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On the Kolmogorov Spectra in the Field of Nonlinear Wind Waves

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Abstract The nature of Kolmogorov spectra formation in the field of nonlinear wind waves are discussed. The currently dominating assertion is that such spectra for wind waves are formed as stationary solutions of the kinetic equation, which describes the conservative nonlinear four-waves resonant interactions. However, this statement here is is refuted by direct numerical calculations of the kinetic integral. Using two independent methods of calculating the kinetic integral for nonlinear surface waves in deep water, it is shown that the theoretical spectra of Kolmogorov type, found by Zakharov and co-authors [4, 7], are not stationary on a limited frequency band. Conditions for the existence of Kolmogorov spectra in real wind waves and possible mechanisms of their formation are discussed.

Keywords Kolmogorov spectra, Kinetic integral, Numerical calculations, Wave energy flux, Wave action flux

1. Introduction

More than 70 years ago, Kolmogorov [1] has expressed the hypothesis that in a random, statistically homogeneous and stationary velocity field of the turbulent flow, some stable statistical forms can exist (structural functions), which are determined only by the flow of certain physical quantity (for example, the kinetic energy dissipation rate, ε, having dimension L^2T^{-3}). In the following paper, Obukhov [2] has applied this hypothesis for derivation of the spatial spectrum of turbulent speed field. He has found the spectrum of the form, $S(k) = ck^{-m}$, in which the dimensional constant, c, is proportional to $\varepsilon^{2/3}$. Thus, the spectrum is really defined by the only constant, ε, which has also the meaning of the flux of kinetic energy through the wave number scale, k. Such spectra are called as Kolmogorov–Obukhov spectra, herein referred to as the Kolmogorov spectra [3]. There are a lot of observations of Kolmogorov's type spectra in real natural processes [3].

25 years later, Zakharov and Filonenko [4] have showed analytically that in the case of infinite axes of wave numbers k, the Kolmogorov's type spectra can exist as an exact stationary solution of the four-wave kinetic equation (KE). For the random field of nonlinear surface waves in water, KE can be written in the form[1]

$$\frac{\partial N(\mathbf{k}_4)}{\partial t} = I_{sy}\{F3\} = \iiint d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\times$$
$$\times\{N(\mathbf{k}_1)N(\mathbf{k}_2)[N(\mathbf{k}_3)+N(\mathbf{k}_4)] - N(\mathbf{k}_4)N(\mathbf{k}_3)[N(\mathbf{k}_1)+N(\mathbf{k}_2)]\} \quad (1)$$

Here I_{st} is the six-fold kinetic integral (KI) over the wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$,$\mathbf{k}_3$, $F3(N)$ means the cubic form of the integrand including positive kernel function T($\mathbf{k}$, $\mathbf{k}_1$, $\mathbf{k}_2$,$\mathbf{k}_3$) , multiplied by the set of the wave action spectra, $N(\mathbf{k}_i)$ (i=1,2,3,4). The wave action spectra is linearly related to the energy spectra by the ratio

$$S(\mathbf{k}_i) \propto \frac{\omega\, N(\boldsymbol{k_i})}{g} \quad (2)$$

(g is the gravitational acceleration), also the spread validity of equation (1) for the energy spectra.

Analytical solutions of KE(1) were found in [4] under the conditions that the frequency-angular spectrum of waves has the power form

$$S(\omega,\theta) = \ cS(\omega) = \ const\ \omega^{-X} \quad (3)$$

in the whole infinite frequency band $\boldsymbol{\omega} \in [0,\infty]$ and they are isotropic in angle $\boldsymbol{\theta}$. In [4] it was found that the solution

$$S_E(\omega) = \ c_1 P_E^{\frac{1}{3}} g^{\frac{4}{3}} \omega^{-4} \quad (4)$$

can be interpreted as the spectrum of a constant non-linear energy flux-up, P_E, (flux forwarded to high frequencies). Dimension of P_E is $[P_E]=L^2T^{-1}$. Later in [7], the another stationary solution of KE was found in the form

$$S_N(\omega) = \ c_2 P_N^{\frac{1}{3}} g \omega^{-\frac{11}{3}}, \quad (5)$$

which was interpreted as the spectrum of a constant nonlinear wave action flux-down, P_N, with dimension

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[1]For the first time it was derived by Hasselmann [5], and re-derived later by Zakharov [6]. The explicit form of the integrand is not principal here.

