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Application of Iteration Perturbation Method and Variational Iteration Method to a Restrained Cargo System Modeled by Cubic-quintic-septic Duffing Equation

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Abstract This paper presents an investigation of the behavior of a restrained cargo modeled by the cubic-quintic-septic Duffing equation using He's iteration perturbation method (IPM) and He's variational iteration method (VIM). We compared our results with the exact result and show the excellent agreement between the approximate methods and the exact result. We also highlight the simplicity and accuracy of IPM and VIM in obtaining analytic approximate solutions to nonlinear differential equations like the cubic-quintic-septic Duffing equation.

Keywords Variational Iteration Method, Iteration Perturbation Method, Cubic-Quintic-Septic Duffing Equation, Restrained Cargo System

1. Introduction

Obviously the study of nonlinear systems and their behavior remains one of the most important aspect of engineering, applied mathematics, physics and other scientific fields. These nonlinear systems are real physical systems which are modeled by nonlinear differential equations, for this reason, one cannot overemphasize the importance of understanding and gaining useful insights towards the behavior of these nonlinear differential equations so as to make accurate and precise decisions while working with real physical systems.

On the other hand, nonlinear differential equations are very difficult and complex to study. Due to their complexity, it is very difficult and most times impossible to obtain an exact solution to these nonlinear differential equations. Over the years, researchers have developed many tools that will aid in the study of these nonlinear differential equations. Methods like perturbation methods[1, 2, 3], numerical methods[4], and most recently the approximate methods have been developed in order to understand the behavior of these nonlinear differential equations. Many approximate methods have evolved recently, among them are, homotopy analysis method (HAM)[5-14], homotopy perturbation method (HPM)[15-24], variational approach method (VAM)[25, 26], energy balance method (EBM)[27, 28], to mention only but a few. Most of these methods mentioned above have been applied to varying physical problems with success.

This paper presents some contributions towards gaining more understanding of the cubic-quintic-septic Duffing equation through the use of He's iteration perturbation method (IPM)[29, 30] and He's variational iteration method[31-39].

The Duffing equation is a well known and extensively studied nonlinear differential equation having the general form (1) and related to many practical engineering systems. Due to the presence of fifth and seventh power nonlinearities, the cubic-quintic-septic Duffing equation becomes more complicated and complex, thus the difficulty in obtaining accurate analysis of the equation. Some practical problems that have been effectively modeled by the Dufffing equation were listed in[40].

$$\begin{gathered}\ddot{\psi}(t) = f[\dot{\psi}(t), \psi(t), F] \\ \psi(0) = p, \quad \dot{\psi}(0) = q\end{gathered} \tag{1}$$

The cubic-quintic-septic Duffing equation which can be represented by (2), turned out to be a useful model for the behavior of a restrained cargo system in[41], where it was noted that even though the cubic and cubic-quintic representations of the Duffing equation were found to adequately represent the restoring force of the restrained cargo, the cubic-quintic-septic approximation of the restoring force is more accurate and efficient as it avoids limitations associated with approximations such as small nonlinearity and low level of excitation.

$$\begin{gathered}\ddot{x} + \alpha x + \beta x^3 + \mu x^5 + \delta x^7 = 0 \\ x(0) = p, \qquad \dot{x}(0) = 0\end{gathered} \tag{2}$$

α is the resonant frequency, β, μ, δ are the nonlinear

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coefficients.

2. Basic idea of the Iteration Perturbation Method (IPM)

Most nonlinear differential equations used to model physical systems can be generally described as (1). Considering (1), we introduce the variable, $\eta = \frac{d\psi}{dt}$ and then (1) can be represented by the system

$$\dot{\psi}(t) = \eta(t) \qquad \dot{\eta}(t) = -f(\psi, \eta, \dot{\eta}, t) \tag{3}$$

Assuming that the initial approximate guess of (3) can be represented as

$$\psi(t) = p\cos(\omega t) \tag{4}$$

where ω is the angular frequency of oscillation and p is the initial amplitude of oscillation. Then we obtain using (4) and (3)

$$\begin{aligned} \dot{\psi}(t) &= -p\omega\sin(\omega t) = \eta(t) \\ \ddot{\psi}(t) &= -p\omega^2\cos(\omega t) = \dot{\eta}(t) \end{aligned} \tag{5}$$

