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Helixon Theory, A New Interpretation of Elementary Particles Gauge Symmetries

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Abstract In this paper, I propose a new model compatible with the internal symmetries of elementary particles to identify the invariant charges of standard model unitary symmetries by physical parameters in an extra spatial dimension. Applying the concepts of zero point energy and center of mass for elementary particles and introducing *helixons* as time-like geodesics in 4+1 dimensional toral manifold, we show that the free motion of particles center of mass on these helixons results in the emergence of gauge symmetries discovered in standard model and presents pure physical interpretations for particles' invariants such as electric charge, isospin, hypercharge and coupling constants of electroweak and strong interactions. This allows to incorporate unitary symmetries without the assumption of internal spaces and several extra dimensions. Higgs vacuum expectation value, and effective radius of Higgs boson are among the prominent results of helixon theory. Moreover it predicts the small deviation from Larmor frequency in very weak field, proves the Montgomery conjecture and emergence of golden ratio in energy levels.

Keywords Gauge symmetries, Standard Model, Higgs Model, Quaternion, Octonion, Helixon Theory

1. Introduction

The *least action principle* has been remaining as the fundamental basis in the construction of various basic classical mechanics equations such as Hamiltonian and Lagrangian formalism and thermodynamics [1, 2]. This method has also been applied in various modern physics theories from general relativity (Einstein-Hilbert action) to quantum field and elementary particle physics and quantum mechanics (as path integral) and string models via introducing a *compatible* Lagrangian (or Hamiltonian) of related system and derivations of equations of motions through the variational principle applied to actions [3, 4, 5]. On the other hand *geodesic equations* result in the same solutions to motion equations of related dynamical system and in principle is equivalent to the least action principle. However the strict concept of geodesics for internal symmetries of particles and subsequent gauge symmetries have not been defined yet because of the abstract notion of internal spaces that lacks any physical counterparts. As an example the Weinberg (mixing) angle has not been realized as a true space-time angle. The rotations and associated unitary symmetries in these spaces do not imply any physical concepts while the resultant invariants interpreted as elementary particles invariant charges. Contrast to least action method we introduce a new model in order to follow a reverse direction, i.e. considering default equations of motion by specifying the *geodesic motion* of C.O.M (center of mass) of a free particle on extended space-time manifolds with an *extra spatial dimension* to derive the governing internal symmetries and invariants of motion and advance toward the realization of abstract spaces parameters. The reason for choosing the C.O.M as a geometrical basis is its applicability to any extension of particles (wave function, fields etc.) moreover to construct a theory with geometrical basis in order to reconcile the realm of quantum and general relativity theories we need a common geometrical concept to entire physics like C.O.M. The close concepts of least action principle and *geodesics equations* allows us to define geodesic motion of C.O.M and deducing the related action and Hamiltonian. New proposed model (*helixon theory*) hypothesizes the free motion of free particles C.O.M on toral manifold geodesics (with an extra spatial dimension) while preserve the effective projected geodetical motion on Minkowskian 3+1 manifold, results in a set of degrees of freedom with certain dynamical invariants. Such a geodesics model, drives the theory to a geometrical interpretation of the related system in comparison with choosing a Lagrangian or Hamiltonian being compatible with the experimental outcomes of the related physical situation. However the geodesics approach may not be sufficient when the physical systems being totally described by field or wave-particles constituents instead of point particles. Perhaps the main

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reason is the lack of a reliable geometric concept in quantum mechanics and subsequent field theories. Actually the main reason for acceptable success of string theories may be indebted to the substitution of a hypothetical pure geometric structure i.e. one dimensional strings as ultimate structures of elementary particles. The priority of geometrical structure in physical models can also be understood by recalling the great success of general relativity and Kaluza-Klein inspired models where the notion of spatio-temporal dimensions of ambient space determines the gravity and electromagnetic forces. In these theories the geometrical structure of the model induces the metric tensors and associated geodesics equations to retrieve the motion or dynamic equations of related physical systems that replaces the least action principle method.

The pure geometrical elements which consist of abstract points, lines etc. without any physical interpretation are not good candidates for presenting the physical issues. However the *center of mass* (C.O.M) as a common physics-geometric concept is valid for both realm of theories and consequently could be applied in a model to encompass both notions. At the level of elementary particles, geodesic motion could be defined as the motion of free particles' C.O.M on geodesics curves on ambient space. Thus for any extension of particle or its spatial distributions in the sense of quantum mechanics either as a particle wave or fields in quantum field theory, it is possible to define the geodesic motion of the particle's C.O.M along geodesics trajectories on the ambient space. In *helixon* theory the center of mass (C.O.M) in conjunction with geodesics has been included in the model as the main geometrical concepts. The significance of C.O.M and *zero point energy* reviewed through sections 2, 3. In subsequent sections it is proved that almost all particle parameters other than *spin* can be recovered by the formalism of C.O.M *helical* motion on the geodesics of toral manifolds in 4+1 spatio-temporal space where spatial dimensions include the 3 regular and 1 extremely small extra dimensions. Similar to string theories, we use a Euclidean metric signature for 4+1 space-time manifold. The helical motions (helixons) of C.O.M as *n*-toral geodesics result in a multidimensional *harmonic oscillators* with a number of degrees of freedom that may exceeds the number of ambient space-time manifold. The excess degrees of freedom allow space-time to adopt more independent parameters which are required for theories of elementary particles. These extra degrees of freedom induces a variety of symmetries that cover electromagnetic and standard model symmetries $U(1), SU(2)_L, SU(3)_c$ and their possible particles multiplets. Consequently the internal symmetries and associated abstract spaces (e.g. isotope spin, hypercharge etc.) could be described and realized by these extra degrees of freedom. The Invariant charges of particles in *helixon theory* being represented as *angular velocities* of helixons which correspond to charges of Lie generators underlying these symmetry groups. In section 11.5, Higgs field potential in helixon model appears as bounding potential energy of helixons. In this sense helixons trace out the points of 4+1 manifold with minimum energy (potential well). On the other hand addition of the unique spatial extra dimension gives a pure complex structures to the extra momentums and the set of $'n'$ independent helical motions in this small extra dimension could be identified as an abstract multidimensional complex manifold $\mathbb{C}^n$. Inspiring the complex *quaternion and octonion algebra* we show that the maximum degrees of freedom bounded to 3 or 7 dimensions. The set of independent complex variables forms a *Kahler* complex manifold where the related closed geodesics are isometric with helixons and this property will help to prove Montgomery conjecture and golden ratio problem.