$[P_N]=L^3T^{-2}$. Both of these solutions mean Kolmogorov's type spectra. The factors in (4) and (5) contain the powers of the fluxes providing the proper dimension of the wave energy spectrum, $[S(\omega)] = L^2T$(for details, see [4, 7, 8]).

Here it is significant to note that in a turbulent flow, a real mechanism of Kolmogorov-Obukhov spectrum formation is not specified explicitly. Its description is a complicated mathematical problem dealing with construction of statistical distributions in the turbulent flow. For solutions (4), (5), such a mechanism has an obvious nature: it is provided by the nonlinear four-wave interactions. At present, this treatment is widely used for understanding the Kolmogorov spectrum formation in the case of waves in water.

However, the analytical theory does not answer the question about possibility of existence of the stationary spectra of forms (4) and (5) in the cases of limited frequency band for spectrum of form (3) and anisotropic angular distribution of spectral energy. Though, this point could be checked directly through numerical calculations of Eq.(1), which is considered as the basis of the explicit mechanism of Kolmogorov spectra formation Such a test that was performed for the first time in [8, 9], where the following equation

$$\partial S / \partial t = In + I_{st} + Dis \qquad (6)$$

is solved numerically. This equation is called as the generalized KE (GKE). A detailed description of numerical calculation of I_{st} in Eq.(6) can be found in [8]. It was numerically shown that the spectra of forms (4) and (5) are actually realized on the finite frequency band both for isotropic [8] and anisotropic [9] angular distribution of the spectra. Later, the similar results were obtained by other authors (for example, [10, 11]).

However, in the subsequent works, the other authors did not explicitly state that the numerical calculations for both types of spectra are realized as the steady solutions of GKE only after averaging these solutions for several tens time-steps, executed in the course of numerical computations. In other words, spectra (4, 5) are not the stationary numerical solutions of the GKE at each moment, but they are ones averaged for the time needed to transfer energy from a source to a sink. This transfer is realized by the nonlinear interactions. This circumstance was provided by the fact that the spectra of forms (4, 5) are not stationary in principal, as they do not put the integral I_{st} to zero. It was left without attention by the other authors.

This communication is devoted to:

1) Demonstrate fact of non-stationarity of the spectra of forms (4, 5) by means of direct numerical calculation the KI (1);
2) Clarify conditions for existence of Kolmogorov spectra for wind waves in water.

2. Computational Details

To address the first part of the task, two independent kinetic integral calculation algorithms are used, described in [12], [13]. As the integrand, the special (academic) spectra of the following frequency-angular presentation are used

$$S(\omega,\theta) = \left[\frac{\exp\left(-\left(\frac{X}{4}\right)*\left(\frac{\omega}{\omega_p}\right)^{-4}\right)}{\omega^X \exp\left(-\frac{X}{4}\right)\omega_p{}^{-X}}\right] \times \cos^n\left(\frac{\theta}{2}\right) \sim \omega^{-X}. \quad (7)$$

Here X is the power for a tail decay of the spectrum, the numerator in the square brackets, later referred as $M(\omega,\omega_p)$, is the well-known cutoff factor at low frequencies [8-12], providing a maximum of spectrum at the fixed peak frequency ω_p (for the general direction of wave propagation, taken as $\theta_p= 0$), n is the exponent defining a certain form for the spectrum distribution in angle θ. In addition, the spectrum is normalized in such manner that its peak value at frequency $\omega = \omega_p$ is $S(\omega_p, 0) = S_p=1$. When $n = 0$, we have the simplest isotropic angular distribution of the spectrum, corresponding to theory [4, 7]. Choosing values of X equal to 4 or 11/3, we get spectra (4) and (5), which are to be checked for stationarity. Variation of index n allows us to expand the research to the case of anisotropic spectra, the shape of which is not critical here.

