR*	0.0645	0.0276	0.0276	0.0034	0.0960	0.3653	0.3910	0.3108	0.2067	0.1054	0.3074	0.2055
		BIAS	0.0100	2.2554	2.2559	1.9956	0.0000	0.0141	0.0100	0.0000	0.0000	0.2355	0.2567	0.3546
0.10	26.7900	*MSE*	0.0920	11.4002	11.4144	9.1378	0.1280	0.5033	0.5251	0.4577	0.3072	0.1935	0.5966	0.5701
	119.7059	*VAR*	0.0919	0.0767	0.0772	0.0025	0.1279	0.5029	0.5246	0.4573	0.3071	0.1526	0.4587	0.3127
		BIAS	0.0100	3.3650	3.3670	3.0224	0.0100	0.0200	0.0223	0.0200	0.0100	0.2022	0.3713	0.5073
0.20	47.6061	*MSE*	0.1016	7.9282	7.9369	6.2745	0.1452	0.5256	0.5756	0.4539	0.3148	0.2075	0.5923	0.6111
	110.2181	*VAR*	0.1015	0.0706	0.0711	0.0024	0.1450	0.5254	0.5751	0.4534	0.3148	0.1722	0.4536	0.3247
		BIAS	0.0100	2.8031	2.8046	2.5044	0.0141	0.0141	0.0223	0.0223	0.0000	0.1878	0.3724	0.5351
0.25	23.2519	*MSE*	0.0815	10.0311	10.0438	7.9206	0.1134	0.4525	0.4867	0.3690	0.2573	0.1781	0.4685	0.4616
	117.4175	*VAR*	0.0815	0.0727	0.0731	0.0024	0.1134	0.4522	0.4865	0.3688	0.2571	0.1286	0.3706	0.2634
		BIAS	0.0000	3.1556	3.1576	2.8139	0.0000	0.0173	0.0141	0.0141	0.0141	0.0495	0.3128	0.4451
0.30	50.2562	*MSE*	0.0677	14.6668	14.6818	11.9490	0.0930	0.4070	0.4243	0.3438	0.2250	0.1582	0.4226	0.3600
	110.4410	*VAR*	0.0676	0.0658	0.0662	0.0025	0.0928	0.4066	0.4239	0.3437	0.2245	0.1021	0.3564	0.2327
		BIAS	0.0100	3.8211	3.8230	3.4563	0.0141	0.0200	0.0200	0.0100	0.0223	0.2368	0.2572	0.3567
0.40	63.3405	*MSE*	0.0702	15.2244	15.2457	12.2458	0.0998	0.3857	0.4031	0.3324	0.2184	0.1640	0.3916	0.3488
	134.6516	*VAR*	0.0700	0.0854	0.0861	0.0024	0.0998	0.3856	0.4028	0.3318	0.2179	0.1084	0.3342	0.2232
		BIAS	0.0141	3.8908	3.8935	3.4990	0.0000	0.0100	0.0173	0.0200	0.0223	0.2357	0.0574	0.1256

Table 12. Case 10: *MSE*, *VAR*, *BIAS* and $\hat{VIF}$ values when data involves leverage-intensifying collinearity and Type 1 outliers

