

On the Behaviour of Second-Order N-Point Equiradial Designs under Varying Model Parameters

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Abstract The behaviour of alternative second-order N-point equiradial designs are studied under variations of model parameters for design radius $\rho = 1.0$. Useful numerical evaluations associated with the designs and the models are presented with respect to A-, D-, E-, G- and T-optimality criteria. Relationships among the optimality criteria are outlined with regards to the designs and the models. Furthermore, D- efficiencies of the equiradial designs are evaluated. The N-point equiradial designs perform correspondingly better for reduced bivariate quadratic model than for the full bivariate quadratic model under A- and D- optimality criteria. The reverse holds for the full bivariate quadratic model under T-optimality criterion as the N-point equiradial designs perform correspondingly better for the full bivariate quadratic model than for the reduced bivariate quadratic model. There is no indication of design preference for the models under E- and G-optimality criteria. On the whole, designs optimal for one model need not be optimal for another model.

Keywords Equiradial designs, Full bivariate quadratic model, Reduced bivariate quadratic model, Optimality criteria, D-efficiency

1. Introduction

First and Second-order Response Surface Methodology designs play vital roles in modelling response functions. Most classical designs such as factorial designs assume the regression model, which is used as an approximation to the true unknown response function, is a full polynomial model. However, models with improper polynomial regression functions also exist. By the flexibility of optimal design theory, it is possible to obtain properties of designs for polynomial regression functions having complete terms present or some terms missing. Iwundu & Albert-Udochukwuka (2014) studied the behaviour of D-optimal exact designs for first-order polynomial models under changing regression polynomials. The polynomials used were with or without intercept terms or with or without interactive terms. The importance of first order designs and models cannot be over-emphasized as the applications are numerous in industrial processes, particularly in screening experiments. On the other hand, second-order response surface methodology designs are particularly very essential in modelling second-order response functions in the presence of curvature. They include the 3^k factorial designs, the central composite designs, the Box-Behnken designs and even D-optimal designs and have been extensively studied.

However, some equiradial designs exist and can serve reasonably well in modelling second-order effects. Myer *et al.* (2009) defines equiradial designs as some special and interesting two-factor designs for modelling second-order response functions and whose design points appear on a common sphere. The class of equiradial designs require not many experimental runs and begins with a pentagon of equally spaced points on the sphere. The design matrix is expressed as

$$\begin{matrix} x_1 & x_2 \\ \{\rho \cos(\theta + 2\pi u/n_1) & \rho \sin(\theta + 2\pi u/n_1)\}; \\ u = 0, 1, 2, \dots, n_1 - 1 \end{matrix}$$

where x_1 and x_2 represent the two controllable variables, ρ is the radius of the design and n_1 represents the number of points on the sphere. n_c center points are usually added to the n_1 radial points of the design. Myer *et al.* (2009) observed that the value of θ could be assumed equal to zero since the information matrix $X^T X$, of the design is not affected by θ . Some considerations for second-order response surface models and designs include Onukogu and Iwundu (2007), who considered the construction of efficient and optimal experimental designs for second-order response surface models. Concerns regarding optimality of designs have been investigated for second-order models (see for example Dette and Grigoriev (2014)). On optimal choices of design points for second-order response surface designs and performances of several types of second-order response surface designs, see Lucas (1976), Chigbu and Nduka (2006),

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Chigbu *et al.* (2009), Ukaegbu and Chigbu (2015), Oyejola and Nwanya (2015) and Iwundu (2015). Graphical techniques have been used in studying the variance properties of second-order response surface designs as seen in Myer *et al.* (1992), Giovannitti-Jensen and Myers (1989), Zahran *et al.* (2003). In a recent study by Iwundu (2016a), the equiradial designs were seen comparable with the standard central composite designs. Particularly, the D-efficiency values reveal that the N-point spherical equiradial designs are better than the inscribed central composite design though inferior to the circumscribed central composite design with efficiency values being less than 50% in all cases studied. We consider in this work the behaviour of the equiradial designs under variations of model parameters for design radius $\rho = 1.0$. We also investigate the assertion that designs optimal for one model need not be optimal for another model.

2. Methodology

It is a common practice to assume some high order interaction terms of a model to be negligible. This practice is seen, for instance, in the area of factorial experiments where the highest order interaction is assumed negligible. It is even possible to assume a no-intercept (or zero-intercept) model as seen in origin regression. In studying the behaviour of alternative second-order N-point equiradial designs under variations of model parameters, a case where the second-order full model is assumed as well as a case where the highest order interaction is assumed negligible and is thus removed from the model shall be considered. The second-order full model (main effects, interaction effects and quadratic effects model) to be employed in this study is

$$y = \beta_0 + \sum_{i=1}^k \beta_i x_i + \left\{ \sum_{i=1}^{k-1} \sum_{j=i+1}^k \beta_{ij} (x_i x_j) \right\} + \sum_{i=1}^k \beta_{ii} x_i^2 + \varepsilon; -\rho \leq x_i, x_j \leq \rho \quad (1)$$

and the second-order reduced (no-interaction) model to be employed in this study is the model

$$y = \beta_0 + \sum_{i=1}^k \beta_i x_i + \sum_{i=1}^k \beta_{ii} x_i^2 + \varepsilon; -\rho \leq x_i, x_j \leq \rho \quad (2)$$

having main effects and quadratic effects only.