In this problem, the following parameters of computational grid are of the most importance: (a) sufficiently fine frequency resolution, and (b) significantly large size of the computational frequency band. Based on a series of preliminary calculations, we have chosen the following grid

$$\omega(i) = \omega_0 q^{i-1} (1 \le i \le N);$$
$$\theta(j) = -\pi + (j-1)\Delta\theta \ (1 \le j \le M) \qquad (8)$$

with the fixed following parameters: $\omega_o= 0.5568374$, $q = 1.05$, $N = 61$ and $M = 18$, whilst $\Delta\theta = 2\pi / M$. In this case, the peak frequency is $\omega_p = 1$, and the upper-edge frequency is $\omega_m = \omega(61) = 10.4$, that provides a sufficiently large frequency range, where the stable calculated values of nonlinear transfer are realized, not depending on the choice of ω_o and ω_m.

It should be noted here that earlier in [12], the basic properties of kinetic integral were studied by the authors for the fast falling spectra ($X> 4$), when the kinetic integral is well defined. This allowed us to restrict ourselves by sufficiently small values of ω_m/ω_p not exceeding 4. But for weakly decaying spectra of forms (4) and (5), the convergence of I_{st} is severely weakened (since the kernel T of I_{st} is depending on the frequency alike ω^{12} [12]), which is manifested by a dependence of the nonlinear transfer at frequency ω on the magnitude of ω_m, especially in the vicinity of ω_m (so called "the edge effect" of a limited bandwidth). However, for large values of ω_m, the amount of nonlinear transfer, $Nl= \partial S(\omega) / \partial t$, is stabilized in the range $\omega_o<\omega/\omega_p<8$, that provides a criterion for selecting the value of ω_m. The choice of ω_0 is justified only by the form of cutting factor $M(\omega, \omega_p)$ in (7).

Following [12], for convenience in calculation of results, the magnitude of nonlinear energy transfer over any spectrum $S(\omega, \theta)$, is defined as

$$Nl\ (\omega,\theta) = \frac{\partial S(\omega,\theta)}{\partial t}, \tag{9}$$

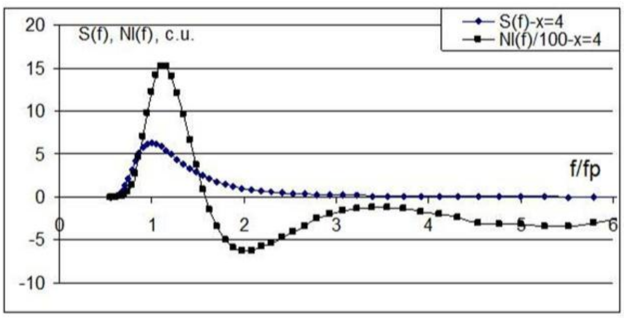
could be renormalized according to the formula

$$Nl(\omega,\theta) = g^{-4} S_p^3 \omega_p^{11} \Phi(\omega,\theta)_{nl} \tag{10}$$