Substituting (5) and (4) into $(3)_2$, we obtain

$$\dot{\eta}(t) = -f(\psi, \eta, \dot{\eta}, t) =$$

$$-\sum_{n=0}^{\infty} \phi_{2n+1}\cos\left[(2n+1)\omega t\right] = -\left[\phi_1\cos(\omega t) + ...\right] \tag{6}$$

Integrating (6), gives

$$\eta(t) = -\frac{\phi_1}{\omega}\sin(\omega t) - ... \tag{7}$$

Comparing $(5)_1$ and (7), we obtain

$$\omega = \sqrt{\frac{\phi_1}{p}}, \qquad T = 2\pi\sqrt{\frac{p}{\phi_1}} \tag{8}$$

T is the period of oscillation.

Then we can represent the solution to (1) as

$$\psi(t) = p\cos\left(\sqrt{\frac{\phi_1}{p}}t\right) \tag{9}$$

3. Application of Iteration Perturbation Method (IPM)

We consider the un-damped and unforced cubic-quintic-septic Duffing equation[41] given by (2), which is equivalent to the two-dimensional system

$$\begin{aligned} &\dot{x} = y, \qquad \dot{y} = -\alpha x - \beta x^3 - \mu x^5 - \delta x^7 \\ &x(0) = p, \qquad y(0) = 0 \end{aligned} \tag{10}$$

We note that setting $\delta = 0$ in (2) gives us the popular cubic-quintic Duffing equation while setting $\delta = \mu = 0$ in (2) as well gives the conventional cubic Duffing equation.

We assume that the solution to (2) can be expressed in the form of (4). Substituting (4) into (10) yields

$$\begin{aligned} &\dot{x}(t) = -p\omega\sin(\omega t) = y \\ &\dot{y}(t) = -p\cos(\omega t)[\alpha + \beta p^2\cos^2(\omega t) + \mu p^4\cos^4(\omega t) + \delta p^6\cos^6(\omega t)] \end{aligned} \tag{11}$$

Expanding (11) in Fourier series, we obtain

$$-p\cos(\omega t)[\alpha + \beta p^2\cos^2(\omega t) + \mu p^4\cos^4(\omega t) + \delta p^6\cos^6(\omega t)] = \phi_1\cos(\omega t) + ... \tag{12}$$

Eq. (12) implies that

$$\phi_1 = -\frac{4}{\pi}\int_0^{\frac{\pi}{2}} p\cos^2\theta[\alpha + \beta p^2\cos^2\theta + \mu p^4\cos^4\theta + \delta p^6\cos^6\theta]d\theta \tag{13}$$

Carrying out the indicated integration in (13), yields

$$\phi_1 = -4p\left[\frac{\alpha}{4} + \frac{3\beta p^2}{16} + \frac{5\mu p^4}{32} + \frac{35\delta p^6}{256}\right] \tag{14}$$

Then we can write $(11)_2$ as

$$\dot{y}(t) = -4p\left[\frac{\alpha}{4} + \frac{3\beta p^2}{16} + \frac{5\mu p^4}{32} + \frac{35\delta p^6}{256}\right]\cos(\omega t) \tag{15}$$

Eq. (15) implies that

$$y(t) = -p\int\left[\alpha + \frac{3\beta p^2}{4} + \frac{5\mu p^4}{8} + \frac{35\delta p^6}{64}\right]\cos(\omega t)dt \tag{16}$$

Simplifying (16) yields

$$y(t) = -\frac{p}{\omega}\left[\alpha + \frac{3\beta p^2}{4} + \frac{5\mu p^4}{8} + \frac{35\delta p^6}{64}\right]\sin(\omega t) \tag{17}$$

Comparing (17) and $(11)_1$ yields

$$\omega_{IPM} = \frac{\sqrt{64\alpha + 48\beta p^2 + 40\mu p^4 + 35\delta p^6}}{8} \tag{18}$$