Since the equiradial designs are alternative second-order N-point spherical response surface methodology designs in two variables, k shall be set at 2. The real value ρ represents the design radius or the axial distance. Interest here lies in knowing what effect the removal of the interactive term would have on the N-point designs as measured by some optimality criteria and efficiency values. The five alphabetic optimality criteria to be employed in this work are A-, D-, E-, G- and T-optimality criteria. Each shall summarize how good the experimental designs are using the two defined models. As can be seen in most standard literatures on optimal designs such as Myer *et al.* (2009) and Rady *et al.* (2009), D-optimality criterion focuses on good model parameter estimation. A D-optimal design is one in which the determinant of the information matrix

$$M = \frac{X^T X}{N}$$

is maximized over all designs, where X represents the design matrix associated with the design and X^T represents its transpose. The criterion of D-optimality is equivalent to minimizing the determinant of M^{-1} . A-optimality criterion is one in which the sum of the variances of the model coefficients is minimized. The criterion of A-optimality thus minimizes the trace of M^{-1} and is defined as

$$\text{Min tr} (M^{-1})$$

where Min implies that minimization is over all designs and tr represents trace.

E-optimality criterion maximizes the minimum eigen value of M or equivalently minimizes the maximum eigen value of M^{-1} . The E-optimality criterion is defined by

$$\text{Max } \lambda_{\min}(M) \equiv \text{Min } \lambda_{\max}(M^{-1})$$

where λ_{\min} and λ_{\max} represent minimum eigen value and maximum eigen value, respectively. G-optimality criterion considers designs whose maximum scaled prediction variance, $v(\underline{x})$, in the region of the design is not too large. Hence, a G-optimal design minimizes the maximum scaled prediction variance and is defined by

$$\text{Min} \{ \max_{\underline{x} \in R} v(\underline{x}) \}$$

The assertion that designs optimal for one model need not be optimal for another model shall be investigated using equiradial designs for the full and reduced bivariate models in equations 1 and 2. Furthermore, the efficiency of designs shall be considered. In comparing two designs, the relative efficiency is seen as the ratio of their separate efficiencies. We shall employ the D-efficiency criterion as the test criterion. This criterion has been extensively used as a single numerical measure of the efficiency of designs. By definition, the D-efficiency of a design $\xi^{(1)}$ is given as

$$D_{eff} = 100 \times (\det M(\xi^{(1)}))^{\frac{1}{p}} \quad (3)$$

and the D-efficiency of a design $\xi^{(1)}$ relative to the design $\xi^{(2)}$ is given as

$$D_{eff} = \left(\frac{\det M(\xi^{(1)})}{\det M(\xi^{(2)})} \right)^{\frac{1}{p}} \quad (4)$$

where $M(\cdot)$ is the information matrix of the design and p is the number of model parameters. For an N -point equiradial design, ξ_N , the information matrix of the design ξ_N is $X_N^T X_N$ and normalized as $\frac{X_N^T X_N}{N}$ to remove the effect of changing design sizes. The $(N \times p)$ matrix, X_N , is the design matrix whose columns are built from the model and the design ξ_N and $(\cdot)^T$ represents transpose. In comparing designs, the best design is one with the largest D-efficiency value. In terms of relative efficiency, the ratio in equation 4 exceeds unity if the design $\xi^{(1)}$ is better than the design $\xi^{(2)}$ and lies between zero and one, otherwise.

3. Illustrative Examples

Since the equiradial designs are defined for two

controllable variables, the bivariate models to be considered are

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \varepsilon$$

and

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{22} x_1^2 + \beta_{22} x_2^2 + \varepsilon$$

The N-point equiradial designs ξ_6 , ξ_7 , ξ_8 and ξ_9 shall be employed. They are defined as

$$\xi_6 = \begin{pmatrix} 1 & 0 \\ 0.31 & 0.95 \\ -0.81 & 0.59 \\ -0.81 & -0.59 \\ 0.31 & -0.95 \\ 0 & 0 \end{pmatrix}$$

$$\xi_7 = \begin{pmatrix} 1 & 0 \\ 0.5 & 0.87 \\ -0.5 & 0.87 \\ -1 & 0 \\ -0.5 & -0.87 \\ 0.5 & -0.87 \\ 0 & 0 \end{pmatrix}$$

$$\xi_8 = \begin{pmatrix} 1 & 0 \\ 0.62 & 0.78 \\ -0.22 & 0.97 \\ -0.90 & 0.43 \\ -0.90 & -0.43 \\ -0.22 & -0.97 \\ 0.62 & -0.78 \\ 0 & 0 \end{pmatrix}$$

$$\xi_9 = \begin{pmatrix} 1 & 0 \\ 0.71 & 0.71 \\ 0 & 1 \\ -0.71 & 0.71 \\ -1 & 0 \\ -0.71 & -0.71 \\ 0 & -1 \\ 0.71 & -0.71 \\ 0 & 0 \end{pmatrix}$$