$n = 50, R^2 = 0.8748, VIF = 17.4806, \beta = [8.0858\ \ 0.0612\ \ 0.8962\ \ -0.0263]$														
η	$\hat{VIF}_G$ $\hat{VIF}_F$		*OLS*	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
0	23.9392	*MSE*	0.0794	3.0339	3.0347	2.3424	0.1166	0.3873	0.4281	0.3567	0.2343	0.1811	0.4595	0.4227
	23.9392	*VAR*	0.0794	0.0175	0.0175	0.0048	0.1165	0.3870	0.4278	0.3564	0.2342	0.1312	0.3670	0.2433
		BIAS	0.0000	1.7367	1.7370	1.5289	0.0100	0.0173	0.0173	0.0173	0.0100	0.2233	0.3041	0.4235
0.01	19.5616	*MSE*	672.6087	67.8598	98.8832	3.8988	1.3040	4.6328	5.0244	3.8504	2.0082	2.6989	9.1949	32.0045
	29.5616	*VAR*	672.1527	61.6650	91.3042	3.3975	1.3038	4.6326	5.0338	3.8473	2.0074	1.3415	3.0344	7.0318
		BIAS	0.6752	2.4889	2.7529	0.7080	0.0141	0.0141	0.0244	0.0556	0.0282	1.1650	2.4820	4.9972
0.05	30.1012	*MSE*	2750.6000	408.7494	687.5647	11.1860	1.2624	4.5882	4.8947	3.6402	1.1454	2.7858	9.8255	60.0734
	68.7780	*VAR*	2719.4000	372.6629	643.8350	7.6444	1.2632	4.5831	4.8937	3.6368	1.1445	1.2635	2.8668	2.0936
		BIAS	5.5856	6.0072	6.6128	1.8819	0.0141	0.0714	0.0316	0.0583	0.0300	1.2215	2.6379	7.6144
0.10	30.6405	*MSE*	5119.7000	844.5322	1425.5000	15.9497	1.0390	3.7400	4.3357	3.0893	0.8565	2.5092	10.0500	64.4927
	70.3800	*VAR*	4994.0000	726.1053	1274.0000	8.1827	1.0386	3.7331	4.3316	3.0864	0.8562	1.0761	2.4230	0.1677
		BIAS	11.2116	10.8824	12.3085	2.7869	0.0200	0.0830	0.0640	0.0538	0.0173	1.1971	2.7617	8.0202
0.20	43.3491	*MSE*	7489.3000	1560.8000	3310.1000	24.8436	0.8550	2.8631	3.1841	1.9910	0.7108	2.1932	10.7809	64.8862
	72.1931	*VAR*	7395.7000	1156.6000	2778.7000	12.5897	0.8548	2.8613	3.1819	1.9895	0.7106	0.9453	1.8237	0.0032
		BIAS	9.6747	20.1047	23.0521	3.5005	0.0141	0.0424	0.0469	0.0387	0.0141	1.1170	2.9928	8.0549
0.25	27.6865	*MSE*	7356.9000	1694.4000	4334.7000	30.8158	0.7442	2.3711	2.4628	1.4607	1.8724	1.7460	10.5504	64.9023
	94.6636	*VAR*	7037.5000	1108.0000	3587.2000	16.5722	0.7437	2.3697	2.4604	1.4595	1.8723	0.8143	1.6390	0.0052
		BIAS	17.8717	24.2156	27.3404	3.7740	0.0223	0.0374	0.0489	0.0346	0.0100	0.9652	2.9851	8.0558
0.30	21.6797	*MSE*	7277.1000	1940.5000	5310.7000	35.4096	18.8904	51.5648	32.6277	260.0006	279.0085	1.8133	12.4053	65.1058
	62.5670	*VAR*	6929.3000	1068.0000	4192.4000	17.2953	18.8807	51.5460	32.5987	259.9710	278.8345	1.0198	2.0905	0.1740
		BIAS	18.6493	29.5381	33.4409	4.2560	0.0984	0.1371	0.1702	0.1720	0.4171	0.8907	3.2116	8.0550
0.40	36.7744	*MSE*	14862.0000	4420.0000	10410.0000	25.6086	806.0672	1455.5000	869.2051	1138.4000	1967.1000	322.9676	193.4065	74.0722
	103.6305	*VAR*	14352.0000	2738.4000	8394.0000	15.5312	770.1222	1421.7000	829.9204	1099.5000	1932.6000	315.5801	187.3735	9.3555
		BIAS	22.5831	41.0073	44.8998	3.1744	5.9954	5.8137	6.2677	6.2369	5.8736	2.7179	2.4562	8.0446

Table 13. Case 11: *MSE*, *VAR*, *BIAS* and $\hat{VIF}$ values when data involves leverage-intensifying collinearity and Type 2 outliers

$n = 50, R^2 = 0.8748, VIF = 17.4806, \beta = [8.0858 \quad 0.0612 \quad 0.8962 \quad -0.0263]$