2. Center of Mass

Center of mass (C.O.M) is a common *pure geometrical* definition in both realms of classical and quantum mechanics. This geometrical point could be defined in quantum mechanics as well as classical mechanics, because it is practically applicable for any mass distribution or particle wave (extension) with a certain spatial distribution. By the assumption that mass distribution of a particle coincides its probability distribution, the center of mass (or briefly center) of the Schrodinger wave (packet) is defined as $\langle \vec{r} \rangle = \int \vec{r} |\psi|^2 dV$, therefor center of mass (C.O.M) could be applied for particle wave in quantum realm. Evidently for systems containing single or multiple particles, the mechanical behavior of system in external fields can be calculated as if all mass distribution concentrated at C.O.M point. The hypothesis of *point particles* for elementary particles is the main assumption for scattering analysis in quantum field theories where aside from any extension (or field) of particles their scatterings of obey the Feynman diagrams illustrated by vertex (points) and propagators (lines). In these diagrams one may substitute the points by the C.O.M of scattered particles. Moreover *center of mass reference frame* is the standard frame of coordinates in particle scattering and cross section analysis because it is applicable even for massless particles as photons. We have exploited this significant concept in our model as a point passing through geodesics in ambient space. Naturally it could be defined as well for single particle or a collection of particles in a system like atoms or molecules.

3. Zero Point Energy

Starting hypotheses in our model has been based on the concept of *zero point energy*. In quantum sense, there is a perpetual vibration corresponding to the lowest energy level $\frac{1}{2}\hbar\omega$ of all particles, fields, atom and molecules, even at absolute zero temperature, well known as zero point energy (ZPE) [6, 7]. It is a natural consequence of the Heisenberg uncertainty principle in quantum realm. ZPE concept is an essential fact to resolve some problems such as Casimir effect and Lamb shift and could be measured in atoms, molecules and elementary particles. Consequently the ZPE could be a starting point in configuring a universal theory compatible with quantum mechanics and experimental data.

ZPE is the source of some conflicts in cosmology such as unresolved *cosmological constant problem* which states that the minute quantity of cosmological constant differs from the suggested value of zero point energy due to quantum field theory. Moreover there are two additional associated facts:

- The standing wave of a particle at the zero thermal velocity is equivalent to zero point fluctuation or energy [8].
- ZPE is Lorentz invariant [9].

4. Kaluza Klein Theory

One of the most influential theories after general relativity and quantum mechanics which makes the substantial background for unifying theories such as *string* and *superstring* theories is Kaluza Klein theory. The original idea in this theory is assumption of a periodic small extra dimension. The addition of this extra degree of freedom brings about the unification of general relativity and electromagnetism. One of the main results of this model states that the charge to mass ratio could be interpreted as the proper time derivative of the fourth spatial dimension [10]. i.e.

$$\omega_4 = \frac{dx_4}{d\tau} = -\frac{1}{2\sqrt{G}}\frac{q}{m} \quad (1)$$

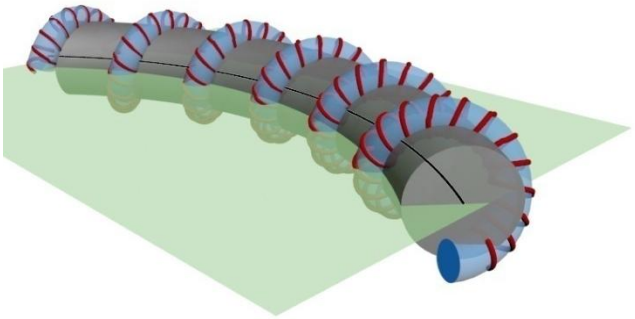

Figure 1. A combined helixon consists of 2 angular frequency depicted by red trajectory is a geodesic on combined torus (blue cylinder) and traces out the C.O.M path. The plane (green) stands for M^{3+1} space time. The fourth spatial dimension regarded as perpendicular to this plane with extreme small range. So the scale of radius of these torus assumed at the range of 10^{-18} meter. The black curve shows the effective world line on M^{3+1}. The radius of gray cylinder denoted by r_1 and radius of blue cylinder by r_2

Where *G* denotes the Newton constant. Helixon model generalizes this notion to all known conserved charges of internal symmetries of elementary particles. These charges include weak and strong hypercharges, weak and strong isospin, Baryon number, etc. with exception of spin as will be discussed later. All conserved charges in the notion of helixon theory corresponds to a set of *angular velocities* ω_j similar to the equation (1). Therefor for a particle with a set of *charges* (electric charge, hypercharge, isospin etc.) helixon composed of a combined helixes (fig1) each corresponds to a unique invariant particle charges. Each angular frequency corresponds to an independent helix as well. For electromagnetic fields this angular velocity is proportional to electric charge to mass ratio i.e. $\frac{q}{m}$. For other charges it will be proportional to $\frac{g}{m}$ where g stands for the coupling constant of the related interacting force and symmetry group. Thus we have the general form for helixes angular velocities:

$$\omega = k\frac{g}{m}$$

k is a constant.

5. Helixon Model Hypothesis

I propose a theory based on a hypothesis backed by ZPE concept and free motion of C.O.M on time like world-line being defined on an extended M^{4+1} manifold as follows:

- A tiny extra spatial dimension with small range is the cornerstone of the theory. For each point of space-time manifold we assume a neighborhood of a small spatial radius extending through an extra dimension x^4 in such a way that C.O.M of all elementary particles involved by a free motion on *helical geodesics* (i.e. helixons) on complex torus in this M^{4+1} space while its trajectory in M^{3+1} results from projection of these geodesics on M^{3+1} manifold fig (1).
- Based on the ZPE concept, we assume that these displacements are *inherent and perpetual oscillations* of particle's C.O.M in M^{4+1} as a result of its motion on geodesics of a combined torus (helixon) in M^{4+1} space-time with a small extra spatial dimension fig (1). The resultant projection of this geodetic motion on M^3 interpreted as a harmonic oscillation.
- All particles *charges* (quantum numbers) could be determined definitely by the winding frequencies ω_j on these geodesics. These charges include all invariants such as electric charge, isospin, hypercharge, lepton and baryonic number etc. We will show the Lorentz invariant properties of associated charges of ω^j.
- As we will show in next sections, Fermions in helixon theory comprise of a set of single helixons and Bosons be represented by those single helixons. Consequently each single helixon corresponds to a boson with associated charges and being represented by a complex variable while a fermion includes a set of complex numbers. We will show that the appropriate complex numbers compatible with fermion statistics are hypercomplex numbers defined in quaternion and octonion algebras.
- In helixon theory, *spin* is an exception among particle's parameters. The spin excluded from other invariants because it is being defined at the center of mass. Particles' spin implies the inherent angular momentum respect to C.O.M of particles. Therefor it excludes from the other invariant charges hypothesized as the winding frequencies of helixons.