The associated design matrices using the designs and the full bivariate quadratic model are

$$X_{(F)6} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0.31 & 0.95 & 0.2945 & 0.0961 & 0.9025 \\ 1 & -0.81 & 0.59 & -0.4779 & 0.6561 & 0.3481 \\ 1 & -0.81 & -0.59 & 0.4779 & 0.6561 & 0.3481 \\ 1 & 0.31 & -0.95 & -0.2945 & 0.0961 & 0.9025 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{(F)7} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0.5 & 0.87 & 0.435 & 0.25 & 0.7569 \\ 1 & -0.5 & 0.87 & -0.435 & 0.25 & 0.7569 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -0.5 & -0.87 & 0.435 & 0.25 & 0.7569 \\ 1 & 0.5 & -0.87 & -0.435 & 0.25 & 0.7569 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{(F)8} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0.62 & 0.78 & 0.4836 & 0.3844 & 0.6084 \\ 1 & -0.22 & 0.97 & -0.2134 & 0.0484 & 0.9409 \\ 1 & -0.9 & 0.43 & -0.387 & 0.81 & 0.1849 \\ 1 & -0.9 & -0.43 & 0.387 & 0.81 & 0.1849 \\ 1 & -0.22 & -0.97 & 0.2134 & 0.0484 & 0.9409 \\ 1 & 0.62 & -0.78 & -0.4836 & 0.3844 & 0.6084 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{(F)9} =$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0.71 & 0.71 & 0.5041 & 0.5041 & 0.5041 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & -0.71 & 0.71 & -0.5041 & 0.5041 & 0.5041 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -0.71 & -0.71 & 0.5041 & 0.5041 & 0.5041 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0.71 & -0.71 & -0.5041 & 0.5041 & 0.5041 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Similarly, the associated design matrices using the designs and the reduced model are

$$X_{(R)6} =$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0.31 & 0.95 & 0.0961 & 0.9025 \\ 1 & -0.81 & 0.59 & 0.6561 & 0.3481 \\ 1 & -0.81 & -0.59 & 0.6561 & 0.3481 \\ 1 & 0.31 & -0.95 & 0.0961 & 0.9025 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{(R)7} =$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0.5 & 0.87 & 0.25 & 0.7569 \\ 1 & -0.5 & 0.87 & 0.25 & 0.7569 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & -0.5 & -0.87 & 0.25 & 0.7569 \\ 1 & 0.5 & -0.87 & 0.25 & 0.7569 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{(R)8} =$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0.62 & 0.78 & 0.3844 & 0.6084 \\ 1 & -0.22 & 0.97 & 0.0484 & 0.9409 \\ 1 & -0.9 & 0.43 & 0.81 & 0.1849 \\ 1 & -0.9 & -0.43 & 0.81 & 0.1849 \\ 1 & -0.22 & -0.97 & 0.0484 & 0.9409 \\ 1 & 0.62 & -0.78 & 0.3844 & 0.6084 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{(R)9} =$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0.71 & 0.71 & 0.5041 & 0.5041 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & -0.71 & 0.71 & 0.5041 & 0.5041 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & -0.71 & -0.71 & 0.5041 & 0.5041 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & 0.71 & -0.71 & 0.5041 & 0.5041 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For the full bivariate quadratic model, the normalized information matrices corresponding to the N-point equiradial designs are as follows;

$$M(\xi_{(F)6}) = \begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0.4174 & 0.4169 \\ 0 & 0.4174 & 0 & 0 & -0.0006 & -0.0007 \\ 0 & 0 & 0.4169 & -0.0007 & 0 & 0 \\ 0 & 0 & -0.0007 & 0.1050 & 0 & 0 \\ 0.4174 & -0.0006 & 0 & 0 & 0.3132 & 0.1050 \\ 0.4169 & -0.0007 & 0 & 0 & 0.1050 & 0.3119 \end{pmatrix}$$

$$M(\xi_{(F)7}) = \begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0.4286 & 0.4325 \\ 0 & 0.4286 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4325 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1081 & 0 & 0 \\ 0.4286 & 0 & 0 & 0 & 0.3214 & 0.1081 \\ 0.4325 & 0 & 0 & 0 & 0.1081 & 0.3274 \end{pmatrix}$$

$$M(\xi_{(F)8}) = \begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0.4357 & 0.4336 \\ 0 & 0.4357 & 0 & 0 & -0.0003 & 0.0010 \\ 0 & 0 & 0.4336 & 0.0010 & 0 & 0 \\ 0 & 0 & 0.0010 & 0.1073 & 0 & 0 \\ 0.4357 & -0.0003 & 0 & 0 & 0.3266 & 0.1073 \\ 0.4336 & 0.0010 & 0 & 0 & 0.1073 & 0.3224 \end{pmatrix}$$

$$M(\xi_{(F)9}) = \begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0.4463 & 0.4463 \\ 0 & 0.4463 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4463 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1129 & 0 & 0 \\ 0.4463 & 0 & 0 & 0 & 0.3352 & 0.1129 \\ 0.4463 & 0 & 0 & 0 & 0.1129 & 0.3352 \end{pmatrix}$$