Then from (4), we write the approximate solution to (2) as

$$x(t)_{IPM} = p\cos\left(\frac{\sqrt{64\alpha + 48\beta p^2 + 40\mu p^4 + 35\delta p^6}}{8}t\right) \tag{19}$$

4. Basic idea of the Variational Iteration Method (VIM)

In[31-39], some of the various literatures that demonstrated the basic idea of the variational iteration method were listed. Depending on the initial conditions given, we choose a suitable guess function $x_0(t) = p\cos(\omega t)$. Following[34], the angular frequency ω is determined by

$$\int_0^{\frac{2\pi}{\omega}} \cos(\omega t)[-\omega^2 \ddot{x}_0 + \alpha x_0 + \beta x_0^3 + \mu x_0^5 + \delta x_0^7]dt = 0 \tag{20}$$

From (20), we obtain the angular frequency as

$$\omega_{VIM} = \left(\frac{\sqrt{64\alpha + 48\beta p^2 + 40\mu p^4 + 35\delta p^6}}{8}\right) \tag{21}$$

According to the linear section of (2), the general Lagrange multiplier can be identified as[38]

$$\lambda = \frac{\sin\omega(\tau - t)}{\omega} \tag{22}$$

Then the second order approximate solution is given by

$$x(t)_{VIM} = x_0(t) + \int_0^t \lambda\left[\ddot{x}_0 + \alpha x_0 + \beta x_0^3 + \mu x_0^5 + \delta x_0^7\right]d\tau \tag{23}$$

Substituting (22) and employing (4) in (23), performing the indicated operation, we obtain

$$x(t)_{VIM} = p\cos(\omega t) + \frac{p}{2}\left[\begin{array}{l} \left(2\cos(\omega t) - 1 - \cos(2\omega t)\right) - \frac{\alpha}{\omega^2}\left(2\cos(\omega t) - 1 - \cos(2\omega t)\right) \\ -\frac{\beta p^2}{4\omega^2}\left(6\cos(\omega t) - 3 - 4\cos(2\omega t) + 2\cos(3\omega t) - \cos(4\omega t)\right) \\ -\frac{\mu p^4}{8\omega^2}\left(10\cos(\omega t) - 5 - \frac{15}{2}\cos(2\omega t) + 5\cos(3\omega t) - 3\cos(4\omega t) + \cos(5\omega t)\right) \\ -\frac{\delta p^6}{32\omega^2}\left(\begin{array}{l} 35\cos(\omega t) - \frac{35}{2} - 28\cos(2\omega t) + 21\cos(3\omega t) - 14\cos(4\omega t) \\ +7\cos(5\omega t) - 4\cos(6\omega t) + \cos(7\omega t) - \frac{\cos(8\omega t)}{2} \end{array}\right) \end{array}\right] \tag{24}$$

5. Discussion of Results

In this section, we compare the results obtained using Iteration Pertubation Method and Variational Iteration Method, to the exact results computed numerically by the use of Runge-Kutta fourth-order method (RK45). The simulations below were obtained using the scilab computer programming package.

