Science and Technology 2015, 5(3): 45-56
DOI: 10.5923/j.scit.20150503.02

Free Convection Flow in Rectangular Enclosures Driven by a Continuously Moving Horizontal Plate

Abdulsalam Ya'u Gital[1,*], Muhammed Abdulhameed[2], Bala Ma'aji Abdulhamid[1], Muhammad Sirajo Aliyu[3], Aliyu Danladi Hina[4]

[1]Deaprtment of Mathematical Sciences, Faculty of Science, Abubakar Tafawa University Bauchi, Nigeria
[2]Department of Mathematics Faculty of Science, Universiti Tun Hussein Onn, Malaysia
[3]Department of Science, Collage of Science and Technology, Hassan Usman Katsina Polytechnic, Katsina Nigeria
[4]Department of Basic Studies, The Federal Polytechnic Bauchi Nigeria

Abstract In this paper, an attempt has been made for the analytical solution of the unsteady mixed convection flow induced by the combined effects of the mechanically driven lid and the buoyancy force within rectangular enclosures. The couple of nonlinear equations describing the fluid behavior and temperature distribution within rectangular enclosures form a very complex system where the analytical solution is not available in the literature. To the best of the author's knowledge, this the first time such an attempt is presented. To drive the solutions of the coupled nonlinear boundary systems, an optimal homotopy asymptotic method was applied in the initial stage, then follow by the method of eigenfunction combined with orthogonality property. Explicit analytical solutions for both fluid velocity and temperature distribution are obtained. The results proved the effective technique for solving the couple nonlinear equations in a rectangular enclosure.

Keywords Optimal homotopy asymptotic technique, Mixed convection, Heat transfer, Stream function, Isotherms

1. Introduction

Considerable attention has been paid in recent years to the problems heat transfer analysis rectangular domain with regular boundary conditions yielding a well-posed problem. The governing couple equations are very complicated and highly nonlinear, with mostly numerical solutions.

The theoretical studies of mixed convective flow in a lid-driven cavity finds applications in flow and heat transfer in solar ponds and solar collectors, dynamics of lakes, reservoirs and cooling ponds, cooling of electronic systems, thermal-hydraulics of nuclear reactors, thermal convection in micropolar fluids, chemical processing equipment, lubricating grooves, crystal growing, materials processing such as float glass production, galvanizing, metal coating and casting, food processing, and industrial processes where a solid ribbon or a solid material is heated as it moves through a furnace, among others [1-10].

The above literature shows that the convective heat transfer and flow in a rectangular enclosure driven by a horizontal wall while being cooled from one horizontal and vertical wall, with the other vertical wall thermally isolated have not been investigated. This configuration finds practical applications in the cooling of an extruded plate in a hot rolling process. The fluid flow and the heat transfer patterns within the enclosure dictate the degree of cooling and hence the quality of the final product. This fact motivates the present study. The purpose of this work is therefore to present a parametric investigation of the Richardson and Prandtl numbers, and the aspect ratio on the flow patterns, energy distribution and heat transfer behaviour for this configuration.

The aim of the present work is to solve the problem of mixed convection flow in rectangular enclosures driven by a continuously moving horizontal plate using an analytical technique, OHAM. In view of this we have established an analytic solution based on the OHAM to venture further into the regime of nonlinear fluids.

2. Analytical approach

We start with the basic idea of OHAM, the governing equations to solved are classified into two velocity Ω and temperature T that is

$$\begin{aligned} L_1\left(\Omega(x,y)\right)+N_1\left(\Omega(x,y)\right)+g_1(x,y)&=0.\\ L_2\left(T(x,y)\right)+N_2\left(T(x,y)\right)+g_2(x,y)&=0.\\ B(\Omega,T)&=0.\end{aligned}\qquad(1)$$

* Corresponding author:
abdulsalamgital@yahoo.com (Abdulsalam Ya'u Gital)
Published online at http://journal.sapub.org/scit

where $L_j\left(j=1,2\right)$, are linear operators, $N_j\left(j=1,2\right)$ are non-linear operators, $g_j\left(j=1,2\right)$ are known function and B is a boundary operator.

Construct an optimal homotopy $\varphi\left(x,y;q\right):\mathrm{R}\times\left[0,1\right]\to\mathrm{R}$ which satisfies the following equations:

$$\begin{aligned}&\left(1-q\right)\left[L_1\left(\varphi_1\left(x,y;\ q\right)\right)+g_1\left(x,y\right)\right]=H\left(q\right)\left[L_1\left(\varphi_1\left(x,y;\ q\right)\right)+g_1\left(x,y\right)+N_1\left(\varphi_1\left(x,y;\ q\right)\right)\right],\\&B_1\left(\varphi_1\left(x,y;\ q\right)\right)=0,\end{aligned}\tag{2}$$

$$\begin{aligned}&\left(1-q\right)\left[L_2\left(\varphi_2\left(x,y;\ q\right)\right)+g_2\left(x,y\right)\right]=H\left(q\right)\left[L_2\left(\varphi_2\left(x,y;\ q\right)\right)+g_2\left(x,y\right)+N_2\left(\varphi_2\left(x,y;\ q\right)\right)\right],\\&B_2\left(\varphi_2\left(x,y;\ q\right)\right)=0.\end{aligned}\tag{3}$$

where $x,y\in\mathrm{R}$ and $0\le q\le 1$ is an embedding parameter, $H\left(q\right)$ is a nonzero auxiliary function for $q\neq 0$ and $H\left(0\right)=0$, and $\varphi_j\left(x,y;\ q\right)$ $\left(j=1,2\right)$ are unknown functions