For the reduced bivariate quadratic model, the normalized information matrices corresponding to the equiradial designs are as follows;

$$M(\xi_{(R)6}) = \begin{pmatrix} 1.0000 & 0 & 0 & 0.4174 & 0.4169 \\ 0 & 0.4174 & 0 & -0.0006 & -0.0007 \\ 0 & 0 & 0.4169 & 0 & 0 \\ 0.4174 & -0.0006 & 0 & 0.3132 & 0.1050 \\ 0.4169 & -0.0007 & 0 & 0.1050 & 0.3119 \end{pmatrix}$$

$$M(\xi_{(R)7}) = \begin{pmatrix} 1.0000 & 0 & 0 & 0.4286 & 0.4325 \\ 0 & 0.4286 & 0 & 0 & 0 \\ 0 & 0 & 0.4325 & 0 & 0 \\ 0.4286 & 0 & 0 & 0.3214 & 0.1081 \\ 0.4325 & 0 & 0 & 0.1081 & 0.3274 \end{pmatrix}$$

$$M(\xi_{(R)8}) = \begin{pmatrix} 1.0000 & 0 & 0 & 0.4357 & 0.4336 \\ 0 & 0.4357 & 0 & -0.0003 & 0.0010 \\ 0 & 0 & 0.4336 & 0 & 0 \\ 0.4357 & -0.0003 & 0 & 0.3266 & 0.1073 \\ 0.4336 & 0.0010 & 0 & 0.1073 & 0.3224 \end{pmatrix}$$

$$M(\xi_{(R)9}) = \begin{pmatrix} 1.0000 & 0 & 0 & 0.4463 & 0.4463 \\ 0 & 0.4463 & 0 & 0 & 0 \\ 0 & 0 & 0.4463 & 0 & 0 \\ 0.4463 & 0 & 0 & 0.3352 & 0.1129 \\ 0.4463 & 0 & 0 & 0.1129 & 0.3352 \end{pmatrix}$$

The numerical evaluations associated with the designs and the models with respect to A-, D-, E-, G- and T-optimality criteria as in Appendices A and B. The evaluations include the determinant value of the information matrix and its inverse, the trace of the information matrix and its inverse, the eigenvalues of the information matrix and its inverse as well as the scaled prediction variances.

4. Results and Discussion

A concise summary of numerical evaluations associated with the equiradial designs and the models is presented in Table 1. D-efficiency values of the equiradial designs for the full and reduced bivariate quadratic models as well as the relative D-efficiency values are presented in Table 2. It is clear as contained in literature that relationships exist among the various optimality criteria. One of such is that a design

that maximizes the determinant value of information matrix equivalently minimizes the determinant value of the inverse of the information matrix. As observed in this research, while seeking to minimize the trace of the inverse of the information matrix, the N-point equiradial designs performed correspondingly better for the reduced model than for the full model. By implication, the average variance of the estimates of the regression coefficients associated with the reduced model is better minimized using the equiradial designs than that observed for estimates of the regression coefficients associated with the full model.

It is also observed that when interest is in maximizing the determinant of the information matrix of the design or equivalently minimizing the determinant of the variance-covariance matrix, the N-point equiradial designs performed correspondingly better for the reduced model than for the full model. By implication, the equiradial designs minimize the generalized variance of the parameter estimates

for the reduced model than for the corresponding full model. However, a different opinion holds for the full model under T-optimality criterion as the N-point equiradial designs performed correspondingly better for the full model than for the reduced model. There was no observed design preference for the models under E- and G-optimality criterion.

For the full model, the 6-point equiradial design was best under A-, E- and G-optimality criteria. The 7-point equiradial design was best under D-optimality criterion and the 9-point equiradial design was best under T-optimality criterion. For the reduced model, the 6-point equiradial design was outstandingly best under A-, D-, E- and G-optimality criteria and the 9-point equiradial design was best under T-optimality criterion. Also for the reduced model, the 7-point equiradial design had maximum determinant value of information matrix too. Hence both 6-point and 7-point equiradial designs have uniform determinant value of

information matrices and consequently have equal preference in terms of D-optimality criterion. The D-efficiency values of the equiradial designs confirm the preferences of the equiradial designs for the reduced model.

It is worthy of note that each of the N-point exact designs had very low efficiency value for the full as well as for the reduced model. This is possibly because the small sized equiradial designs may be far from being orthogonal. Although relative D-efficiency values have been computed for the designs under study, it would be more informative to obtain the efficiency of each design relative to the global D-optimal design for the full as well as for the reduced model. Global D-optimal design as used here refers to the D-optimal design whose maximum scaled prediction variance approximately equals the number of model parameters. Investigations carried out in this work confirm that optimal designs are model dependent.