η	$\hat{VIF}_G$ / $\hat{VIF}_F$		OLS	RR1	RR2	RR3	RLS	LMS	LTS	M	S	RTRLS	RTM	RTS
0.01	28.5851	MSE	0.0638	5.3235	5.3251	4.2614	0.0929	0.3938	0.4058	0.3322	0.2128	0.0979	0.3782	0.3219
	33.9596	VAR	0.0636	0.0217	0.0218	0.0038	0.0922	0.3929	0.4051	0.3309	0.2116	0.0927	0.3298	0.2128
		BIAS	0.0141	2.3025	2.3028	2.0633	0.0264	0.0300	0.0264	0.0360	0.0346	0.0721	0.2200	0.3303
0.05	19.6146	MSE	82.4347	34.5118	42.1973	30.7589	0.1118	0.4596	0.4396	0.3689	0.0865	0.1302	0.4991	64.4276
	56.1883	VAR	0.0862	0.0522	0.0208	0.0025	0.1117	0.4589	0.4392	0.3681	0.0865	0.1132	0.3843	0.0000
		BIAS	9.0746	5.8702	6.4943	5.5458	0.0100	0.0264	0.0200	0.0894	0.0100	0.1303	0.3388	8.0266
0.10	42.0109	MSE	276.0651	12.0242	172.5823	9.6274	0.1288	0.4833	0.5141	0.4042	0.0989	0.1528	0.6572	64.8285
	79.2778	VAR	0.0916	0.0984	0.0229	0.0653	0.1288	0.4829	0.5136	0.4038	0.0989	0.1302	0.4283	0.0000
		BIAS	16.6124	3.4533	13.1361	3.0922	0.0000	0.0000	0.0223	0.0200	0.0000	0.1503	0.4784	8.0516
0.20	54.9583	MSE	5781.2000	621.7742	1758.9000	19.1197	0.1007	0.3428	0.3546	0.2474	0.0899	0.1296	0.6410	64.9181
	102.2074	VAR	0.1000	0.0102	0.0000	0.0000	0.1006	0.3416	0.3545	0.2473	0.0899	0.1017	0.2732	0.0000
		BIAS	76.0335	24.9351	41.9392	4.3726	0.0100	0.0346	0.0100	0.0100	0.0000	0.1670	0.6064	8.0571
0.25	86.8766	MSE	1644.9000	512.8800	754.0734	7.7226	0.1585	0.5162	0.5208	0.3294	0.1381	0.2035	1.1086	64.8777
	114.3630	VAR	0.1000	0.0051	0.0315	0.0006	0.1579	0.5139	0.5204	0.3281	0.1378	0.1626	0.3798	0.0001
		BIAS	40.5561	22.6467	27.4598	2.7788	0.0244	0.0479	0.0200	0.0360	0.0173	0.2022	0.8536	8.0546
0.30	31.1363	MSE	2472.5000	1005.0000	1271.0000	29.6379	0.1333	0.3916	0.4229	0.2274	0.1239	0.1762	1.2320	64.9940
	149.2460	VAR	0.1000	0.0000	0.0000	0.0001	0.1332	0.3916	0.4227	0.2274	0.1238	0.1370	0.2721	0.0001
		BIAS	49.7232	31.7017	35.6510	5.4440	0.0100	0.0000	0.0141	0.0000	0.0100	0.1979	0.9797	8.0618
0.40	101.0830	MSE	4492.7000	2071.4000	7211.8000	7.9805	0.1427	0.3661	0.3848	0.1662	0.1371	0.2009	3.2372	65.0338
	158.3028	VAR	0.1000	0.0000	0.0000	0.0000	0.1427	0.3658	0.3846	0.1661	0.1371	0.1469	0.3293	0.0001
		BIAS	67.0268	45.5126	84.9223	2.8249	0.0000	0.0173	0.0141	0.0100	0.0000	0.2323	1.7052	8.0643

Table 14. Case 12: *MSE*, *VAR*, *BIAS* and $\hat{VIF}$ values when data involves leverage-intensifying collinearity and Type 1 and 2 outliers

$n = 50, R^2 = 0.8748, VIF = 17.4806, \beta = [8.0858 \quad 0.0612 \quad 0.8962 \quad -0.0263]$