6. Helixons

6.1. Stationary Reference Frames

Considering a flat space-time manifold M^{3+1} with coordinates x^j, $j = 0,1,2,3$ and a small extra spatial dimension x^4 along an axis perpendicular to M^{3+1}, allows us to define world-lines as time-like geodesics on 2-dimensional torus manifolds. Let us imagine a time-like world-line in M^{3+1} then construct a 2-dimensional torus expanded with axis along time-like world-line that its circular cross section expanded on the plane $x^j - x^4$ and radius r_1 limited to the extra dimension range fig (1). The maximum range of extra dimension have to be limited to small value ϵ then:

$$-\epsilon \leq x^4 \leq \epsilon \tag{2}$$

Motion of a particle C.O.M on helical geodesics of this 2-dimensional torus while the axis coinciding the time axis x^0, leaves a helical trajectory (world-line) $\mathcal{C}_1$ embedded in M^{4+1} space. For an observer $\mathcal{O}_1$ with the reference frame co-moving with the center of that circle on time axis, the motion of C.O.M will be recorded as a circular motion with radius r_1 and frequency ω_1, perpendicular to M^{3+1}. This circular motion can be identified by a complex number z^1 which is living in the plane $x^j - x^4$ with x^j as an arbitrary spatial axis of M^3 and x^4 as the extra fourth spatial dimension regarded as *imaginary* axis. Let imaginary basis in this plane be denoted by *i*. We call this helical world line as *Helixon*. If we construct another tubular neighborhood of this trajectory with radius r_2 and longitudinal axis coinciding the first world-line $\mathcal{C}_1$ and name it as torus $\mathcal{T}^2$, then another Helixon $\mathcal{C}_2$ could be made as geodesics on this new torus. If the C.O.M is moving on this new world line, the observer $\mathcal{O}_1$ detects a combined circular motions $\mathcal{C}_1$ and $\mathcal{C}_2$ with one center on time axis and the other on $\mathcal{C}_1$. However the observer $\mathcal{O}_2$ on the world-line attached to the center of second circle fig (1) records merely a circular motion with radius r_2 and frequency ω_2. Reference frame of Observer $\mathcal{O}_2$ is an *inertial frame* while following the helical world-line. Therefor the plane of circular motion in this reference frame will be $x^{j'} - x^4$ with $x^{j'}$ as another spatial axis and x^4 as *another imaginary basis j* which is independent of the imaginary basis $\boldsymbol{i}$ on the plane $x^j - x^4$. This circular motion can be identified by a complex number z^2 which is living in the plane $x^{j'} - x^4$. Thus the resultant world-line $\mathcal{C}_2$ includes two distinct frequencies ω_1 and ω_2 and two independent radius r_1 and r_2 which have been defining on the torus $\mathcal{T}^2$. The projection of Helixon $\mathcal{C}_1$ on M^3 represents minimally a 1-dimensional harmonic oscillator. Subsequently the projection of the new Helixon $\mathcal{C}_2$ on M^3 will read minimally as 2-dimensional harmonic oscillator with frequencies ω_1 and ω_2. Although the Iteration of this procedure will give rise to a complex Helixon $\mathcal{C}_N$ with *N* independent frequencies and radiuses and a corresponding *N*- dimensional harmonic oscillator in M^3 however in next section we show that the allowed dimension restricted to the 3 and 7 that is the number of independent imaginary basis of *Quaternion* and Octonion. These division algebras represent the definition of *cross product* and are isomorphic to Lie algebras $su(2)$ and $su(3)$. The latter algebras form the governing symmetries of 2 and 3 dimensional homogenous harmonic oscillators [11]. The dimension of oscillator may exceed the number of ambient space because there is not any limit on the degree of freedoms of these oscillator with *N* independent r_j and ω_j. Moreover these oscillations while be projected onto M^{3+1} interpreted as ZPE OR Zitterbewegung. This is in parallel with Lorentz invariance of ZPE as mentioned in sec (3), and supports the notion of ZPE origin for this helixon hypothesis.

6.2. Complex Planes and Hypercomplex Numbers

The planes $x^j - x^4$ and $x^{j'} - x^4$ ..., carrying circular motions which result in $\mathcal{C}_1, \mathcal{C}_2$ and contain independent complex structures z^1 and z^2 ..., each is living in independent 1-dimensional complex plane $\mathbb{C}$. Consequently the set of helixons and the position of C.O.M could be determined by these set of different complex number basis. For an observer located on center of first helixon $\mathcal{C}_1$, C.O.M could be represented by a complex number z^1. If this observer dislocated to the center of second helixon the location of C.O.M will be determined by another complex number z^2. Generally the planes of helixons $\mathcal{C}_1, \mathcal{C}_2$ are not parallel then their complex number should contain the different hypercomplex (imaginary) basis as has been described in the context of quaternion algebra. Transition from usual complex number to quaternions ($\mathbb{C} \rightarrow \mathbb{H}$) could be achieved by introducing another independent complex basis $\boldsymbol{j}$ in the form $\mathbb{H} \sim \mathbb{C} + \boldsymbol{j}\mathbb{C}$. Consequently there exist three independent imaginary basis denotes by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ i.e. Quaternionic basis. Accordingly transition from quaternion algebra to the next algebra can be achieved by $\mathbb{H} + \boldsymbol{l}\,\mathbb{H}$ to form another divisible algebra known as Octonions i.e. $\mathbb{O} \sim \mathbb{H} + \boldsymbol{l}\,\mathbb{H}$. Subsequently in order to distinguish the imaginary elements of different complex planes of helixons $\mathcal{C}_1, \mathcal{C}_2$ ..., we need a notion of hypercomplex numbers. We know the possible hypercomplex number that merely represents a vector space with definition of *cross product* include Quaternions and Octonions. The ordinary complex numbers and Quaternions defined as:

$$\mathcal{C} = a + b\boldsymbol{i}$$
$$\mathcal{Q} = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \tag{3}$$

For single helixon $\mathcal{C}_1$ the appropriate representation admits $\mathcal{C} = a + b\boldsymbol{i}$ while for the triple helixon $\mathcal{C}_3$ we need extra imaginary basis which are realized as quaternions *Q* with independent imaginary bases $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. Now we claim a simple proposition: *Any closed curve in a plane (2-dimensional flat manifold) could be traced out by a complex series as [12]:*

$$Z_1 = \sum_j r_j e^{\boldsymbol{i}\omega_j \tau} \tag{4}$$

Where τ stands for time parameter. Then for closed curves in 3 and 4 dimensional space we need additional complex series with another independent imaginary basis $\boldsymbol{j}$

and $\boldsymbol{k}$ *respectively:*

$$Z_2 = \sum_m r_m e^{\boldsymbol{j}\omega_m \tau}, \qquad Z_3 = \sum_l r_l e^{\boldsymbol{k}\omega_l \tau} \tag{5}$$

Therefor closed curves in 4-dimensional space will be determined by the triplet of complex numbers:

$$\{\sum_n r_n e^{\boldsymbol{i}\omega_n \tau}, \quad \sum_m r_m e^{\boldsymbol{j}\omega_m \tau}, \quad \sum_l r_l e^{\boldsymbol{k}\omega_l \tau}\}$$

We may replace the parametric function characterizing the position of C.O.M on closed curve by quaternion $\mathcal{Q}$:

$$\mathcal{Q} = Z_1 + Z_2 + Z_3 \tag{6}$$

Apparently the single helixon comprises of an ordinary complex number $\mathcal{C}$ while the triple helixon being determined by a quaternion. We will show in the next section that imaginary parts of z^1, z^2, ... are interpreted as extra momentums of C.O.M on helixons.