Table 1. Optimality Constants for the Equiradial Designs

Model Type	Design Size N	Optimality Constant						
		A-Optimality (Trace of M^{-1})	D-Optimality		E-Optimality		G-Optimality (maximum scaled predictive variance)	T-Optimality (Trace of M)
			(Determinant of M)	(Determinant of M^{-1})	(minimum Eigen value of M)	(max Eigen value of M^{-1})		
Second-order full model	6	39.5017	2.6398e-004	3.7881e+003	0.0509	19.6363	6	2.5644
	7	41.7031	2.6782e-004	3.7339e+003	0.0445	22.4650	7.0000	2.6180
	8	45.0559	2.3756e-004	4.2095e+003	0.0387	25.8116	8	2.6255
	9	46.9166	2.4890e-004	4.0177e+003	0.0352	28.3730	8.9988	2.6758
Second-order no interaction model	6	29.98130	0.0025	397.8996	0.0509	19.6363	6	2.4594
	7	32.4548	0.0025	403.7416	0.0445	22.4650	7.0000	2.5099
	8	35.7356	0.0022	451.6479	0.0387	25.8116	8	2.5182
	9	38.0624	0.0022	453.7635	0.0352	28.3730	8.9988	2.5629

Table 2. D-Efficiency Values and Relative D-Efficiency Values of the Equiradial Designs

Design size N	Full model		Reduced model	
	D-efficiency	Relative D-efficiency	D-efficiency	Relative D-efficiency
6	0.2533	0.9976	0.3017	1.0000
7	0.2539	1.0000	0.3017	1.0000
8	0.2489	0.9802	0.2941	0.9748
9	0.2508	0.9879	0.2941	0.9748

Appendix A: Computations for the Full Model

$N=6$

$(A'A)/6$

ans =

1.0000	-0.0000	0.0000	0	0.4174	0.4169
-0.0000	0.4174	0	0.0000	-0.0006	-0.0007
0.0000	0	0.4169	-	0.0007	0.0000
0	0.0000	-0.0007	0.1050	-0.0000	0
0.4174	-0.0006	0.0000	-	0.0000	0.3132
0.4169	-0.0007	0.0000	0	0.1050	0.3119

$\det((A'A)/6) = 2.6398e-004$

$\text{trace}((A'A)/6) = 2.5644$

$\text{eig}((A'A)/6) =$

0.0509
0.1050
0.2075
0.4169
0.4174
1.3667

$6*\text{inv}(A'A) =$

6.0000	-0.0184	0.0000	-0.0000	-5.9816	-6.0050
-0.0184	2.3959	-0.0000	-0.0000	0.0209	0.0231
0.0000	-0.0000	2.3989	0.0166	-0.0000	-0.0000
-0.0000	-0.0000	0.0166	9.5203	0.0000	0.0000
-5.9816	0.0209	-0.0000	0.0000	9.5623	4.7745
-6.0050	0.0231	-0.0000	0.0000	4.7745	9.6243

$\det(6*\text{inv}(A'A)) = 3.7881e+003$

$\text{trace}(6*\text{inv}(A'A)) = 39.5017$

$\text{eig}(6*\text{inv}(A'A)) =$

19.6363
0.7317
4.8187
2.3958
9.5204
2.3988

$((A'A)/6)*(6*\text{inv}(A'A)) =$

1.0000	0	0.0000	-0.0000	0	0
0.0000	1.0000	0.0000	-0.0000	-0.0000	-0.0000
-0.0000	-0.0000	1.0000	0.0000	0.0000	0.0000
0.0000	-0.0000	-0.0000	1.0000	-0.0000	-0.0000
0.0000	0.0000	-0.0000	-0.0000	1.0000	0
0.0000	-0.0000	0.0000	-0.0000	-0.0000	1.0000

$\text{trace}(((A'A)/6)*(6*\text{inv}(A'A))) = 6$

$B*(6*\text{inv}(A'A))*B' = 6.0000$

$C*(6*\text{inv}(A'A))*C' = 6.0000$

$$D*(6*\text{inv}(A'*A))*D'=6$$

$$E*(6*\text{inv}(A'*A))*E'=6.0000$$

$$F*(6*\text{inv}(A'*A))*F'=6$$

$$G*(6*\text{inv}(A'*A))*G'=6.0000$$

$$N=7$$

$$(A'*A)/7=$$

1.0000	0	0	0	0.4286	0.4325
0	0.4286	0	0	0	0
0	0	0.4325	0	0	0
0	0	0	0.1081	0	0
0.4286	0	0	0	0.3214	0.1081
0.4325	0	0	0	0.1081	0.3274

$$\det((A'*A)/7)=2.6782\text{e-}004$$

$$\text{trace}((A'*A)/7)=2.6180$$

$$\text{eig}((A'*A)/7)=$$

0.0445
0.1081
0.2163
0.4286
0.4325
1.3880

$$7*\text{inv}(A'*A)=$$

7.0000	0	0	0	-7.0000	-6.9362
0	2.3333	0	0	0	0
0	0	2.3121	0	0	0
0	0	0	9.2482	0	0
-7.0000	0	0	0	10.5000	5.7802
-6.9362	0	0	0	5.7802	10.3094