η	$\hat{VIF}_G$ / $\hat{VIF}_F$		OLS	RR1	RR2	RR3	RLS	LMS	LTS	M	S	RTRLS	RTM	RTS
0.05	29.8696	MSE	3037.2000	391.5414	581.3001	10.3888	1.3054	5.2700	5.2297	4.2141	1.1573	2.8058	10.2667	60.0901
	46.5099	VAR	3037.0000	357.4223	537.8448	7.1430	1.3036	5.2686	5.2295	4.2120	1.1541	1.2963	3.1629	2.1626
		BIAS	0.4472	5.8411	6.5920	1.8016	0.0424	0.0374	0.0141	0.0458	0.0565	1.2286	2.6652	7.5805
0.10	23.5355	MSE	4214.3000	652.0077	1072.6000	16.9534	0.9021	3.4753	3.6330	2.7538	0.6913	1.9970	8.2075	64.5545
	40.0408	VAR	4163.7000	549.7566	943.0000	7.5629	0.9014	3.4739	3.6316	2.7525	0.6910	0.9297	2.3872	0.1521
		BIAS	7.1133	10.1119	11.3841	3.0643	0.0264	0.0374	0.0374	0.0360	0.0173	1.0331	2.4125	8.0251
0.20	20.9433	MSE	1034.0000	1743.8000	2762.1000	20.0085	1.2498	3.7758	3.8461	2.5991	1.0711	3.2174	14.6686	64.9261
	53.0619	VAR	998.1000	1341.8000	2271.1000	11.0153	1.2490	3.7728	3.8430	2.5966	1.0703	1.2566	2.0636	0.0027
		BIAS	5.9916	20.0499	22.1585	2.9988	0.0282	0.0547	0.0556	0.0500	0.0282	1.4002	3.5503	8.0575
0.25	27.4871	MSE	7792.1000	1715.9000	3694.9000	29.6453	0.7228	2.1786	2.4584	1.4346	0.6373	1.6582	10.2578	64.9369
	56.1844	VAR	7362.8000	1114.5000	2930.8000	14.3138	0.7223	2.1748	2.4566	1.4336	0.6368	0.7823	1.7945	0.0030
		BIAS	20.7195	24.5234	27.6423	3.9155	0.0223	0.0616	0.0424	0.0316	0.0223	0.9358	2.9091	8.0581
0.30	29.1562	MSE	13568.0000	2910.0000	5985.0000	24.6414	3.0536	29.7233	26.3566	9.6463	50.4712	3.8986	21.5070	65.1654
	100.0755	VAR	13189.0000	1997.7000	4901.3000	13.6960	3.0534	29.7141	26.3224	9.6450	50.4679	1.3778	2.0258	0.1352
		BIAS	19.4679	30.2043	32.9195	3.3083	0.0141	0.0959	0.1849	0.0360	0.0574	1.5877	4.4137	8.0641
0.40	27.9464	MSE	11147.0000	2965.4000	6812.5000	28.7826	769.4040	1251.5000	2214.7000	912.1713	3228.3000	299.5366	182.3799	74.4559
	96.1681	VAR	10364.0000	1555.3000	5079.6000	19.4185	738.7035	1229.2000	2201.6000	889.3722	3197.5000	293.5102	175.7464	9.7082
		BIAS	27.9821	37.5512	41.6281	3.0600	5.5408	4.7222	3.6193	4.7748	5.5497	2.4548	2.5755	8.0465

Table 15. Case 13: *MSE*, *VAR*, *BIAS* and $\hat{VIF}$ values when data has high collinearity and Type 2 outliers (no leverage observations)

$n = 50, R^2 = 0.8527, VIF = 501.3193, \beta = [5.0298 \quad 0.2375 \quad 0.2375 \quad 0.4726]$