7. Properties of Helixon Momentum

7.1. Ground State

Let a particle in a reference frame $\mathcal{K}$ be in stationary state and thus the particle is tracing out along time axis. Now let C.O.M of the particle in a co-moving frame is orbiting with a constant angular velocity ω_j (clockwise) on a circle in the plane $x^j - x^4$ (for fixed j) which is perpendicular to the M^{3+1} and moving along the time axis tracking a spiral geodesic on a $\mathcal{T}^2$ (2-torus) while the torus axis coincides the time axis x^0. If the momentum on circular motion denoted by p^j and its projection on x^j in M^{3+1} by $p_\perp^j$, for an observer in M^{3+1} (regular space-time) the measured linear momentum will be the projection of p^j on corresponding spatial axis x^j in M^{3+1}. For linear momentum p^j on circular motion and its projection $p_\perp^j$ on x^j we have:

$$p^j = m\omega_j r_j$$
$$p_\perp^j = p^j \sin\theta = m\omega_j r_j \sin\theta \tag{7}$$

We define a complex value in plane $x^j - x^4$ of circle with imaginary axis along x^4 and real axis along x^j:

$$z^j = x^j + ix^4 \; ; \qquad j = 1,2,3 \tag{8}$$

By the identity $x^4 = r_j \sin\theta$ and equation (8) we get:

$$z^j = x^j + ir_j \sin\theta = x^j + \frac{i}{m\omega_j} p_\perp^j \tag{9}$$

This proves the equality of:

$$x^4 = \frac{p_\perp^j}{m\omega_j} \tag{10}$$

If we replace x^j and $p_\perp^j$ with corresponding operators $\hat{x}^j$ and $\hat{p}_\perp^j$, equation (9) transforms to an annihilation operator:

$$\hat{a}_j = \sqrt{\tfrac{m\omega_j}{2\hbar}}\,(\hat{x}^j + \tfrac{i}{m\omega_j}\hat{p}_\perp^j) = \sqrt{\tfrac{m\omega_j}{2\hbar}}\,\hat{z}^j \tag{11}$$

Up to an appropriate coefficient for dimensional adjustment, the equality will be valid. Hence the complex variable z^j which defined in $x^j - x^4$ plane, could be understood as an operator $\hat{z}^j$ equivalent to *annihilation* operator in a quantum harmonic oscillator, and its conjugate $\hat{\bar{z}}^j$ as *creation* operator as well. Moreover energy eigenvalues of this n-dimensional harmonic oscillator, normally can be determined by $\hat{a}_j \hat{a}_j^\dagger$ (number operator) eigenvalues i.e. integer n_j numbers:

$$E_j = \hbar\omega_j \left(\hat{a}_j \hat{a}_j^\dagger + \tfrac{1}{2}\right) = \hbar\omega_j (n_j + \tfrac{1}{2}) \tag{12}$$

The lowest level for each Helixon corresponds to the $n_j = 0$, and consequently for total ground level energy (ZPE) of a particle with these Helixons we obtain:

$$\mathbb{E}_0 = \sum_j E_j = \tfrac{1}{2}\sum_j \hbar\omega_j \;\Rightarrow \tfrac{1}{2}\sum_j \hbar\omega_j \;\Rightarrow\; \sum_j \omega_j = \text{cte} \tag{13}$$

In contrast to the usual definition of ZPE, the summation is carried out on finite numbers of modes which is determined by the total modes of particle helixons. $\mathbb{E}_0$ should contribute to the total mass of a particle. Hence for particles in a multiplet (as hadron octet or decuplet etc.) with approximately the same mass, $\mathbb{E}_0$ should be considered as a common constant. Therefore the equation (13) indicates the equation of a plane (simplex) in an abstract N-dimensional space spanned by orthogonal coordinates ω_j where all particles belonging to this multiplet should be presented by some points on this simplex (hyperplane) in a flat space spanned by ω_j coordinates. This resembles the $SU(n)$ representation multiplets in the Gell-Mann Nishijima model for hadron multiplet symmetries. ω_j in our model corresponds to invariant charges such as Isospin, and Hypercharge in Gell-Mann Nishijima model. The underlying reason for this consistency originated from the $SU(N)$ symmetries of N-dimensional harmonic oscillators embedded in helixons of a particle. This reveals a connection between the frequencies ω_j and the weights of adjoint representation of $SU(N)$ which reduces to Gell-Mann Nishijima model for $N = 3$.

All particle's invariant charges should be represented by ω_j. Electrical charge, Baryonic and Leptonic numbers, Hypercharge of hadrons, weak hyper charge and Isospin all included in this hypothesis. Helixons with ω_j as frequency and z^j as the state of the charge carrying particle represents an *orbital* angular momentum while the *spin* of particle is an exception among these quantum numbers because it represents an *intrinsic* angular momentum that is the angular momentum respect to the C.O.M of the particle. Therefor the spin is not included in the set of helixons and should be considered as a vector with parallel transport along the geodesics determined by helixons fig (2). This hypothesis unifies the origin of all these invariant charges in a set of various helixon and their modes. In other words all invariant charges originate from a helical motion on helixon geodesics extended through a unique extra spatial dimension. Equation (13) implies a *linear* relation among invariant charges of a multiplet (i.e. with the same $\mathbb{E}_0$). A well known example is the relation of electrical charge *Q*, baryonic number *B*, strangeness *S*, in Gell-Mann–Nishijima

equation:

$$Q = I_3 + \frac{1}{2}(B + S) \tag{14}$$

7.2. Excited States (non-commensurate frequency)

For next energy levels (i.e. $n_j \neq 0$ and non-commensurate frequency) the total energy reads as:

$$\mathbb{E} = \sum_j \hbar\omega_j \left(n_j + \frac{1}{2}\right) = \hbar \sum_j \omega_j n_j + \mathbb{E}_0 \tag{15}$$

For eigenvalues of $\hat{a}_j \hat{a}_j^{\dagger}$, equation (11) gives:

$$n_j = \frac{m\omega_j}{2\hbar}|z^j|^2 \tag{16}$$

Substitution of n_j from (16) into equation (15) converts it to:

$$\mathbb{E} = \sum_j \frac{1}{2} m \sum_j {\omega_j}^2 |z^j|^2 + \mathbb{E}_0 \tag{17}$$

The charges ω_j of slightly excited particles in a multiplet should satisfy equation (17) as well. Hence the locations of particles on simplex hyper-plane which defined by equation (13) are the intersection of this simplex with surface defined by (17). This property explains the topology of points on simplex characterizing the particles in a multiplet.