$$\det(7*\text{inv}(A'*A))=3.7339\text{e+}003$$

$$\text{trace}(7*\text{inv}(A'*A))=41.7031$$

$$\text{eig}(7*\text{inv}(A'*A))=$$

22.4650
0.7204
4.6240
2.3333
2.3121
9.2482

$$((A'*A)/7)*(7*\text{inv}(A'*A))=$$

1.0000	0	0	0	-0.0000	0
0	1.0000	0	0	0	0
0	0	1.0000	0	0	0
0	0	0	1.0000	0	0

-0.0000	0	0	0	1.0000	0.0000
0.0000	0	0	0	-0.0000	1.0000

$$\text{trace}(((A'*A)/7)*(7*\text{inv}(A'*A)))=6$$

$$B*(7*\text{inv}(A'*A))*B'=5.8333$$

$$C*(7*\text{inv}(A'*A))*C'=5.8333$$

$$D*(7*\text{inv}(A'*A))*D'=5.8333$$

$$E*(7*\text{inv}(A'*A))*E'=5.8333$$

$$F*(7*\text{inv}(A'*A))*F'=5.8333$$

$$G*(7*\text{inv}(A'*A))*G'=5.8333$$

$$H*(7*\text{inv}(A'*A))*H'=7.0000$$

$$N=8$$

$$(A'*A)/8=$$

1.0000	0.0000	0.0000	0	0.4357	0.4336
0.0000	0.4357	0	0	-0.0003	0.0010
0.0000	0	0.4336	0.0010	0	-0.0000
0	0	0.0010	0.1073	0	0
0.4357	-0.0003	0	0	0.3266	0.1073
0.4336	0.0010	-0.0000	0	0.1073	0.3224

$$\det((A'*A)/8)=2.3756\text{e-}004$$

$$\text{trace}((A'*A)/8)=2.6255$$

$$\text{eig}((A'*A)/8)=$$

0.0387
0.1073
0.2172
0.4336
0.4357
1.3930

$$8*\text{inv}(A'*A)=$$

8.0000	0.0116	-0.0000	0.0000	-8.0157	-8.0903
0.0116	2.2952	-0.0000	0.0000	-0.0065	-0.0202
-0.0000	-0.0000	2.3066	-0.0204	0.0000	0.0000
0.0000	0.0000	-0.0204	9.3203	-0.0000	-0.0000
-8.0157	-0.0065	0.0000	-0.0000	11.4698	6.9620
-8.0903	-0.0202	0.0000	-0.0000	6.9620	11.6640

$$\det(8*\text{inv}(A'*A))=4.2095\text{e+}003$$

$$\text{trace}(8*\text{inv}(A'*A))=45.0559$$

$$\text{eig}(8*\text{inv}(A'*A))=$$

25.8116
0.7179
4.6044
2.2951

9.3204
2.3065

$$\det(8*\text{inv}(A'*A))= 4.2095\text{e}+003$$

$$B*(10*\text{inv}(A'*A))*B'= 7.1796$$

$$B*(8*\text{inv}(A'*A))*B'= 5.7436$$

$$C*(8*\text{inv}(A'*A))*C'= 5.7079$$

$$D*(8*\text{inv}(A'*A))*D'= 5.7045$$

$$E*(8*\text{inv}(A'*A))*E'= 5.7158$$

$$F*(8*\text{inv}(A'*A))*F'= 5.7158$$

$$G*(8*\text{inv}(A'*A))*G'= 5.7045$$

$$H*(8*\text{inv}(A'*A))*H'= 5.7079$$

$$I*(8*\text{inv}(A'*A))*I'= 8.0000$$

$$N=9$$

$$(A'*A)/9 =$$

1.0000	0	0	0	0.4463	0.4463
0	0.4463	0	0	0	0
0	0	0.4463	0	0	0
0	0	0	0.1129	0	0
0.4463	0	0	0	0.3352	0.1129
0.4463	0	0	0	0.1129	0.3352

$$\det((A'*A)/9)= 2.4890\text{e}-004$$

$$\text{trace}((A'*A)/9)= 2.6758$$

$$\text{eig}((A'*A)/9)=$$

0.0352
0.1129
0.2222
0.4463
0.4463
1.4129

$$9*\text{inv}(A'*A)=$$

8.9988	0	0	0	-8.9619	-8.9619
0	2.2408	0	0	0	0
0	0	2.2408	0	0	0
0	0	0	8.8542	0	0
-8.9619	0	0	0	12.2910	7.7910
-8.9619	0	0	0	7.7910	12.2910

$$\det(9*\text{inv}(A'*A))= 4.0177\text{e}+003$$

$$\text{trace}(9*\text{inv}(A'*A))=46.9166$$

$$\text{eig}(9*\text{inv}(A'*A))=$$

0.7078

2.2408
 2.2408
 4.5000
 8.8542
 28.3730

$((A'A)/9)*(9*inv(A'A))=$

1.0000	0	0	0	-0.0000	-0.0000
0	1.0000	0	0	0	0
0	0	1.0000	0	0	0
0	0	0	1.0000	0	0
-0.0000	0	0	0	1.0000	-0.0000
-0.0000	0	0	0	-0.0000	1.0000