Clearly, when $q=0$ and $q=1$ it holds that

$$\begin{aligned}&\varphi_1\left(x,y;\ 0\right)=\Omega_0\left(x,y\right),\quad \varphi_2\left(x,y,\ 0\right)=T_0\left(x,y\right),\\&\varphi_1\left(x,y,\ 1\right)=\Omega\left(x,y\right),\quad \varphi_2\left(x,y,\ 1\right)=T\left(x,y\right).\end{aligned}\tag{4}$$

Choose the auxiliary function $H\left(x,y;q\right)$ in the form

$$H\left(q\right)=qC_1+q^2C_2+\ldots+q^nC_n.\tag{5}$$

where $C_1,C_2\ldots,C_n$ are constants to be determined.

Construct a Taylor's series solution of equations (2) and (3) in the form

$$\varphi_j\left(x,y,\ q,C_i\right)=\Omega_0\left(x,y\right)+\sum_{k=1}^{\infty}\Omega_{jk}\left(x,y;C_i\right)q^k,\quad j=1,2.\text{ and } i=1,2,..,n\tag{6}$$

substituting equation (6) into equations (2) and (3) and boundary and equating the coefficients of like powers of q to obtain

$$\begin{aligned}&L_j\left(\Omega_{j0}\left(x,y\right)+g_j\left(x,y\right)\right)=0,\quad B\left(\Omega_{j0}\right)=0,\\&L_j\left(\Omega_{j1}\left(x,y\right)\right)=C_1N_{j0}\left(\Omega_{j0}\left(x,y\right)\right),\quad B\left(\Omega_{j1}\right)=0,\end{aligned}\tag{7}$$

and the k th order equation is defined by

$$\begin{aligned}L_j\left(\Omega_k\left(x,y\right)\right)-L_j\left(\Omega_{k-1}\left(x,y\right)\right)=C_kN_{j0}\left(\Omega_0\left(x,y\right)\right)+\sum_{i=1}^{k-1}C_i\Big[L_j\left(\Omega_{k-i}\left(x,y\right)\right)\\+N_{j(k-i)}\left(\Omega_0\left(x,y\right),\Omega_1\left(x,y\right),\ldots\Omega_{k-i}\left(x,y\right)\right)\Big],\\B\left(\Omega_k\right)=0;\qquad k=2,3,4,\ldots\end{aligned}\tag{8}$$

where $N_{j(k-i)}\left(\Omega_0\left(x,y\right),\Omega_1\left(x,y\right),\ldots\Omega_{k-i}\left(x,y\right)\right)$ is the coefficient of q^{k-i}, obtained by expanding $N_j\left(\Omega_j\left(x,y\right)\right)$ in series with respect to the embedding parameter q.

$$N_j\left(\Omega_j\left(x,y\right)\right)=N_{j0}\left(\Omega_{j0}\left(x,y\right)\right)+\sum_{k=1}^{\infty}N_{jk}\left(\Omega_{j0}\left(x,y\right),\Omega_{j1}\left(x,y\right),\Omega_{j2}\left(x,y\right),\ldots\Omega_{jk}\left(x,y\right)\right)q^k.\tag{9}$$

The solution of equation (1) can be approximately obtained in the form

$$\Omega^m\left(x,y;C_i\right)=\Omega_0\left(x,y\right)+\sum_{k=1}^{m}\Omega_k(x,y;C_i) \tag{10}$$

$$T^m\left(x,y;C_i\right)=T_0\left(x,y\right)+\sum_{k=1}^{m}T_k(x,y;C_i) \tag{11}$$

Substitute equations (10) and (11) into equation (1) it results the following expression for residual:

$$R_1\left(x,y;C_i\right)=L_1\left(\Omega^m\left(x,y;C_i\right)\right)+g_j\left(x,y\right)+N_2\left(\Omega^m\left(x,y;C_i\right)\right) \tag{12}$$

$$R_2\left(x,y;C_i\right)=L_2\left(\Omega^m\left(x,y;C_i\right)\right)+g_j\left(x,y\right)+N_2\left(\Omega^m\left(x,y;C_i\right)\right) \tag{13}$$