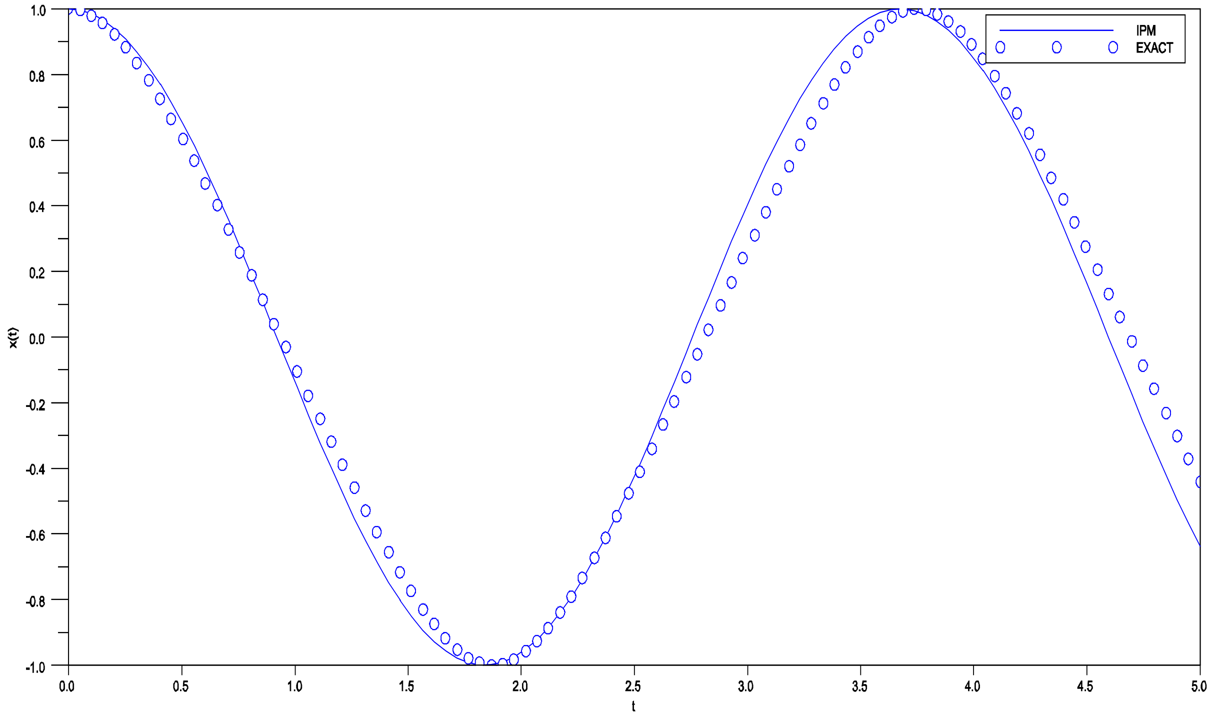

Figure 1. presents the behavior of the displacement of the cubic-quintic-septic Duffing equation with increasing time for the parameter values $p = \alpha = \beta = \mu = \delta = 1$