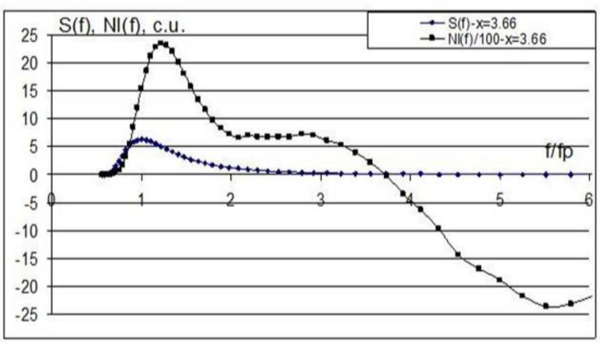
In (10), the dimensional coefficient is given in the units of ω_p, S_p and g, whilst the shape of Nl is represented by the dimensionless function, $\Phi(\omega,\theta)_{nl}$. It is the analysis of shape for function $\Phi(\omega,\theta)_{nl}$ (which on angle-averaging is denoted as $Nl(f)$), arranged on the axis of the dimensionless frequency (ω/ω_p), is of the primary interest here, regardless of quantitative values of the initial spectrum, peak frequency, and the nonlinear transfer.

Note that this is the most effective way for comparing computational results obtained by using different calculating methods.

3. Numerical Results



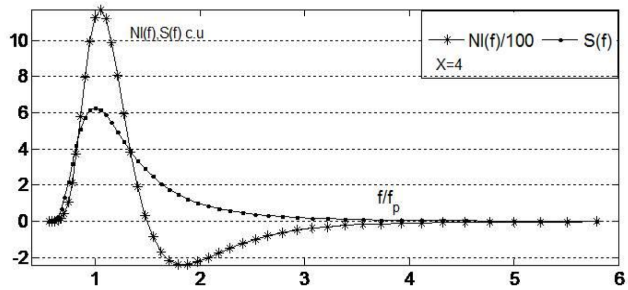
(a)



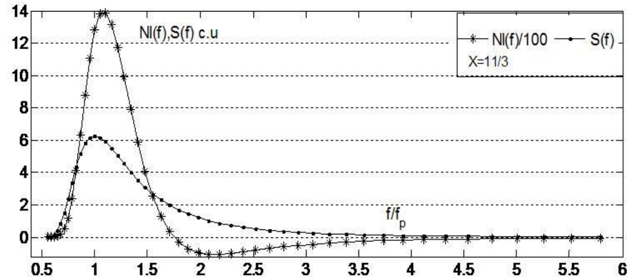
(b)

Figure 1. One-dimensional spectrum (diamonds) and nonlinear transfer (triangles) for an angular isotropic spectrum calculated by algorithm [12]: a) $X = 4$; b) $X = 11/3$. Spectra and transfers are given in conventional units

The results of calculations for the case of isotropic angular distribution of the spectra, obtained by two different methods of calculation for the KI [12, 13], are presented in Figs. 1 and 2. Regardless of the method of computing it is shown that for each of parameters, $X = 4$ and $X = 11/3$, the nonlinear transfer, $Nl(f)$, substantially differs from the zero in the practically important frequency range, $\omega_0<\omega/\omega_p<6$. These results give a clear and unambiguous conclusion about non-stationarity of spectra (4) and (5) on a limited frequency band, even in the case of an isotropic angular distribution of the spectra. Earlier in [12], it was shown that these spectra are also non-stationary for anisotropic angular distributions. (all results of our calculations are presented in detail on the site ArXiv.org [14]).



(a)



(b)

Figure 2. Thesame as in Fig. 1, but by the WRT method [13]: a) $X = 4$; b) $X = 11/3$. Spectra and transfers are given in conventional units

4. Analysis

In connection to the above discussions in Secs. 2 and 4, the question arises: whether the spectra of forms (4) and (5) can be found in nature, for example, in real wind waves. According to the theory and calculations [4, 7, 8, 9], the formation of Kolmogorov spectra in the framework of the GKE demands a fulfillment of the following three conditions:1) existence of the localized source, *In*, and sink, *Dis*, spaced on frequency band, to make the so-called "inertial range" [3]; 2) absence of sources and sinks (or negligible value of them compared with *Nl*), distributed within the inertial range; and 3) spectra must be averaged over a significant period of time (see Sec. 2). Let us make a brief analysis for feasibility of the above conditions in real wind waves.