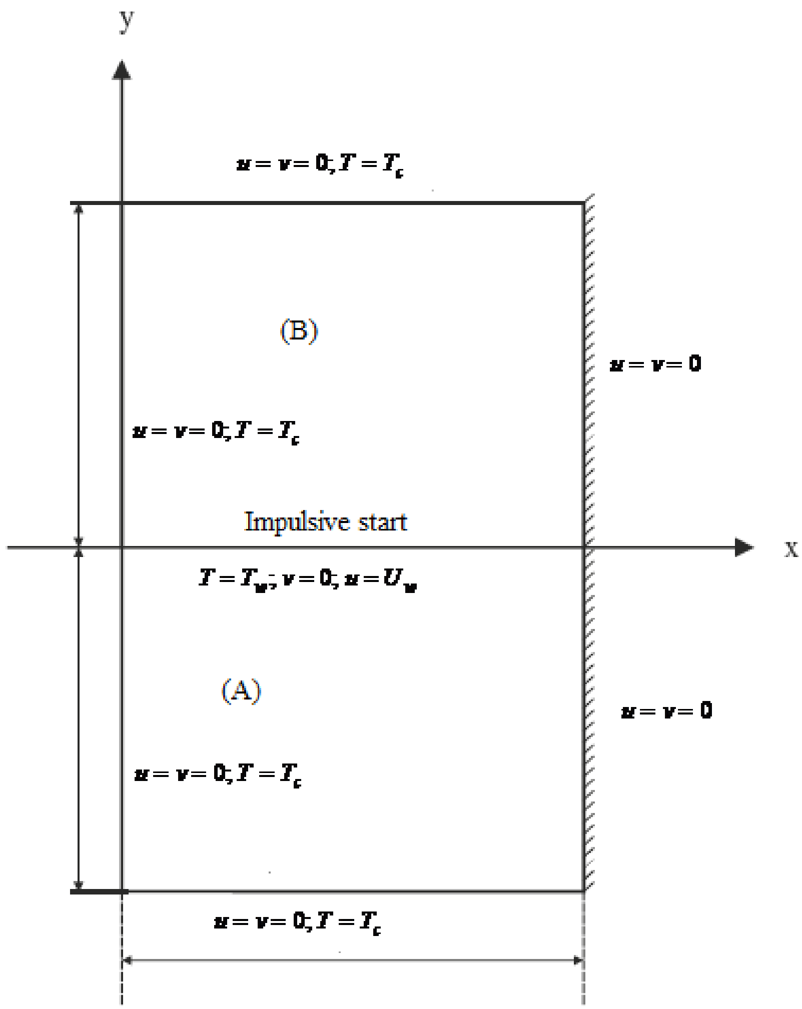

Figure 1. Physical configuration

If $R_j\left(x,y;C_i\right)=0$, then (10) and (11) will be the exact solution. Generally such a case will not happen for nonlinear problems, but we can minimize the functional by the method of lease squares

$$J_j\left(C_i\right)=\int_{a_1}^{b_1}\int_{a_2}^{b_2}R_j^2\left(x,y;C_i\right)dxdy \tag{14}$$

where a_1, a_2, b_1 and b_2 belong to the domain of the problem. Finally, the unknown constants $C_i\left(i=1,2,...m\right)$ can be optimally identified from the conditions

$$\frac{\partial J_j(C_i)}{\partial C_1} = \frac{\partial J_j(C_i)}{\partial C_2} = \dots \frac{\partial J_j(C_i)}{\partial C_m} = 0. \tag{15}$$

With these known values of $C_i\,(i=1,2,\dots m)$, the approximate solution of equation (10) and (11) are well determined.

Statement of the problem

In the present investigation, we consider convective heat transfer and flow in a rectangular enclosure driven by a horizontal wall, the horizontal plate divides the rectangular enclosure into equal halves while being cooled from one horizontal and vertical wall, with the other vertical wall thermally isolated as described by Waheed [1] Figure 1 show the physical configuration.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{16}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{17}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \upsilon\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \beta g\left(T - T_c\right), \tag{18}$$

$$\rho c_p\left(u\frac{\partial T}{\partial u} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{19}$$

where T_c is the temperature of ambient medium

In the above equations we will describe the velocity components u and v as the derivatives of the stream function

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{20}$$

the vorticity equation

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{21}$$

Introduced the following non-dimensional variables

$$x^* = \frac{x}{b},\ y^* = \frac{y}{l},\ u^* = \frac{u}{U_w},\ v^* = \frac{v}{U_w},\ T^* = \frac{T - T_c}{T_c - T_h}, \qquad \psi^* = \frac{\psi}{U_w L},\ \ \Omega = \frac{\omega}{U_w / L}. \tag{22}$$

where ψ is the stream function, ω is the vorticity and b is the weight of the enclosure.

The results of the analysis (after dropping the $*$-notation) are the stream function equation

$$\Omega = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) \tag{23}$$

and the vorticity transport equation

$$u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = -\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2}\right) + Ri\frac{\partial T}{\partial x}, \tag{24}$$

and also, the energy equation

$$u\frac{\partial T}{\partial u}+v\frac{\partial T}{\partial y}=\frac{1}{\operatorname{Re}\operatorname{Pr}}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{25}$$

where Re is the Reynolds number, R_i is the Richardson number and Pr is the Prandtl number.