$trace(((A'A)/9)*(9*inv(A'A)))= 6.0000$

$B*(9*inv(A'A))*B'= 5.6068$

$C*(9*inv(A'A))*C'= 5.6435$

$D*(9*inv(A'A))*D'= 5.6068$

$E*(9*inv(A'A))*E'= 5.6435$

$F*(9*inv(A'A))*F'= 5.6068$

$G*(9*inv(A'A))*G'= 5.6435$

$H*(9*inv(A'A))*H'= 5.6068$

$I*(9*inv(A'A))*I'= 5.6435$

$J*(9*inv(A'A))*J'= 8.9988$

Appendix B: Computations for the Reduced Model

$N=6$

$(A'A)/6=$

1.0000	-0.0000	0.0000	0.4174	0.4169
-0.0000	0.4174	0	-0.0006	-0.0007
0.0000	0	0.4169	0.0000	0.0000
0.4174	-0.0006	0.0000	0.3132	0.1050
0.4169	-0.0007	0.0000	0.1050	0.3119

$det((A'A)/6)=0.0025$

$trace((A'A)/6)=2.4594$

$eig((A'A)/6)=$

0.0509
 0.2075
 0.4169
 0.4174
 1.3667

$6 * \text{inv}(A' * A) =$

6.0000	-0.0184	0.0000	-5.9816	-6.0050
-0.0184	2.3959	-0.0000	0.0209	0.0231
0.0000	-0.0000	2.3988	-0.0000	-0.0000
-5.9816	0.0209	-0.0000	9.5623	4.7745
-6.0050	0.0231	-0.0000	4.7745	9.6243

$\det(6 * \text{inv}(A' * A)) = 397.8996$

$\text{trace}(6 * \text{inv}(A' * A)) = 29.9813$

$\text{eig}(6 * \text{inv}(A' * A)) =$

19.6363
0.7317
4.8187
2.3958
2.3988

$((A' * A) / 6) * (6 * \text{inv}(A' * A)) =$

1.0000	0	0	-0.0000	0
-0.0000	1.0000	-0.0000	0.0000	0
0.0000	-0.0000	1.0000	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	1.0000	-0.0000
0.0000	-0.0000	0	-0.0000	1.0000

$\text{trace}(((A' * A) / 6) * (6 * \text{inv}(A' * A))) = 5$

$B * (6 * \text{inv}(A' * A)) * B' = 6$

$C * (6 * \text{inv}(A' * A)) * C' = 5.1650$

$D * (6 * \text{inv}(A' * A)) * D' = 3.8350$

$E * (6 * \text{inv}(A' * A)) * E' = 3.8350$

$F * (6 * \text{inv}(A' * A)) * F' = 5.1650$

$G * (6 * \text{inv}(A' * A)) * G' = 6.0000$

$N = 7$

$(A' * A) / 7 =$

1.0000	0	0	0.4286	0.4325
0	0.4286	0	0	0
0	0	0.4325	0	0
0.4286	0	0	0.3214	0.1081
0.4325	0	0	0.1081	0.3274

$\det((A' * A) / 7) = 0.0025$

$\text{trace}((A' * A) / 7) = 2.5099$

$\text{eig}((A' * A) / 7) =$

0.0445
0.2163
0.4286
0.4325
1.3880

$$7*\text{inv}(A'*A)=$$

7.0000	0	0	-7.0000	-6.9362
0	2.3333	0	0	0
0	0	2.3121	0	0
-7.0000	0	0	10.5000	5.7802
-6.9362	0	0	5.7802	10.3094

$$\det(7*\text{inv}(A'*A))= 403.7416$$

$$\text{trace}(7*\text{inv}(A'*A))= 32.4548$$

$$\text{eig}(7*\text{inv}(A'*A))=$$

22.4650
0.7204
4.6240
2.3333
2.3121

$$((A'*A)/7)*(7*\text{inv}(A'*A))=$$

1.0000	0	0	-0.0000	0
0	1.0000	0	0	0
0	0	1.0000	0	0
-0.0000	0	0	1.0000	0.0000
0.0000	0	0	-0.0000	1.0000