η	$\hat{VIF}_G$ / $\hat{VIF}_F$		OLS	RR1	RR2	RR3	RLS	LMS	LTS	M	S	RTRLS	RTM	RTS
0	512.4558	MSE	2.3796	24.6229	24.7196	21.9792	3.2373	12.6230	13.1339	10.8013	7.1478	5.6698	10.1465	11.5084
	512.4558	VAR	2.3775	0.3749	0.4638	0.0033	3.2367	12.6187	12.1247	10.7960	7.1363	1.9506	2.6371	1.1702
		BIAS	0.0458	4.9242	4.9250	4.6878	0.0244	0.0655	0.0959	0.0728	0.0115	1.9285	2.7403	3.2153
0.01	555.3845	MSE	9452.6000	1050.8000	243.4547	18.2171	22.6924	75.8010	88.3182	64.9796	34.3113	14.4203	17.2176	20.0683
	555.3845	VAR	9446.0000	1020.6000	211.6657	5.5795	22.6739	75.7253	88.0943	64.8417	34.2788	2.4226	2.0550	1.2293
		BIAS	2.4494	5.4954	5.6381	3.5549	0.1360	0.5250	0.4731	0.3713	0.1802	3.4637	3.8939	4.3397
0.05	544.2039	MSE	65962.0000	5515.9000	772.1803	16.7031	35.8354	132.9150	142.0878	110.2073	28.6163	15.8720	18.2338	23.6330
	544.2039	VAR	65951.0000	5405.7000	684.8276	14.2513	35.8012	132.9119	141.9569	110.0991	28.5946	2.3432	2.0382	0.3285
		BIAS	3.3166	10.4976	9.3462	1.5658	0.1849	0.0556	0.3618	0.3289	0.1473	3.6781	4.0243	4.8274
0.10	558.6257	MSE	119000.0000	11945.0000	1592.0000	23.2165	29.9124	112.7232	123.0658	91.4191	24.1987	15.2253	18.1326	24.1651
	558.6257	VAR	118900.0000	11778.0000	1408.6000	21.7034	29.8993	112.6723	112.9995	91.3823	24.1857	2.0945	1.5596	0.0881
		BIAS	10.0000	12.9228	13.5646	1.2300	0.1144	0.2256	0.2574	0.1918	0.1140	3.6236	4.0709	4.9069
0.20	502.6033	MSE	155390.0000	15439.0000	3429.8000	36.4314	24.8525	81.8411	82.6778	61.8523	20.9572	15.1952	18.2976	24.4440
	502.6033	VAR	155130.0000	14905.0000	2898.9000	35.3767	24.8284	81.7611	82.6132	61.8073	20.9369	1.7559	0.9534	0.0143
		BIAS	16.1245	23.1084	23.0412	1.0269	0.1552	0.2828	0.2541	0.2121	0.1424	3.6659	4.1646	4.9426
0.25	545.7461	MSE	156540.0000	17223.0000	4287.3000	50.2846	25.9521	79.1611	85.4764	52.8965	23.5767	15.5680	19.0394	24.4659
	545.7461	VAR	156430.0000	16510.0000	3533.3000	46.7674	25.9344	79.0304	85.3951	52.8725	23.5532	1.6013	0.5831	0.0175
		BIAS	10.4880	26.7020	27.4590	1.8754	0.1330	0.3615	0.2851	0.1549	0.1532	3.7372	4.2960	4.9445
0.30	483.5043	MSE	208390.0000	20476.0000	5946.5000	54.1917	360.7619	2619.2000	2255.6000	944.6106	214.8049	15.9674	19.7264	24.6551
	483.5043	VAR	208340.0000	19516.0000	4947.5000	50.6020	360.4067	2617.1000	2253.7000	943.0894	214.2132	2.5349	1.0735	0.1955
		BIAS	70.0710	30.9838	31.6069	1.8946	0.5959	1.4491	1.3784	1.2333	0.7492	3.6650	4.3189	4.9456
0.40	538.5732	MSE	236450.0000	24902.0000	8669.2000	89.2239	9288.4000	112600.0000	85155.0000	116900.0000	66094.0000	61.6352	63.5830	42.0822
	538.5732	VAR	235920.0000	23255.0000	7002.6000	76.5959	9252.8000	112520.0000	85111.0000	116560.0000	65439.0000	52.4548	43.5377	18.3731
		BIAS	23.0217	40.5832	40.8240	3.5535	5.9665	8.9442	6.6332	18.4390	25.5929	3.0299	4.4771	4.8691

Table 16. Case 14: *MSE*, *VAR*, *BIAS* and $\hat{VIF}$ values when data has low/no collinearity and Type 2 outliers (no leverage observations)