The boundary conditions and initial condition for two enclosure A and B are as follows:

$$\begin{aligned}
&u=v=\psi=0, \quad \Omega=\left(\frac{\partial v}{\partial x}\right)_{x=0}, \quad T=0 \quad \text{at } x=0,\\
&u=v=\psi=0, \quad \Omega=\left(\frac{\partial v}{\partial x}\right)_{x=1}, \quad \frac{\partial T}{\partial x}=0 \quad \text{at } x=1,\\
&u=1, \quad v=\psi=0, \quad \Omega=-\left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad T=1 \quad \text{at } y=0.
\end{aligned} \tag{26}$$

Special cases for the boundary condition:

Case 1: Upper wall of enclosure A

$$u=v=\psi=0, \quad \Omega=-\left(\frac{\partial u}{\partial y}\right)_{y=1}, \quad T=0 \quad \text{at } y=1, \tag{27}$$

Case 2: Lower wall of enclosure B

$$u=v=\psi=0, \quad \Omega=-\left(\frac{\partial u}{\partial y}\right)_{y=-1}, \quad T=0 \quad \text{at } y=-1, \tag{28}$$

where ρ is the fluid density, p is the pressure, c_p is the specific heat capacity at constant pressure, β is the volumetric coefficient of thermal expansion, μ is the fluid viscosity, k is the thermal conductivity, t is the time, T is the temperature and u, v are the fluid velocity components, in the $x-$ and $y-$ directions, respectively.

The aim of the present investigation is to discuss the analytical approach for the steady two-dimensional incompressible and laminar in rectangular domain.

Applications

Choosing the linear and nonlinear operator defined by

$$L_1\left(\varphi_1(x,y;q)\right)=-\frac{1}{\operatorname{Re}}\left(\frac{\partial^2\Omega(x,y;q)}{\partial x^2}+\frac{\partial^2\Omega(x,y;q)}{\partial y^2}\right) \tag{29}$$

$$L_2\left(\varphi_2(x,y;q)\right)=-\frac{1}{\operatorname{Re}\operatorname{Pr}}\left(\frac{\partial^2 T(x,y;q)}{\partial x^2}+\frac{\partial^2 T(x,y;q)}{\partial y^2}\right) \tag{30}$$

$$N_1\left(\varphi_1(x,y;q)\right)=u(x,y;q)\frac{\partial\Omega(x,y;q)}{\partial x}-Ri\frac{\partial T(x,y;q)}{\partial x}+v(x,y;q)\frac{\partial\Omega(x,y;q)}{\partial y} \tag{31}$$

$$N_2\left(\varphi_2(x,y;q)\right)=u(x,y;q)\frac{\partial T(x,y;q)}{\partial x}+v(x,y;q)\frac{\partial T(x,y;q)}{\partial y} \tag{32}$$

and the stream function

$$\Omega_j=-\left(\frac{\partial^2\psi_j}{\partial x^2}+\frac{\partial^2\psi_j}{\partial y^2}\right) \tag{33}$$

The correspond boundary conditions are

$$\begin{aligned}
\varphi_1(0,y;q) &= \psi_1 = 0, \quad \varphi_2(0,y;q) = 0,\\
\varphi_1(1,y;q) &= \psi_1 = 0, \quad \frac{\partial \varphi_2(1,y;q)}{\partial x} = 0,\\
\varphi_1(x,0;q) &= \psi_1 = 1, \quad \varphi_2(x,0;q) = 1,\\
\varphi_1(x,-1;q) &= \psi_1 = 0, \quad \varphi_2(x,-1;q) = 0,\\
\varphi_1(x,1;q) &= \psi_1 = 0, \quad \varphi_2(x,1;q) = 0,
\end{aligned} \tag{34}$$

The zeroth order deformation:

$$\frac{\partial^2 \Omega_0}{\partial x^2} + \frac{\partial^2 \Omega_0}{\partial y^2} = 0, \tag{35}$$

$$\begin{aligned}
&\Omega_0(0,y) = \Omega_0(1,y) = \Omega_0(x,-1) = \Omega_0(x,1) = 0,\\
&\Omega_0(x,0) = 1.
\end{aligned} \tag{36}$$

$$\frac{\partial^2 T_0}{\partial x^2} + \frac{\partial^2 T_0}{\partial y^2} = 0, \tag{37}$$

$$\begin{aligned}
&T_0(0,y) = \frac{\partial T_0(1,y)}{\partial x} = T_0(x,-1) = T_0(x,1) = 0,\\
&T_0(x,0) = 1.
\end{aligned} \tag{38}$$

whose solutions are

$$\Omega_0(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\sinh\left[n\pi(y-1)\right], \tag{39}$$

$$T_0(x,y) = \sum_{m=0}^{\infty} A_m \sin\left[\left(\frac{2m+1}{2}\right)\pi x\right]\sinh\left[\left(\frac{2m+1}{2}\right)\pi(y-1)\right],. \tag{40}$$

Where $A_n = \dfrac{2}{\sinh(-n\pi)}\int_0^1 \sin(n\pi x)\,dx$ and $A_m = \dfrac{2}{\sinh\left[-\left(\frac{2m+1}{2}\right)\pi\right]}\int_0^1 \sin\left[\left(\frac{2m+1}{2}\right)\pi x\right]dx$

Using equation (39), the stream function (33) becomes

$$\frac{\partial^2 \psi_0}{\partial x^2} + \frac{\partial^2 \psi_0}{\partial y^2} = Q(x,y), \tag{41}$$

subject to the the boundary conditions

$$\psi_0(0,y) = \psi_0(1,y) = \psi_0(x,-1) = \psi_0(x,1) = 0 \tag{42a}$$

$$\psi_0(x,0) = 1. \tag{42b}$$

where $Q(x,y) = -\sum_{n=1}^{\infty} A_n \sin(n\pi x)\sinh\left[n\pi(y-1)\right]$