8. Corresponding Complex Space

Respect to definition of z^j and $\bar{z}^j$, canonical coordinates of Hamiltonian formalism can be obtained by the variables:

$$p^j = \frac{1}{2}(z^j - \bar{z}^j) \qquad q^j = \frac{1}{2}(z^j + \bar{z}^j) \tag{18}$$

These variables constitute the canonical coordinates of real space $\mathbb{R}^{2n} \sim \mathbb{C}^n$ with the phase space properties. As we have shown before all conjugate pairs $(z^j, \bar{z}^j)$ and (p^j, q^j) are independent coordinates in $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ respectively. $\mathbb{C}^n$ endowed with a Euclidean metric which induces a Kahler metric on $\mathbf{T}^n$ with the fundamental Kahler form:

$$\omega_0 = i \sum_j d\, z^j \wedge d\bar{z}^j \tag{19}$$

With this setting we obtain $\hat{z}^j$ and $\hat{\bar{z}}^j$ operators as creation an annihilation operators for bosons with frequencies ω_j where index j running from one to the number of Helixons on the world-line of a particle. On the other hand z^j and $\bar{z}^j$ denote complex numbers configuring the position of C.O.M of particle on corresponding Helixon structures on $\mathcal{T}^n$. These complex values are time dependent while their modules considered to be constants. Thus each Helixon contain a finite set of z^j and corresponding operators $\hat{z}^j$. Consequently each particle specifies a unique Helixon with its associated operators $\hat{z}^j$ (or $\hat{\bar{z}}^j$) as creation (or annihilation) bosonic operators which restricts the interaction of this particle with certain type of bosons with creation (or annihilation) operators $\hat{z}^j$ (or $\hat{\bar{z}}^j$). Let label this set of bosons by $\mathfrak{B}_1, \mathfrak{B}_2 \ldots, \mathfrak{B}_N$. Particles defined by Helixon $\mathfrak{C}_N$ possess the set of charges $\mathcal{Q}_j$ each corresponding to boson $\mathfrak{B}_j$. As an example, the electric charge $\mathfrak{q}$ corresponds to the photon.

9. Torus Action and Moment Map

By definition a 2-Torus: $\mathbf{T}^2 = \{(t_1, t_2) \in \mathbb{C}^2 : |t_i| = 1, \forall i\}$ or in a simplified form (e^{it_1}, e^{it_2}) acting on a $\mathbb{C}^2$ manifold (z^1, z^2) by:

$$(e^{i\omega_1 t}, e^{i\omega_2 t}).(z^1, z^2) = (e^{i\omega_1 t} z^1, e^{i\omega_2 t} z^2) \tag{20}$$

Where $\omega_1, \omega_2 \in \mathbb{Z}$ (actually here ω_1, ω_2 can be considered as fixed integers) and $z^1 = x^1 + ix^4$, $z^2 = x^2 + ix^4$. Naturally this action is a Hamiltonian with moment map:

$$\mu(z^1, z^2) = -\frac{1}{2}(\omega_1 |z^1|^2, \omega_2 |z^2|^2) \tag{21}$$

The moment map μ due to its definition maps a point from $\mathbb{C}^n$ to $\mathbb{R}^n$.

In generalized form with: $\mathbf{T}^n = \{(t_1, t_2, \ldots t_n) \in \mathbb{C}^n : |t_i| = 1, \forall i\}$. If we set all $t_i = e^{it}$, the action of *n*-torus on $\mathbb{C}^n$ reads as:

$$(e^{i\omega_1 t} z^1, e^{i\omega_2 t} z^2, \ldots\ e^{i\omega_n t} z^n) \tag{22}$$

And related moment map reads as [14]:

$$\mu(z^1, z^2, \ldots, z^n) = -\frac{1}{2}(\omega_1 |z^1|^2, \omega_2 |z^2|^2, \ldots \omega_n |z^n|^2) \tag{23}$$

Thus moment map $\mu(z^1, z^2, \ldots, z^n)$ identifies as "angular momentum" when $|z^n|^2$ realized as squared radius of *n-th* branch of combined torus. The set of points in equation (23) refers to the *weighted (twisted) projective space* [13, 14].

10. Hamiltonian and Action

Hamiltonian for n-dimensional Harmonic oscillator is given by [15]:

$$H = \frac{1}{2}\hbar\omega_j \sum_j (p_j^2 + q_j^2)$$

p_j and q_j denote the momentum and position of oscillation with frequency ω_j along dimension 'j'. Respect to equation (11) and definition of creation an annihilator operators $a_j, a_j^{\dagger}$ we obtain:

$$a_j = \sqrt{\frac{m\omega_j}{2\hbar}}\, z^j \quad , a_j^{\dagger} = \sqrt{\frac{m\omega_j}{2\hbar}}\, \bar{z}^j \tag{24}$$

$$H = \hbar\omega_j \sum_j (a_j a_j^{\dagger} + \frac{1}{2}) = \frac{1}{2} m \sum_j {\omega_j}^2 |z^j|^2 + ZPE \tag{25}$$

Let the motion of C.O.M on Helixon geodesics Γ (i.e. $\mathcal{C}_N$) being mapped to a trajectory γ on n-tori $\mathbf{T}^n$ as follows:

$$\Gamma : t \to \mathcal{T}^n \subset M^4 \qquad \Phi : \Gamma \to \gamma \subset \mathbb{C}^n \tag{26}$$

Φ maps any point p on geodesic $\Gamma \subset \mathcal{T}^n$ to a point p' with coordinates $(z^1, z^2, \ldots, z^n)$ on curve $\gamma \subset \mathbb{C}^n$. The geodesic Γ parametrized by t (time), determines the trajectory of particle C.O.M (helixon) in M^4. The

continuous mapping results in a curve γ in $\mathbb{C}^n$ which could be a closed or open curve. If the frequencies ω_j are rational i.e. $\omega_j \in \mathbb{Q}$ ($\mathbb{Q}$ denotes rational number set) then Γ will be periodic in time and the curve γ will be a *closed contour* in $\mathbb{C}^n$. With at least one irrational ω_j, Γ is not periodic and γ will not be closed forever. In this case Γ represents a *dense* curve (orbit) bounded in Ω.

Proposition 1: The map Φ is an *isometric* map.