$n = 50, R^2 = 0.9902, VIF = 1.2121, \beta = [4.3545 \quad 0.7061 \quad 0.3894 \quad 0.1909]$														
η	$\hat{VIF}_G$ $\hat{VIF}_F$		*OLS*	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
0	1.0900	*MSE*	0.1175[a]	29.2821	29.2822	28.6880	0.1641[a]	0.0006	0.0007	0.5340[a]	0.3817[a]	0.0198	0.5338[a]	0.3807[a]
	1.0900	*VAR*	0.1175[a]	0.0000	0.0000	0.0001	0.1640[a]	0.0006	0.0007	0.5336[a]	0.3812[a]	0.0002	0.5333[a]	0.3804[a]
		BIAS	0.0000[a]	5.4112	5.4113	5.3561	0.0100[a]	0.0000	0.0000	0.0200[a]	0.0223[a]	0.1400	0.0223[a]	0.0173[a]
0.01	1.0956	*MSE*	118.5589	38.0426	50.7075	24.2791	0.0219	0.0767	0.0874	0.0702	0.0339	0.0414	0.0736	7.0986
	1.0956	*VAR*	118.2735	6.1389	6.9422	6.2147	0.0219	0.0767	0.0873	0.0701	0.0339	0.0224	0.0698	4.2013
		BIAS	0.5342	5.6483	6.6155	4.2502	0.0000	0.0000	0.0100	0.0100	0.0000	0.1378	0.0616	1.7021
0.05	1.2855	*MSE*	311.1166	80.0950	136.6830	18.2300	0.0105	0.0397	0.0422	0.0336	0.0093	0.0301	0.0356	16.7200
	1.2855	*VAR*	308.5961	33.5817	39.7513	15.7765	0.0105	0.0397	0.0422	0.0336	0.0093	0.0107	0.0337	1.3778
		BIAS	1.5876	6.8200	9.8453	1.5663	0.0000	0.0000	0.0000	0.0000	0.0000	0.1392	0.0435	3.9169
0.10	1.3593	*MSE*	1133.6000	196.2361	288.7994	15.9575	0.0197	0.0726	0.0781	0.0586	0.0171	0.0392	0.0671	18.2507
	1.3593	*VAR*	1109.6000	68.3964	74.8974	15.6342	0.0197	0.0725	0.0779	0.0586	0.0171	0.0204	0.0597	0.1250
		BIAS	4.8989	11.3066	14.6253	0.5685	0.0000	0.0100	0.0141	0.0000	0.0000	0.1371	0.0860	4.2574
0.20	1.1865	*MSE*	2393.9000	480.7275	690.2081	29.3182	0.0237	0.0747	0.0830	0.0508	0.0213	0.0430	0.0699	18.4937
	1.1865	*VAR*	2356.9000	129.3801	127.0547	28.6830	0.0237	0.0747	0.0826	0.0507	0.0212	0.0249	0.0525	0.0046
		BIAS	6.0827	18.7442	23.7308	0.8277	0.0000	0.0000	0.0200	0.0100	0.0100	0.1345	0.1319	4.2998
0.25	1.4299	*MSE*	2226.3000	668.4250	966.7046	28.3098	0.1343	22.2761	77.9125	31.4638	27.7888	0.0795	0.2056	18.6269
	1.4299	*VAR*	2049.8000	161.1371	169.5724	27.9398	0.1342	22.2751	77.9070	31.4630	27.7881	0.0616	0.1823	0.0886
		BIAS	13.2853	22.5230	28.2335	0.6082	0.0100	0.0316	0.0741	0.0282	0.0264	0.1337	0.1526	4.3056
0.30	1.2636	*MSE*	2913.5000	817.5727	1165.8000	34.9610	0.6785	3.0410	9.0535	1.6179	0.9066	0.0919	0.2218	18.5919
	1.2636	*VAR*	2793.5000	162.0080	149.5000	33.9334	0.6785	3.0400	9.0520	1.6179	0.9065	0.0747	0.1671	0.0813
		BIAS	10.9544	25.6039	31.8794	1.0137	0.0000	0.0316	0.0387	0.0000	0.0100	0.1311	0.2338	4.3023
0.40	1.1387	*MSE*	5319.3000	1450.7000	1890.5000	35.9136	488.7645	2995.1000	1831.2000	1726.0000	1273.7000	98.3034	94.1499	27.6250
	1.1387	*VAR*	4993.1000	213.0000	175.6000	33.0685	478.9987	2988.1000	1820.5000	1708.2000	1265.4000	93.4260	91.4289	9.2201
		BIAS	18.0610	35.1809	41.4113	1.6867	3.1250	2.6457	3.2710	4.2190	2.8809	2.2084	1.6495	4.2900

[a] 1.0e-003

Table 17. Case 15: *MSE*, *VAR*, *BIAS* and $\hat{VIF}$ values when data has moderate collinearity and Type 2 outliers (no leverage points)