Our first approach to find the stream function is to split problem (41) in two problems that are easier to solve:

$$\psi_0 = \psi_{0_1} + \psi_{0_2}, \tag{43}$$

where ψ_{0_1} is a particular solution of the stream function and ψ_{0_2} is the solution of the corresponding homogeneous stream function. That is,

$$\frac{\partial^2 \psi_{0_1}}{\partial x^2}+\frac{\partial^2 \psi_{0_1}}{\partial y^2}=Q(x,y) \tag{44}$$

non-homogeneous equation (44) corresponding to the homogeneous boundary condition in (42a)

$$\frac{\partial^2 \psi_{0_2}}{\partial x^2}+\frac{\partial^2 \psi_{0_2}}{\partial y^2}=0 \tag{45}$$

homogeneous equation (45) corresponding to the nonhomogeneous boundary condition in (42b)

Using the method of eigenfunction expansion we expand ψ_{0_1} in a series of $x-$ eigenfunctions

$$\psi_{0_1}(x,y)=\sum_{p=1}^{\infty}\varphi_p(y)\sin(p\pi x) \tag{46}$$

By differentiating (46) twice with respect to y then replacing in (44) we obtain

$$\sum_{p=1}^{\infty}\frac{d^2\varphi_p(y)}{dy^2}\sin(p\pi x)+\frac{\partial^2 \psi_0}{\partial x^2}=Q(x,y) \tag{47}$$

Again, differentiating term-by-term of (46) twice with respect to x we obtain:

$$\frac{\partial^2 \psi_{0_1}}{\partial x^2}=\sum_{p=1}^{\infty}(p\pi)^2\varphi_p(y)\sin(p\pi x) \tag{48}$$

so that equation (47) becomes

$$\sum_{p=1}^{\infty}\left(\frac{d^2\varphi_p(y)}{dy^2}-\omega_p^2\varphi_p(y)\right)\sin(p\pi x)=Q(x,y) \tag{49}$$

where $\omega_p = p\pi$

Multiplying equation (49) by $\sin(q\pi x)$ and integrating with respect to x from $x=0$ to $x=1$ and interchanging the order of summation and integration, we get

$$\sum_{p=1}^{\infty}\left(\frac{d^2\varphi_p(y)}{dy^2}-\omega_p^2\varphi_p(y)\right)\int_0^1\sin(p\pi x)\sin(q\pi x)\,dx=\int_0^1 Q(x,y)\sin(q\pi x)\,dx \tag{50}$$

Using the orthogonality property, this equation reduce to

$$\frac{d^2\varphi_q(y)}{dy^2}-\omega_q^2\varphi_q(y)=Q_q(x,y) \tag{51}$$

where

$$Q_q(y)=2\int_0^1 Q(x,y)\sin(q\pi x)\,dx \tag{52}$$

The series (46) satisfy the boundary condition (42a) if

$$\varphi_q(-1)=\varphi_q(1)=0 \tag{53}$$

Equation (51) subject to the condition (53) is a second order linear ODE which can be solve by using the method of variation of parameters.

The complementary function for the homogeneous function is $B_1\cosh(q\pi y)+B_2\sinh(q\pi y)$. Taking B_1 and B_2 as function of y, let

$$\varphi_q(y) = B_1(y)\cosh\omega_q y + B_2(y)\sinh\omega_q y \tag{54}$$

$$\frac{d\varphi_q}{dy} = \frac{dB_1}{dy}\cosh\omega_q y + \frac{dB_2}{dy}\sinh\omega_q + B_1\omega_q \sinh\omega_q y + B_2\omega_q \cosh\omega_q y \tag{55}$$

we choose B_1 and B_2 such that

$$\frac{dB_1}{dy}\cosh\omega_q y + \frac{dB_2}{dy}\sinh\omega_q y = 0 \tag{56}$$

Therefore,

$$\frac{d^2\varphi_q}{dy^2} = B_1\omega_q^2 \cosh\omega_q y + B_2\omega_q^2 \sinh\omega_q y + \frac{dB_1}{dy}\omega_q \sinh\omega_q y + \frac{dB_2}{dy}\omega_q \cosh\omega_q y \tag{57}$$

Substituting equations (54) and (57) into equation (51), we obtain

$$\omega_q\left(\frac{dB_1}{dy}\sinh\omega_q y + \frac{dB_2}{dy}\cosh\omega_q y\right) = Q_q(y) \tag{58}$$

Solving equations (56) and (58) for $\frac{dB_1}{dy}$ and $\frac{dB_2}{dy}$, we obtain

$$\frac{dB_1}{dy} = \frac{Q_q(y)\sinh\omega_q y}{\omega_q} \tag{59}$$

$$\frac{dB_2}{dy} = \frac{Q_q(y)\cosh\omega_q y}{\omega_q} \tag{60}$$

Integrating, we get

$$B_1 = \frac{1}{\omega_q}\int_{-1}^{1} Q_q(\xi)\sinh\omega_q\xi d\xi \tag{61}$$

$$B_2 = \frac{1}{\omega_q}\int_{-1}^{1} Q_q(\xi)\cosh\omega_q\xi d\xi \tag{62}$$

Thus

$$\varphi = \frac{1}{\omega_q}\int_{-1}^{1} Q_q(\xi)\sinh\left[\omega_q(y+\xi)\right]d\xi \tag{63}$$

where ξ is a dummy variable.