Let the Hermitian metric defined on $\mathbb{C}^n$ manifold given by:

$$d\boldsymbol{l}^2 = \sum_{i\bar{j}} g_{i\bar{j}}\ dz^j\, d\bar{z}^j$$

Because $\mathbb{C}^n$ has been considered as a Euclidean (flat) manifold we have:

$$g_{i\bar{j}} = \delta_{ij}$$

Taking into consideration $|d\boldsymbol{l}|^2 = |z^j|^{\mathbf{2}}{\omega_j}^2 dt^2$ as the line element on Γ, for world-line element we obtain:

$$d\boldsymbol{s}^2 = d\boldsymbol{l}^2 - c^2dt^2 = \sum_j |z^j|^{\mathbf{2}}{\omega_j}^2 dt^2 - c^2dt^2 \qquad (27)$$

Therefore the spatial elements $d\boldsymbol{l}^2$ is the same for γ:

Because:

$$d\boldsymbol{l}^2 = \sum_j |z^j|^{\mathbf{2}}{\omega_j}^2 dt^2 = \sum_{i,j} \delta_{ij}\, dz^i\, d\bar{z}^j = \sum_j\ dz^j\, d\bar{z}^j \qquad (28)$$

This means that Φ is an *isometric* map. Consequently as long as Γ be considered as a minimum length curve between 2 certain points on $\mathcal{T}^n$ (with certain tangent vector Γ'), its isometric map γ treats as a local geodesic. Therefore Φ is a geodesic map as well. To find the motion equations, the variation of γ length $L(\gamma)$ should be vanished:

$$\delta L(\gamma) = \delta \int_{t_1}^{t_2} \left(\sum_j |z^j|^{\mathbf{2}}{\omega_j}^2\right)^{\frac{1}{2}} dt = 0 \qquad (29)$$

Leaving the $|z^j|^{\mathbf{2}}$ as constants, total energy for each mode represented by $\frac{1}{2}|z^j|^{\mathbf{2}}{\omega_j}^2$. Hence the action (Energy functional) could be defined (mass assumed to be constant) as:

$$S(\gamma) = \tfrac{1}{2}\int_{t_1}^{t_2} |z^j|^{\mathbf{2}}{\omega_j}^2\ dt \qquad (30)$$

The extremum path for this integral is the same as for equation (29). In other words the geodesics for both length *L* and action *S* are the same. Respect to Cauchy-Schwarz theorem an inequality relation holds in this case:

$$L(\gamma)^2 \le 2(t_2 - t_1)S(\gamma) \qquad (31)$$

Since the velocity of C.O.M on Γ (and on γ) is constant, then $L \sim (t_2 - t_1)$ and one concludes:

$$L(\gamma) \le \kappa S(\gamma) \qquad (32)$$

With κ as a constant coefficient. This result proves the *Montgomery conjecture* in section 12. On $\mathbf{T}^n$ with a set of metrics (including flat metrics), there exists *closed geodesics* of the types of *elliptic or parabolic*, i.e. *stable closed geodesics (orbits)* [16]. This is a corollary by Ballmann et.al which proves the existence of *stable closed geodesics*. These closed orbits on $\mathbf{T}^n$ correspond to *periodic* geodesics on $\mathcal{T}^n$ (in M^{4+1}).

11. Consequences of Helixon Model

Let geodesic motion of a particle be restricted to $\mathcal{C}_1$ as is defined in section (5). In Helixon theory this helical motion around the world-line axis represents the *particle charge*. This assumption suggests a dynamical meaning for electric charge of elementary particles. In this section we bring some of fundamental outcomes of this theory.

11.1. Vector Potential

C.O.M of a free charged particle (e.g. electron) is passing on the *charge Helixon* $\mathcal{C}_1$. The total momentum of such particle is the sum of linear momentum and oscillatory momentum as a result of harmonic oscillation on $\mathcal{C}_1$ with frequency ω_1. The oscillatory component of total momentum is perpendicular to linear momentum and its frequency ω_1 is proportional to charge to mass ratio $\frac{q}{m}$. Obviously the oscillatory component is small relative to linear component. Let consider this charged particle in an electromagnetic field with local four vector potential $\vec{A}$. This implies an interaction between the particle and photons of related fields. Without loss of generality the final effect of this interaction will be shifting the frequency ω_1 of Helixon $\mathcal{C}_1$ to a new frequency mode $\omega'_1 = \omega_1 + \Delta\omega_1$. This interaction results in a change in total momentum to a generalized momentum:

$$\vec{\boldsymbol{P}} = \vec{\boldsymbol{p}}_{\mathbf{0}} + m(\omega_1 + \Delta\omega_1)\vec{z}^1 e^{i(\omega_1+\Delta\omega_1)t}$$
$$= \vec{\boldsymbol{p}}_{\mathbf{0}} + m\omega_1\vec{z}^1 e^{i(\omega_1+\Delta\omega_1)t} + m\Delta\omega_1\vec{z}^1 e^{i(\omega_1+\Delta\omega_1)t} \qquad (33)$$

The second term on R.H.S of (25) denotes the oscillatory part of free charged particle momentum in M^3 and $\boldsymbol{P}$ denotes the total momentum while $\boldsymbol{p}_{\mathbf{0}}$ stands for non-oscillatory part of particle in M^3. $\vec{z}^1$ denotes the position vector in $x^j - x^4$ plane with x^j as the oscillation axis and x^4 as the extra dimension. If $\vec{\boldsymbol{p}}$ stands for the total momentum in the absence of field $\vec{A}$, then (33) could be read as:

$$\vec{\boldsymbol{P}} = \vec{\boldsymbol{p}} + m\Delta\omega_1\vec{z}^1 e^{i(\omega_1+\Delta\omega_1)t} \qquad (34)$$

Where $\vec{\boldsymbol{p}}$ denotes the free charged particle momentum in M^3. Vector $\vec{z}^1$ can be described in terms of $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$ as polarization vectors living on $x^j - x^4$ plane perpendicular to $\vec{\boldsymbol{p}}_{\mathbf{0}}$. Taking into account the relation $\omega_1 = k\frac{q}{m}$ (from section 4), or $m = \frac{kq}{\omega_1}$ we have:

$$\vec{\boldsymbol{P}} = \vec{\boldsymbol{p}} + kq\frac{\Delta\omega_1}{\omega_1}(\xi^1\vec{\epsilon}_1 + \xi^2\vec{\epsilon}_2)e^{i(\omega_1+\Delta\omega_1)t} \qquad (35)$$

Using the familiar definition of $\vec{A} = \vec{\epsilon}_\lambda e^{i(\omega_1+\Delta\omega_1)t}$ we reach the *total momentum* in the sense of quantum mechanics (or classic):

$$\vec{\boldsymbol{P}} = \vec{\boldsymbol{p}} + q\vec{A} \qquad (36)$$

Where $\frac{k\Delta\omega_1}{\omega_1}$ in (35) included in the term $\vec{A}$. This result reveals the compatibility of basic assumptions in Helixon theory.