$n = 50, R^2 = 0.8748, VIF = 17.4806, \beta = [8.0858 \quad 0.0612 \quad 0.8962 \quad -0.0263]$														
η	$\hat{VIF}_G$ $\hat{VIF}_F$		*OLS*	*RR1*	*RR2*	*RR3*	*RLS*	*LMS*	*LTS*	*M*	*S*	*RTRLS*	*RTM*	*RTS*
0	19.1051	*MSE*	00888	3.5067	3.5075	2.7791	0.1212	0.4680	0.5145	0.4150	0.2787	0.1877	0.5083	0.4716
	19.1051	*VAR*	0.0886	0.0142	0.0142	0.0051	0.1209	0.4652	0.5134	0.4141	0.2778	0.1358	0.4211	0.2834
		BIAS	0.0141	1.8688	1.8690	1.6655	0.0173	0.0529	0.0331	0.0300	0.0300	0.2278	0.2952	0.4338
0.01	17.4849	*MSE*	418.3974	36.7016	61.3033	5.6493	0.8949	3.1115	3.4415	2.7444	1.4334	1.6706	5.8226	30.1839
	17.4849	*VAR*	418.3276	30.7838	53.7940	5.4619	0.8941	3.1092	3.4393	2.7431	1.4328	0.9804	2.5077	9.0082
		BIAS	0.2441	2.4326	2.7403	0.4328	0.0282	0.0479	0.0469	0.0360	0.0244	0.8307	1.8206	4.6017
0.05	15.9841	*MSE*	1798.5000	183.3509	282.0157	13.0788	0.8117	3.1484	3.0236	2.5783	0.6967	1.5449	6.2177	61.1083
	15.9841	*VAR*	1791.8000	151.8558	237.0014	6.4130	0.8109	3.1465	3.0260	2.5769	0.6960	0.8955	2.4108	2.0885
		BIAS	2.5884	5.6120	6.7092	2.5818	0.0282	0.0435	0.0244	0.0374	0.0264	0.8058	1.9511	7.8079
0.10	21.1748	*MSE*	4397.5000	466.5543	646.1131	12.8336	1.0480	3.7458	4.3136	3.1194	0.8697	2.6431	9.9077	64.3819
	21.1748	*VAR*	4335.5000	381.3028	535.4876	6.5446	1.0470	3.7426	4.3109	3.1186	0.8694	1.1435	2.5239	0.1438
		BIAS	7.8740	9.2331	10.5178	2.5077	0.0316	0.0565	0.0519	0.0282	0.0173	1.2245	2.7173	8.0148
0.20	19.1100	*MSE*	8571.4000	1038.4000	1396.0000	17.7605	1.1368	3.8754	4.0969	2.7088	1.0275	3.4012	14.8909	64.8884
	19.1100	*VAR*	8323.4000	711.4000	997.9000	7.9317	1.1358	3.8723	4.0935	2.7075	1.0269	1.2527	2.1836	0.0028
		BIAS	15.7480	18.0831	19.9524	3.1350	0.0316	0.0556	0.0583	0.0360	0.0244	1.4657	3.5647	8.0551
0.25	20.0922	*MSE*	9223.0000	1307.4000	1815.4000	18.9726	24.6927	35.7905	14.0655	8.1390	6.7753	8.5473	18.8689	65.0591
	20.0922	*VAR*	9059.6000	854.6000	1251.0000	11.0319	24.6866	35.7751	14.0545	8.1330	6.7702	6.4924	3.8415	0.1428
		BIAS	12.8062	21.2790	23.7571	2.8179	0.0781	0.1240	0.1048	0.0774	0.0714	1.4334	3.8765	8.0570
0.30	21.3360	*MSE*	10679.0000	1648.3000	2272.9000	21.9163	175.9073	1291.2000	828.4259	1050.8000	886.1710	14.2368	24.9498	65.4004
	21.3360	*VAR*	10175.0000	948.7000	1437.2000	11.0634	175.8164	1290.7000	827.6520	1050.3000	885.7006	12.3109	7.3982	0.5455
		BIAS	22.4499	26.4499	28.9084	3.2943	0.3014	0.7071	0.7791	0.7071	0.6858	1.3877	4.1894	8.0532
0.40	19.4039	*MSE*	13223.0000	2217.9000	2904.3000	20.4069	1023.4000	6865.0000	6193.4000	4601.2000	4346.7000	225.0664	143.7477	73.3702
	19.4039	*VAR*	12261.0000	1018.6000	1490.5000	13.6066	984.1000	6828.8000	6125.6000	4565.2000	4290.2000	223.2017	131.9560	9.0425
		BIAS	31.0161	34.6309	37.6005	2.6077	6.2689	6.0827	8.2462	6.0000	7.5166	1.3655	3.4339	8.0204

REFERENCES

[1] O. Arslan and N. Billor, "Robust Liu estimator for regression based on a M-estimator," J. Appl. Stat., vol 27 pp. 39-47, 2000.

[2] H. Ertas and S. Toker, and S. Kaciranlar, "Robust two parameter ridge M-estimator for linear regression" J. Appl. Stat., vol 42, pp. 1490-1502, 2015.