Hence, the solution of the nonhomogeneous equation describe by equations (41) and (42a), using the superposition principle is

$$\psi_{0_1} = \sum_{p=1}^{\infty}\left\{\frac{1}{\omega_p}\int_{-1}^{1} Q_q(\xi)\sinh\left[\omega_q(y+\xi)\right]d\xi\right\}\sin(p\pi x) \tag{64}$$

By using the superposition principle, the general solution of equation (45) is found to be

$$\psi_{0_2} = \sum_{p=1}^{\infty}\left(D_{1p}\cosh\omega_p y + D_{2p}\sinh\omega_p y\right)\sin(p\pi x) \tag{65}$$

Using the nonhomogeneous boundary condition in (42b)

$$\psi_{0_2}\left(x,0\right)=1 \tag{66}$$

It follows from (66)

$$\frac{\partial\psi_{0_2}\left(x,0\right)}{\partial y}=0 \tag{67}$$

Equation (67) reduces to

$$\psi_{0_2}=\sum_{p=1}^{\infty}\frac{1}{\sin\left(p\pi x\right)}\cosh\omega_p y \tag{68}$$

Hence, the series solution of the stream function (41) is

$$\psi_0=\sum_{p=1}^{\infty}\frac{1}{\sin\left(p\pi x\right)}\cosh\omega_p y+\sum_{p=1}^{\infty}\left\{\frac{1}{\omega_p}\int_{-1}^{1}Q_q\left(\xi\right)\sinh\left[\omega_q\left(y+\xi\right)\right]d\xi\right\}\sin\left(p\pi x\right) \tag{69}$$

After evaluating the integral, we have

$$\begin{aligned}\psi_0=&\sum_{n=1}^{\infty}\left[\frac{1}{\sin\left(n\pi x\right)}\cosh\left(n\pi y\right)\right]\\&-\sum_{n=1}^{\infty}\left\{\frac{16}{3n^2\pi^2}\left[\left(2+\cos\left(n\pi\right)\right)\csc h\left(n\pi\right)\sin\left(\frac{n\pi}{2}\right)^4\left[\begin{array}{l}\frac{\sinh(1-2n\pi-y)+\sinh(1+y)}{n\pi-1}\\+\frac{\sinh(1+2n\pi-y)+\sinh(1+y)}{n\pi+1}\end{array}\right]\right]\right\}\sin\left(n\pi x\right)\end{aligned} \tag{70}$$

In view of equations (20) and (21), the velocity components are the derivatives of the stream function (70)

$$\begin{aligned}u_0=&\sum_{n=1}^{\infty}\pi\csc\left(n\pi x\right)\sinh\left(n\pi y\right)\\&-\sum_{n=1}^{\infty}\left\{\left[\frac{16}{3n^2\pi^2}\left(2+\cos\left(n\pi\right)\right)\csc h\left(n\pi\right)\sin\left(\frac{n\pi}{2}\right)^4\right]\left[\begin{array}{l}\frac{-\cosh(1-2n\pi-y)+\cosh(1+y)}{n\pi-1}\\+\frac{-\cosh(1+2n\pi-y)+\cosh(1+y)}{n\pi+1}\end{array}\right]\right\}\sin\left(n\pi x\right)\end{aligned} \tag{71}$$

$$\begin{aligned}v_0=&\sum_{n=1}^{\infty}n\pi\cosh\left(n\pi y\right)\cot\left(n\pi x\right)\csc\left(n\pi x\right)\\&-\sum_{n=1}^{\infty}\left\{\left[\frac{16}{3n\pi}\left(2+\cos\left(n\pi\right)\right)\csc h\left(n\pi\right)\sin\left(\frac{n\pi}{2}\right)^4\right]\left[\begin{array}{l}\frac{\sinh(1-2n\pi-y)+\sinh(1+y)}{n\pi-1}\\+\frac{\sinh(1+2n\pi-y)+\sinh(1+y)}{n\pi+1}\end{array}\right]\right\}\end{aligned} \tag{72}$$