$$\text{trace}(((A'*A)/7)*(7*\text{inv}(A'*A)))= 5$$

$$B*(7*\text{inv}(A'*A))*B'= 5.8333$$

$$C*(7*\text{inv}(A'*A))*C'= 4.0833$$

$$D*(7*\text{inv}(A'*A))*D'= 4.0833$$

$$E*(7*\text{inv}(A'*A))*E'= 5.8333$$

$$F*(7*\text{inv}(A'*A))*F'= 4.0833$$

$$G*(7*\text{inv}(A'*A))*G'= 4.0833$$

$$H*(7*\text{inv}(A'*A))*H'= 7.0000$$

$$N=8$$

$$(A'*A)/8=$$

1.0000	0.0000	0.0000	0.4357	0.4336
0.0000	0.4357	0	-0.0003	0.0010
0.0000	0	0.4336	0	-0.0000
0.4357	-0.0003	0	0.3266	0.1073
0.4336	0.0010	-0.0000	0.1073	0.3224

$$\det((A'*A)/8)=0.0022$$

$$\text{trace}((A'*A)/8)= 2.5182$$

$$\text{eig}((A'*A)/8)=$$

0.0387
0.2172
0.4336

0.4357
1.3930

$8 \cdot \text{inv}(A' \cdot A) =$

8.0000	0.0116	-0.0000	-8.0157	-8.0903
0.0116	2.2952	-0.0000	-0.0065	-0.0202
-0.0000	-0.0000	2.3065	0.0000	0.0000
-8.0157	-0.0065	0.0000	11.4698	6.9620
-8.0903	-0.0202	0.0000	6.9620	11.6640

$\det(8 \cdot \text{inv}(A' \cdot A)) = 451.6479$

$\text{trace}(8 \cdot \text{inv}(A' \cdot A)) = 35.7356$

$\text{eig}(8 \cdot \text{inv}(A' \cdot A)) =$

25.8116
0.7179
4.6044
2.2951
2.3065

$\det(8 \cdot \text{inv}(A' \cdot A)) = 451.6479$

$B \cdot (8 \cdot \text{inv}(A' \cdot A)) \cdot B' = 5.7436$

$C \cdot (8 \cdot \text{inv}(A' \cdot A)) \cdot C' = 3.5435$

$D \cdot (8 \cdot \text{inv}(A' \cdot A)) \cdot D' = 5.2716$

$E \cdot (8 \cdot \text{inv}(A' \cdot A)) \cdot E' = 4.3131$

$F \cdot (8 \cdot \text{inv}(A' \cdot A)) \cdot F' = 4.3131$

$G \cdot (8 \cdot \text{inv}(A' \cdot A)) \cdot G' = 5.2716$

$H \cdot (8 \cdot \text{inv}(A' \cdot A)) \cdot H' = 3.5435$

$I \cdot (8 \cdot \text{inv}(A' \cdot A)) \cdot I' = 8.0000$

$N=9$

$(A' \cdot A)/9 =$

1.0000	0	0	0.4463	0.4463
0	0.4463	0	0	0
0	0	0.4463	0	0
0.4463	0	0	0.3352	0.1129
0.4463	0	0	0.1129	0.3352

$\det((A' \cdot A)/9) = 0.0022$

$\text{trace}((A' \cdot A)/9) = 2.5629$

$\text{eig}((A' \cdot A)/9) =$

0.0352
0.2222
0.4463
0.4463
1.4129

$$9 * \text{inv}(A' * A) =$$

$$\begin{array}{ccccc} 8.9988 & 0 & 0 & -8.9619 & -8.9619 \\ 0 & 2.2408 & 0 & 0 & 0 \\ 0 & 0 & 2.2408 & 0 & 0 \\ -8.9619 & 0 & 0 & 12.2910 & 7.7910 \\ -8.9619 & 0 & 0 & 7.7910 & 12.2910 \end{array}$$

$$\det(9 * \text{inv}(A' * A)) = 453.7635$$

$$\text{trace}(9 * \text{inv}(A' * A)) = 38.0624$$

$$\text{eig}(9 * \text{inv}(A' * A)) =$$

$$\begin{array}{c} 0.7078 \\ 2.2408 \\ 2.2408 \\ 4.5000 \\ 28.3730 \end{array}$$

$$((A' * A) / 9) * (9 * \text{inv}(A' * A)) =$$

$$\begin{array}{ccccc} 1.0000 & 0 & 0 & -0.0000 & -0.0000 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 \\ -0.0000 & 0 & 0 & 1.0000 & -0.0000 \\ -0.0000 & 0 & 0 & -0.0000 & 1.0000 \end{array}$$

$$\text{trace}(((A' * A) / 9) * (9 * \text{inv}(A' * A))) = 5.0000$$

$$B * (9 * \text{inv}(A' * A)) * B' = 5.6068$$

$$C * (9 * \text{inv}(A' * A)) * C' = 3.3935$$

$$D * (9 * \text{inv}(A' * A)) * D' = 5.6068$$

$$E * (9 * \text{inv}(A' * A)) * E' = 3.3935$$

$$F * (9 * \text{inv}(A' * A)) * F' = 5.6068$$

$$G * (9 * \text{inv}(A' * A)) * G' = 3.3935$$

$$H * (9 * \text{inv}(A' * A)) * H' = 5.6068$$

$$I * (9 * \text{inv}(A' * A)) * I' = 3.3935$$

$$J * (9 * \text{inv}(A' * A)) * J' = 8.9988$$

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