[3] A.E. Hoerl and R.W. Kennard, "Ridge regression: biased estimation for non orthogonal problems," Technometrics vol 12, pp. 55-68, 1970a.

[4] A.E. Hoerl, R.W. Kennard, and K.F. Baldwin, "Ridge regression: some simulations," Commun. Statist., vol 4, pp. 105-123, 1975.

[5] R.A. Maronna, "Robust ridge regression for high-dimensional data," Technometrics, vol 53, pp. 44-53, 2011.

[6] M.J. Silvapulle, Robust ridge regression based on an M estimator, Austral. J. Stat., vol 33, pp. 319-333, 1991.

[7] J.R. Simpson and D.C. Montgomery, "A biased robust regression technique for combined outlier multicollinearity problem," J. Stat. Comput. and Sim., vol 56, pp. 1-22, 1996.

[8] F.A. Graybill, Introduction to Linear Statistical Models, New York-Toronto-London: Mc. Graw-Hill Series in Probability & Statistics, 1961.

[9] A.V. Dorugade, "On comparison of some ridge parameters in ridge regression," Sri Lanka J. Appl. Stat., vol 15, pp. 31-46, 2014.

[10] J.F. Lawless and P. Wang, "A simulation study of ridge and other regression estimators," Commun. Stat. Theory Methods vol 14, pp. 1589-1604, 1976.

[11] G. Muniz and B.M.G. Kibria, "On some ridge regression estimators: an emprical comparison," Commun. Stat. Simul. Comput., vol 38, pp. 621-630, 2009.

[12] L. Davies, "The asymptotics of S-estimators in the linear regression model," Ann. Stat., vol 18, pp. 1651-1675, 1990.

[13] P.J. Rousseeuw and A.M. Leroy, Robust Regression and Outlier Detection, United States: Wiley, 1987.

[14] B. Kan and O. Alpu, and B. Yazici, "Robust ridge and robust Liu estimator for regression based on the LTS estimator," J. Appl. Stat., vol 40, pp. 644-655, 2013.

[15] O.U. Ekiz and M. Ekiz, "A small-sample correction factor for S-estimators," J. Stat. Comput. and Sim., vol 85, pp. 794-801, 2015.

[16] M. Ekiz and O.U. Ekiz, "Outlier detection with Mahalanobis square distance: incorporating small sample correction factor," J. Appl. Stat., vol 44, pp. 2444-2457, 2017.

[17] A. Bagheri, Robust estimation methods and robust multicollinearity diagnostics for multiple regression model in the presence of high leverage collinearity-influential observations, Unpublished doctorial thesis. University Putra, 2011.

[18] A. Bagheri, M. Habshah, and R.H.M.R. Imon, "A novel collinearity-influential observation diagnostic measure based on a group deletion approach," Commun. Stat. Simul. Comput., vol 41, pp. 1379-1396, 2012.

[19] D.A. Belsley, E. Kuh, and R.E. Welsch, Regression Diagnostics: Identifying Influential Data and Sources of Collinearity, New York: John Wiley & Sons, 1980.

[20] S. Chatterjee and B. Price, Regression Analysis by Example, New York: John Wiley & Sons, 1977.

[21] D.K. Guilkey and J.L. Murphy, "Directed ridge regression techniques in cases of multicollinearity," J. Am. Stat. Assoc., vol 70, pp. 769-775, 1975.

[22] B.M.G. Kibria, "Performance of some new ridge regression estimators," Commun. Stat. Simul. Comput., vol 32, pp. 419-435, 2003.

[23] G.C. McDonald and D.I. Galarneau, "A Monte Carlo evaluation of some ridge-type estimators," J. Am. Stat. Assoc., vol 70, pp. 407-416, 1975.

[24] R.A. Maronna, D.R. Martin, and V.J. Yohai, Robust Statistics: Theory and Methods, New York: Wiley, 2006.

[25] P. Rousseeuw and V. Yohai, "Robust and nonlinear time series analysis," Lecture Notes in Statistics, vol 26, pp. 256-272, 1984.

[26] M. Amini and M. Roozbeh, "Least trimmed squares ridge estimation in partially linear regression models," J. Stat. Comput. and Sim., vol 86, pp. 2766-2780, 2016.

[27] D.M. Hawkins, D. Bradu, and G.V. Kass, "Location of several outliers in multiple regression data using elemental sets," Technometrics, vol 26, pp. 197-208, 1984.