The first order deformation:

$$\frac{\partial^2\Omega_1}{\partial x^2}+\frac{\partial^2\Omega_1}{\partial y^2}=\left(1+C_1\right)\left[\left(\frac{\partial^2\Omega_0}{\partial x^2}+\frac{\partial^2\Omega_0}{\partial y^2}\right)\right]-C_1\operatorname{Re}\left[u_0\frac{\partial\Omega_0}{\partial x}+v_0\frac{\partial\Omega_0}{\partial y}-Ri\frac{\partial T_0}{\partial x}\right] \tag{73}$$

$$\begin{aligned}&\Omega_1\left(0,y\right)=\Omega_1\left(1,y\right)=\Omega_1\left(x,-1\right)=\Omega_1\left(x,1\right)=0,\\&\Omega_1\left(x,0\right)=0.\end{aligned} \tag{74}$$

$$\frac{\partial^2T_1}{\partial x^2}+\frac{\partial^2T_1}{\partial y^2}=\left(1+C_1\right)\left[\left(\frac{\partial^2T_0}{\partial x^2}+\frac{\partial^2T_0}{\partial y^2}\right)\right]-C_1\operatorname{Re}\Pr\left[u_0\frac{\partial T_0}{\partial x}+v_0\frac{\partial T_0}{\partial y}\right] \tag{75}$$

$$T_1(0,y) = \frac{\partial T_1(1,y;q)}{\partial x} = T_1(x,-1) = T_1(x,1) = 0, \\ T_1(x,0) = 0. \tag{76}$$

where Ω_0, T_0, u_0 and v_0 are defined by equations (39), (40), (71), and (72) , respectively.

We assume the solutions of equations (73) and (75) of the form

$$\Omega_1 = \Omega_{1_1} + \Omega_{1_2}, \tag{77}$$

$$T_1 = T_{1_1} + T_{1_2}, \tag{78}$$

where Ω_{1_1} and T_{1_1} are the particular solutions while Ω_{1_2} and T_{1_2} are the solutions of the corresponding homogeneous equations. That is

$$\frac{\partial^2 \Omega_{1_1}}{\partial x^2} + \frac{\partial^2 \Omega_{1_1}}{\partial y^2} = f(x,y) \\ \frac{\partial^2 \Omega_{1_2}}{\partial x^2} + \frac{\partial^2 \Omega_{1_2}}{\partial y^2} = 0 \tag{79}$$

$$\frac{\partial^2 T_{1_1}}{\partial x^2} + \frac{\partial^2 T_{1_1}}{\partial y^2} = g(x,y) \\ \frac{\partial^2 T_{1_2}}{\partial x^2} + \frac{\partial^2 T_{1_2}}{\partial y^2} = 0 \tag{80}$$

using the method of eigenfunction and variation of parameters the solution of Ω_{1_1} and T_{1_1} satisfying the homogeneous boundary condition (74) and (76) respectively, are express in the following form the following form

$$\Omega_{1_1} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n\pi} \int_{-1}^{1} f_{1n}(\xi) \sinh\left[\omega_n (y+\xi)\right] d\xi \right\} \sin(n\pi x) \tag{81}$$

$$T_{1_1} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n\pi} \int_{-1}^{1} g_{1n}(\xi) \sinh\left[\omega_m (y+\xi)\right] d\xi \right\} \sin(n\pi x) \tag{82}$$

where

$$f_{1n}(\xi) = 2\int_{-1}^{1} f(x,\xi) \sin(n\pi x)\, dx \tag{83}$$

$$g_{1n}(\xi) = 2\int_{-1}^{1} g(x,\xi) \sin(n\pi x)\, dx \tag{84}$$

Now, we shall find Ω_{1_2} and T_{1_2} from equation (80) and (82) satisfying

$$\Omega_{1_2}(0,y) = -\Omega_{1_1}(0,y) = 0 \\ \Omega_{1_2}(1,y) = -\Omega_{1_1}(1,y) = 0 \\ \Omega_{1_2}(x,0) = -\Omega_{1_1}(x,0) = -F_1(x) \\ \Omega_{1_2}(x,1) = -\Omega_{1_1}(x,1) = -F_2(x) \tag{85}$$

$$\begin{aligned} T_{1_2}(0,y) &= -T_{1_1}(0,y) = 0 \\ T_{1_2}(1,y) &= -T_{1_1}(1,y) = 0 \\ T_{1_2}(x,0) &= -T_{1_1}(x,0) = -G_1(x) \\ T_{1_2}(x,1) &= -T_{1_1}(x,1) = -G_2(x) \end{aligned} \tag{86}$$

where $F_{1,2}$ and $G_{1,2}$ are the only non-zero boundary conditions.

The above boundary conditions are obtained by using equations (77) and (78) and the given boundary condition (74) and (76).

By using the superposition principle, the general solution of equations (80) and (82) are found

$$\Omega_{1_2} = \sum_{n=1}^{\infty} \sin(n\pi x)\left[k_1 \exp(n\pi y) + k_2 \exp(-n\pi y)\right] \tag{87}$$

$$T_{1_2} = \sum_{n=0}^{\infty} \sin(n\pi x)\left[l_1 \exp(n\pi y) + l_2 \exp(-n\pi y)\right] \tag{88}$$

where $k_{1,2}$ and $l_{1,2}$ are constants to determined.