The modern understanding of mechanisms for the evolution of wind waves, in terms of the GKE [15], indicates that in the right hand side of equation (1), in addition to the term $I_{st.}$, there are two source terms: the wave energy input, $In(\omega,\theta)$, and the sink, $Dis(\omega,\theta)$, describing the dissipation of wave energy. However, the typical distribution of intensities of these terms, averaged over angle, (for example, taken from [15] and shown in Fig. 3) are far from the fulfillment of the above conditions: 1) and 2). Note that the explicit expressions for *In* and *Dis* are not critical here, due to the fact that their numerous versions differ insignificantly in quantity [15].

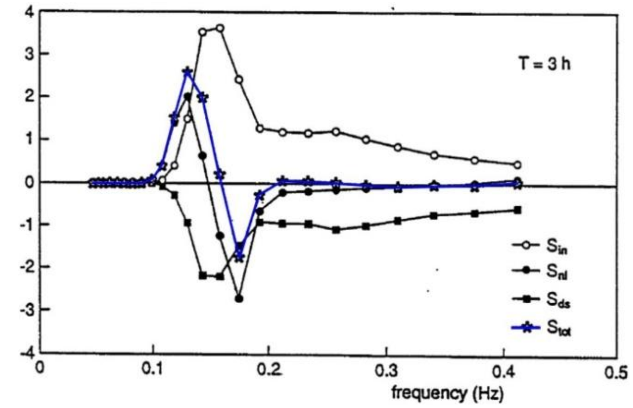


Figure 3. Typical ratio of contributions from different mechanisms of evolution of wind involved in the generalized kinetic equation (model WAM)

Nevertheless, in recent paper [16], advocates of the kinetic nature of Kolmogorov spectra in wind waves have tried to decompose the single, relatively small term *Nl* in the region of the spectrum tail, $\omega/\omega_p>3$ (Fig. 3), into two large independent parts of different signs, Nl^+ and Nl^- (see Eq. (1)). They did it in order to support the decisive role of the KI in the dynamics of wind waves. In such a case, the latter, Nl^+ and Nl^-, are much greater than typical magnitudes of *In* and *Dis*, respectively, in the domain $\omega/\omega_\mathbf{p}>>1$.

In our mind, such an artificial "enrichment" of the weak term *Nl* is incorrect. From physical point of view, it is incorrect to replace a single mechanism by two ones. From a strict mathematical point of view, it is incorrect, since the parts Nl^+, Nl^- are the similar terms cubic in spectrum *S*. Thus, they must be summed before making comparison with the other terms, *In* and *Dis*.

In our mind, in order to estimate relative contribution of the source function terms (i.e. the terms in the right hand side of GKE), it is more appropriate to perform comparison of the full *Nl*-term with the balance of the total energy income to waves, *B* = (*In* + *Dis*), though in this case, the above conditions, 1) as well as 2), are not fulfilled. The typical distribution *B(f)*, for example growth of waves, is everywhere positive (Fig. 4), and its values are $B(f) \approx |Nl(f)|$ in the domain, $\omega/\omega_\mathbf{p} >1$.[2]

From physical point of view, it needs to be taken into account that the pumping-dissipation processes are much faster than the nonlinear interactions in waves (since *B (S), Dis(S)* ~ *S*, but *Nl* ~ S^3, see [15]). Therefore, nonlinear mechanism cannot follow the instant spectrum shape, rather it works on the scales of hundreds of dominant periods of waves, when the balance of income, *B(f)*, is already realized.