11.2. Helixon of Electrical Charge

It is also noteworthy to see that the steep of geodesic $\mathcal{C}_1$ on 2-torus (specified by particle electric charge) embedded in whole spatial space (3 regular and 1 extra) is a constant. In present model this steep $\frac{dx^4}{d\tau}$ regarded as a constant for fermions with usual ZPE $=\frac{1}{2}\hbar\omega_j$ in the absence of electromagnetic field (or other fields). Consequently for an arbitrary charge $\mathcal{Q}_j$ we have a corresponding mode ω_j for geodesic $\mathcal{C}_N$ as the particle's Helixon. In other words motion of C.O.M of particle on geodesic $\mathcal{C}_j$ with frequency ω_j is equivalent or the origin of charge $\mathcal{Q}_j$ i.e.

$$\omega_j = \frac{\mathcal{Q}_j}{m} \tag{37}$$

m in this equation denotes the total mass of particle. Hence the relativistic correction naturally imposes on ω_j as any other internal frequencies in especial relativity. Hence $\mathcal{Q}_j$ charges are Lorentz invariant while the frequencies ω_j are not.

11.3. Larmor Precession

Respect to the notion of general relativity it is well known that displacement of a spin (gyroscope) while moving along geodesics in space time, generally preserves its magnitude and direction i.e. parallel transporting along geodesics [17]. Thus the angle between spin and tangent vector (speed vector) on geodesics remains constant. By imposing a vector field $\vec{A}$ on Helixon $\mathcal{C}_1$ and changing the frequency mode to ω'_1, $\mathcal{C}_1$ deforms to a new geodesics say $\mathcal{C}'_1$. As explained in section 10.b respect to parallel transport of spin along the $\mathcal{C}_1$, spin direction remains unchanged and will be parallel to world-line of C.O.M. After imposing of field $\vec{A}$ (i.e. absorption of photons) C.O.M shifts to another geodesics $\mathcal{C}'_1$. Because of parallel transport, the spin vector preserves its direction and magnitude and angle with tangent vector on new geodesics $\mathcal{C}'_1$ and consequently deviates from world-line alignment in M^3 fig (2). This bring about the precession of spin vector with $(\omega_1 + \Delta\omega_1)$ frequency.

Taking into account the relative smallness of ω_1 which stands for one of zero point energy modes, we conclude that except for extremely weak field $\vec{A}$, we have:

$$\omega_1 \ll \Delta\omega_1 \tag{38}$$

This means that after imposing the electromagnetic field $\vec{A}$ the frequency of helixon shifts from ω_1 to $\Delta\omega_1 + \omega_1$ but respect to (38) $\Delta\omega_1 \cong \Delta\omega_1 + \omega_1$ the precession frequency takes the value $\Delta\omega_1$. The effective frequency and related extra momentum in M^3 should be the second term of equation (33). The corresponding vector potential in a homogenous magnetic field could be obtained by:

$$\vec{A} = \frac{1}{2}\vec{B} \times \vec{r} \tag{39}$$

The extra momentum appearing after imposing the magnetic field $\vec{A}$ could be compared with the extra momentum of particle relative to a rotating coordinate system, i.e. $m\Delta\vec{\boldsymbol{\omega}}_l \times \vec{r}$, thus the second term in equation (33) could be read as:

$$\vec{\boldsymbol{P}} = \vec{\boldsymbol{p}} + q\vec{A} = \vec{\boldsymbol{p}} + \frac{1}{2}q\vec{B} \times \vec{r} = \vec{\boldsymbol{p}} + m\Delta\vec{\boldsymbol{\omega}}_l \times \vec{r} \tag{40}$$

This immediately results in the Larmor precession frequency:

$$\Delta\omega_l = \frac{qB}{2m} \tag{41}$$

However in the case of very weak magnetic field when $\omega_1 \sim \Delta\omega_1$ (are at the same scale) we expect some deviation from equation (40) for Larmor frequency. At this scale Larmor precession will be modified in the form:

$$\omega_l = \Delta\omega_l + \omega_1 \tag{42}$$

This can be figured out in recent experiment results [18] where the Berlin group found some small deviation of longitudinal relaxation rate at lower Larmor precession $(< 100Hz)$ which suggests an as-yet-unknown mechanism in very weak fields. This results emerges because of the equation (42) for very weak fields where the Larmor frequency is not in strict proportion with $\vec{B}$. This prediction of helixon theory will be discussed in detail in another paper.

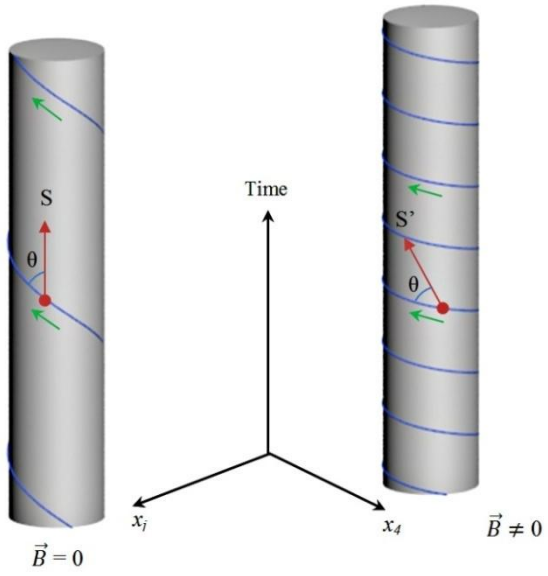

Figure 2. In the absence of magnetic field, C.O.M of a charged particle (red bullet) traces out the helixon $\mathcal{C}_1$ (left side) with a spin S along time axis (or world-line) while parallel displacement along geodesic ($\mathcal{C}_1$) with constant angle θ without any precession. After imposing a magnetic field, and increasing the angular velocity, the geodesic $\mathcal{C}_1$ transforms to helixon $\mathcal{C}'_1$ as spin continuing the parallel displacement on $\mathcal{C}'_1$ (right side) preserving the same angle θ and a precession equal to the angular velocity of helixon $\mathcal{C}'_1$. Emergence of precession is due to the deviation of spin from time axis (or world-line)

11.4. Vector Bosons and Electroweak Force

Unlike the photons, massive vector bosons $W^{\mp}$ and Z^0 as the mediators of electroweak force, require an extra degree of freedom namely *longitudinal polarization*. This additional degree of freedom endows *mass* to these vector bosons. Actually the main difference between massless bosons (photons, gluons) and massive bosons ($W^{\mp}$,Z^0) is merely this longitudinal polarization. In *helixon* theory these

extra degree of freedom achieves through the geodesic motion of C.O.M along additional helixons. For a photon field there are two *transverse polarization vectors* as described in equation (35). The plane containing these polarization vectors is $x^j - x^4$ perpendicular to the time axis in rest frame. The motion of C.O.M on the related helixon in a rest frame leaves a circular path that coincides plane $x^j - x^4$ without any oscillation along the time axis. Both polarization vectors bounded to this plane. Additional helixons which get constructed on the world line of C.O.M, result in other circular path that are no longer perpendicular to time axis. This *inclination* brings about an extra oscillation with a new degree of freedom. In helixon theory this extra oscillation is equivalent to degrees of freedom contained in Higgs field of standard model. Consequently the mass of vector bosons in helixon model comes from this longitudinal polarization. The origin of fermion masses in helixon theory is not included in this mechanism. Similarly it is not clear that the all fermion masses in standard model originated from Higgs like mechanisms.