Now, applying the nonhomogeneous boundary conditions (85) and (86) into equations (87) and (88), respectively, we get, after renaming the constants, the equations

$$\begin{aligned} \Omega_{1_2} &= -F_1 = \sum k \sin(n\pi x) \\ T_{1_2} &= -G_1 = \sum l \sin(n\pi x) \end{aligned} \tag{89}$$

Also, applying homogeneous boundary conditions (85) and (86), equations (87) and (88), respectively can be written as

$$\begin{aligned} -F_2(x) &= \sum_{n=1}^{\infty}\left[k_1 \cosh(n\pi) + k_2 \sinh(n\pi)\right]\sin(n\pi x) \\ -G_2(x) &= \sum_{n=1}^{\infty}\left[l_1 \cosh(n\pi) + l_2 \sinh(n\pi)\right]\sin(n\pi x) \end{aligned} \tag{90}$$

which gives

$$\begin{aligned} k_1 &= -2\int_0^1 F_2(x)\sin(n\pi x)\,dx \\ l_1 &= -2\int_0^1 G_2(x)\sin(n\pi x)\,dx \end{aligned} \tag{91}$$

Also, from equations (87) and (88), we have

$$\begin{aligned} k_1 \cosh(n\pi) + k_2 \sinh(n\pi) &= -2\int_0^1 F_2(x)\sin(n\pi x)\,dx = k_1 \\ l_1 \cosh(n\pi) + l_2 \sinh(n\pi) &= -2\int_0^1 G_2(x)\sin(n\pi x)\,dx = l_1 \end{aligned} \tag{92}$$

therefore,

$$\begin{aligned} k_2 &= \frac{k_1\left[1-\cosh(n\pi)\right]}{\sinh(-n\pi)} \\ l_2 &= \frac{l_1\left[1-\cosh(n\pi)\right]}{\sinh(-n\pi)} \end{aligned} \tag{93}$$

Hence, the series solutions of equations is

$$\Omega_1(x,y)=\sum_{n=1}^{\infty}\left[k_1\cosh(n\pi y)+k_2\sinh(n\pi y)\right]\sin(n\pi x)+\sum_{n=1}^{\infty}\left\{\frac{1}{n\pi}\int_{-1}^{1}f_{1n}(\xi)(\xi)\sinh\left[\omega_n(y+\xi)\right]d\xi\right\}\sin(n\pi x) \quad (94)$$

$$T_1(x,y)=\sum_{n=1}^{\infty}\left[l_1\cosh(n\pi y)+l_2\sinh(n\pi y)\right]\sin(n\pi x)+\sum_{n=1}^{\infty}\left\{\frac{1}{n\pi}\int_{-1}^{1}g_{1n}(\xi)(\xi)\sinh\left[\omega_n(y+\xi)\right]d\xi\right\}\sin(n\pi x) \quad (95)$$

3. Conclusions Remarks

In this paper, an optimal homotopy asymptotic method is applied for solving rectangular enclosures driven by a continuously moving horizontal plate. This procedure is explicit, efficient and has a distinct advantage over usual approximation methods in that the approximate solution obtained here is valid not only for weakly nonlinear equations, but also for strongly nonlinear ones. This approach seems to be useful and can be used to obtain other analytical solutions for other couple flow problems.

REFERENCES

[1] Waheed, M. A. (2009). Mixed convective heat transfer in rectangular enclosures driven by a continuously moving horizontal plate. International Journal of Heat and Mass Transfer, 52(21--22), 5055-5063.

[2] Abu-Nada, E., and Chamkha, A. J. (2010). Mixed convection flow in a lid-driven inclined square enclosure filled with a nanofluid. European Journal of Mechanics - B/Fluids, 29(6), 472-482.

[3] Bhattacharya, M., Basak, T., Oztop, H. F., and Varol, Y. (2013). Mixed convection and role of multiple solutions in lid-driven trapezoidal enclosures. International Journal of Heat and Mass Transfer, 63(0), 366-388.

[4] Teamah, M. A., and El-Maghlany, W. M. (2010). Numerical simulation of double-diffusive mixed convective flow in rectangular enclosure with insulated moving lid. International Journal of Thermal Sciences, 49(9), 1625-1638.

[5] Teamah, M. A., Elsafty, A. F., and Massoud, E. Z. (2012). Numerical simulation of double-diffusive natural convective flow in an inclined rectangular enclosure in the presence of magnetic field and heat source. International Journal of Thermal Sciences, 52(0), 161-175.

[6] Alinia, M., Ganji, D. D., and Gorji-Bandpy, M. (2011). Numerical study of mixed convection in an inclined two sided lid driven cavity filled with nanofluid using two-phase mixture model. International Communications in Heat and Mass Transfer, 38(10), 1428-1435.

[7] Oosthuizen, P. H., and Paul, J. T. (2005). Natural convection in a rectangular enclosure with two heated sections on the lower surface. International Journal of Heat and Fluid Flow, 26(4), 587-596.

[8] Ece, M. C., and Büyük, E. (2006). Natural-convection flow under a magnetic field in an inclined rectangular enclosure heated and cooled on adjacent walls. Fluid Dynamics Research, 38(8), 564-590.

[9] Bocu, Z., and Altac, Z. (2011). Laminar natural convection heat transfer and air flow in three-dimensional rectangular enclosures with pin arrays attached to hot wall. Applied Thermal Engineering, 31(16), 3189-3195.

[10] Kuznetsov, G. V., and Sheremet, M. A. (2010). Numerical simulation of turbulent natural convection in a rectangular enclosure having finite thickness walls. International Journal of Heat and Mass Transfer, 53(1--3), 163-177.