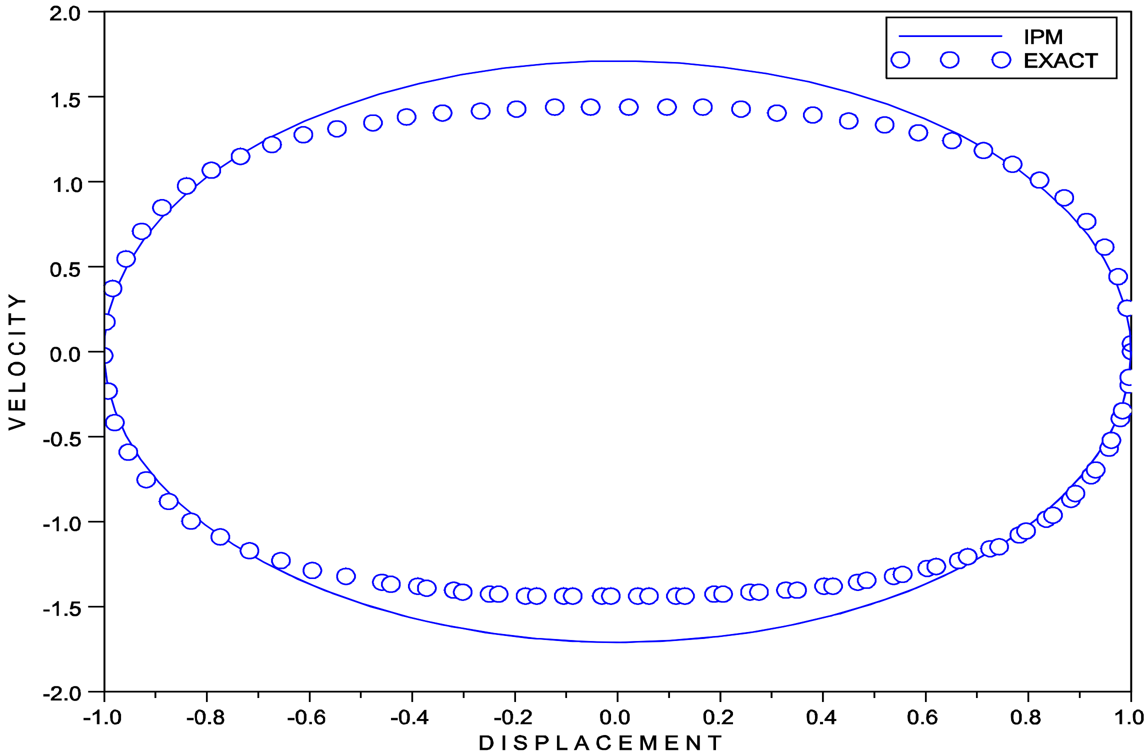

Figure 2. presents the phase plot of the cubic-quintic-septic Duffing equation for the parameter values $p = \alpha = \beta = \mu = \delta = 1$

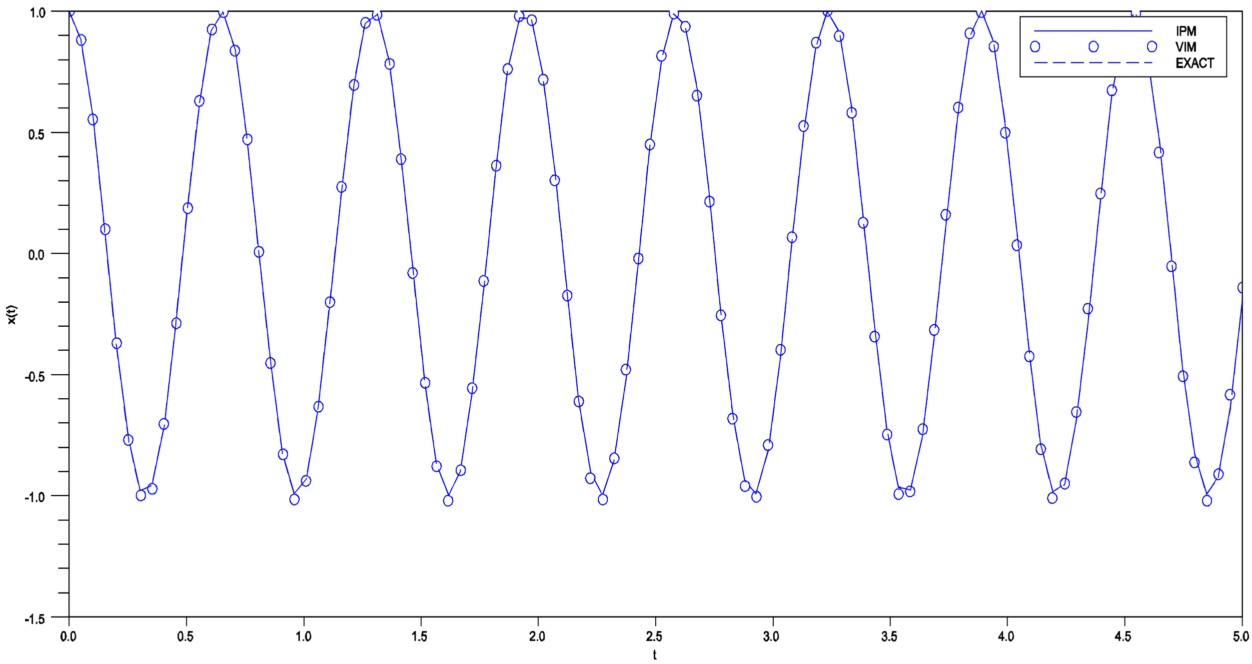

Figure 3. presents the behavior of the displacement of the cubic-quintic-septic Duffing equation with increasing time for the parameter values $p = \delta = 1, \beta = -3.5, \alpha = 96.6289, \mu = -0.8$