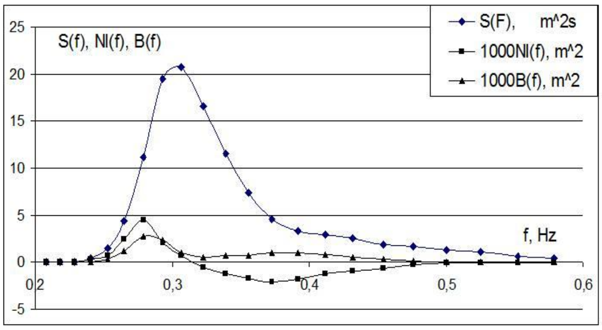


Figure 4. Typical ratio of deposits terms Nl (f) and B (f) (in units. m^{2}s) at a particular spectrum S (f) (in units. m^{2}s) for developing waves (wind W = 10m / s)

Thus, from the point of view of the wind-wave dynamics, the existence of Kolmogorov spectra, in the frame of the generalized kinetic equation, as the stationary solution of the GKE is not real due to violation of conditions 1) and 2).

5. Discussion

However, existence of the spectra with the tail of form (4) are often observed (e.g., [17-19]), the question arises about the nature of formation of such a spectra. Discarding the possibility of realization the constant nonlinear fluxes, P_E or P_N, due to the reasons said in Sec. 4, the only plausible justificating existence of the spectrum of form (4) can be obtained from the random-walk theory applied for vertical displacement of water surface in the phase-space. According to this theory, described by Golitsyn [3], it is sufficient to assume that the acceleration of the free surface, $\ddot{\eta}(t)$, is affected by the random force of turbulent air pressure, having the white noise spectrum, *P(ω)*. For a qualitative description of this process, it is necessary to represent the Euler equations for waves on liquid surface η(t) in a form of proper single equation., The final form of this equation in **k**-space has the kind

$$\ddot{\eta}(t) = -gk\eta(t) + P(t)/\rho_w + \ldots \tag{11}$$

Here $P(t)/\rho_w$ means the random perturbations of the free surface due to turbulent air pressure at the interface, normalized by the water density, which provides the sought surface acceleration, $\ddot{\eta}(t)$. Therefore, if the perturbation has a frequency spectrum of the white noise, $S_P(\omega)=const$, , by assuming the other terms in the right-hand side of (11) to be relatively small, we get the same spectrum for the acceleration of liquid surface $S_{\ddot{\eta}}(\omega)$ = const. This implies immediately the spectrum elevations $S_\eta(\omega)$ of form (4) (for details, see [3]). Herewith, the dimension of coefficient in (4) is completely determined by the dimension of the disturbing force.

[2] It should be noted here that the distribution of balance *B(f)* can vary significantly, depending on the stage of wave development, characterized by the wave age, *A*, given by the ratio of the phase velocity for the dominant waves to the wind speed. Thus, wave age *A* may also affect the law of decay for the tail of the spectrum (see discussion in chapter 6.2 of book [3]).

In this regard, it is appropriate to recall Kitaigorodskii hypothesis[20], who obtained the spectrum with the tail of form (4) from the simplest dimensional considerations, taking three dimensional variables as a basis: frequency ω, friction wind speed u_* and the acceleration of gravity, g. In such a case, the expression for the wind-wave frequency spectrum becomes,

$$S_k(\omega) = const. gu_*\omega^{-4}, \qquad (12)$$

the tail of which is coinciding with form (4). The proper spectra were observed experimentally [17-19]. As expected, the value (gu_*) has the dimension of energy flux, ε (i.e. L^2T^{-3}), which allows us to treat (11) as the spectrum of Kolmogorov type. Here the dependence of energy flux on the wind indicates clearly that this flux is formed by near-surface wind, and (apparently) it is uniformly distributed throughout the frequency spectrum, in full analogy with Fig. 4. This explains the possibility of observing the spectra of Kolmogorov type in wind waves, without accepting concept of solution of the GKE.

6. Conclusions

In conclusion, it remains only to note that the spectra of form (10) are not always realized in the wind waves, even under stationary wind [21]. Apparently, some additional conditions are needed for realization the Kolmogorov-type spectra in wind waves, existence of which requires further investigations.

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