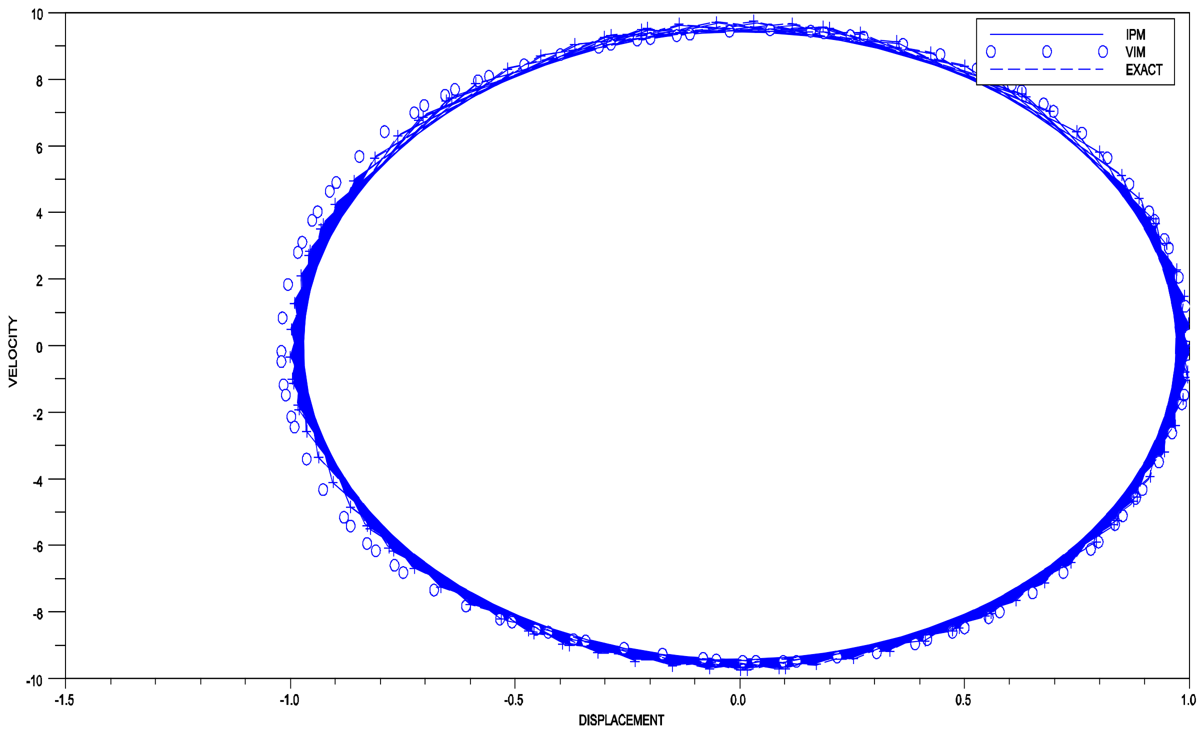

Figure 4. presents the phase plot of the cubic-quintic-septic Duffing equation for the parameter values $p = \delta = 1, \beta = -3.5, \alpha = 96.6289, \mu = -0.8$

We note that, setting $\delta = 0$ in (19) yields

$$x(t)_{IPM} = p\cos\left(\frac{\sqrt{64\alpha + 48\beta p^2 + 40\mu p^4}}{8}t\right) \quad (25)$$

Eq. (25) is the approximate solution to the popular cubic-quintic Duffing equation. Likewise one can also obtain the approximate solution to the cubic Duffing equation using the two given methods employed in this paper.

6. Conclusions

In this paper, we have shown the effectiveness and efficiency of the Iteration Perturbation Method (IPM) and the Variational Iteration Method (VIM) in obtaining analytic approximate solutions to nonlinear differential equations such as the cubic-quintic-septic Duffing equation. We compared our results with the exact result obtained numerically and our comparison shows that the two methods considered in this paper give accurate results. Moreover, the two methods showcased in this paper, are very easy and simple to handle as they do not involve rigorous calculation processes as well as complex mathematical ideas. Though more research is required in the light of gaining more information as to how these approximate methods affects real physical systems, this paper presents a step towards a successful and positive implementation of the two methods.

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