In helixon theory fermions helixons have been formed by bosons helixons. The C.O.M of fermions passes through helixon geodesics which comprise of all boson helixons allowed to participate with this fermion.

11.5. Higgs Vacuum Expectation Value

The relation of contravariant vector potential and metric tensor of Kaluza-Klein theory:

$$A^{\mu} = Kg^{\mu 4} \tag{43}$$

Where $g^{\mu 4}$ denotes the component of Kaluza-Klein metric tensor and K is a constant defined by the equation:

$$K = \frac{2}{c}\sqrt{4\pi G\varepsilon_0} \tag{44}$$

As we have shown in section (11.1) vector potential components are the components of the tangent vector on helixon $\mathcal{C}_1$. Since the plane of $\mathcal{C}_1$ is orthogonal to time (or world-line) the base vector of fourth spatial dimension $e^4 = \frac{\partial}{\partial x_4}$ is parallel to $\vec{A}$. Therefor the components of $\vec{A}$ may be derived by scaler product:

$$A^{\mu} \cong e^4.e^{\mu} = g^{\mu 4} \tag{45}$$

Equation (45) implies the compatibility of helixon model with Kaluza-Klein result (43). $g^{\mu 4}$ stands for the velocity components parallel to A^{μ} and we have $g^{\mu 4}\sim\omega$, while the radius is the Higgs helixon radius r_H. Therefor we have the relation $A^{\mu} = r_H\omega = r_H g^{\mu 4}$. Moreover calculating K in MKS system respect to appendix, gives the value $K = 0.57 \times 10^{-18}m$. The effective radius for Higgs boson r_H estimated by Lehnert, Bo et al about $0.54 - 1 \times 10^{-18}m$ [20]. Comparing this value with K shows the compatibility of the helixon model results with experiments. Similar results will be mentioned in section 11.6.3.

11.6. Standard Model

11.6.1. Linear Momentum of Helixons and Gauge Covariant Derivatives

In this section we prove the equivalence of total momentum of helixons in M^{4+1} and gauge covariant derivative of standard model. Let a set of helixons $\mathcal{C}_0, \mathcal{C}_1 \ldots \mathcal{C}_N$ associated to a particle (fermion) with $\vec{z}^0, \vec{z}^1, \ldots \vec{z}^N$ instantaneous vectors and angular velocities $\omega_1, \omega_2, \ldots \omega_N$. Let work it out for $N = 4$ with ω_0 of helixon $\mathcal{C}_0$ associated with the charged particles and photons involved in an electromagnetic interaction and $\omega_1 = \omega_2 = \omega_3 = \omega$ of helixon $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ associated with $\vec{z}^1, \vec{z}^2, \vec{z}^3$ as state vectors and related new bosons. The frequencies of the latter helixons assumed to be the same to impose degeneracy for harmonic oscillators associated with $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ with resulting total displacement vector of C.O.M that can be obtained by:

$$V_{\mu} = V_{\mu}^1 + V_{\mu}^2 + V_{\mu}^3 = \dot{z}_{\mu}^1 + \dot{z}_{\mu}^2 + \dot{z}_{\mu}^3 \tag{46}$$

Helixons $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ induces a 2-dimensional *isotropic harmonic oscillator* on M^3 associated with $SU(2)$ symmetry and its $su(2)$ algebra with generators $\sigma_1, \sigma_2, \sigma_3$ (Pauli matrices). The amplitudes of this isotropic harmonic oscillator are independent and could be denoted by z_{μ}^j. The velocity vector $\vec{z}^j$ (for an observer on center of helixon) results in *tangent vectors* W_{μ}^j orthogonal to the instantaneous rotation axis $\vec{\boldsymbol{\Omega}}^j$ with a common angular frequency ω in M^3 (because of isotropic property). Then to obtain $\vec{z}^j$ in terms of generator elements recall the relation for linear velocity $\vec{z}_{\mu}^j$ vector and angular velocity vector:

$$(\vec{z}^j)_{\mu} = (\vec{\boldsymbol{\Omega}}^j \times \vec{z}^j)_{\mu} \tag{47}$$

Let restrict the rotation of $\vec{z}^3$ to one axis (Z-axis) for the first helixon with angular velocity $\vec{\boldsymbol{\Omega}}^3$. It has been proved that $\vec{\boldsymbol{\Omega}}^3$ is equivalent to $\vec{L}_z = \vec{L}_3$, the Z-component of angular momentum vector. Then the cross product (47) could be replaced by:

$$(\vec{z}^3)_{\mu} = (\vec{L}_3 \times \vec{z}^3)_{\mu} \tag{48}$$

$\vec{L}_3$ in the operator sense is equivalent to the operator $(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$ and this operator is also equivalent to a tangent vector to the circle of helixon for the observer on the center of helixon. Given component of this tangential vector by W_{μ}^3, it is easy to prove that $W_{\mu}^3 = \kappa g_{4\mu}$ with $g_{4\mu}$ as a metric tensor element defined by $g_{4\mu} = \boldsymbol{e}_4.\boldsymbol{e}_{\mu}$. Hence (48) will be replaced by:

$$(\vec{z}^3)_{\mu} = (\vec{W}^3 \times \vec{z}^3)_{\mu} \tag{49}$$

This yields the cross product in 3 or 7 dimensional Euclidean space with its definition:

$$(\vec{z}^3)_{\mu} = (\vec{W}^3 \times \vec{z}^3)_{\mu} = \varepsilon_{\mu\nu\lambda}\, W_{\nu}^3 \vec{z}_{\lambda}^3 \tag{50}$$

Respect to the identity $(\hat{L}_{\mu})_{\nu\lambda} = i\varepsilon_{\mu\nu\lambda}$ we get:

$$(\vec{z}^3)_{\mu} = -i(\hat{L}_{\mu})_{\nu\lambda}\, W_{\nu}^3 \vec{z}_{\lambda}^3 \tag{51}$$

Without loss of generality vector $\vec{z}_k^3$ could be replaced by z^3 as a complex variable and $(\hat{L}_i)_{jk}$ by $\hat{L}_3$:

$$(\vec{z}^3)_{\mu} = -i\hat{L}_3\, W_{\mu}^3 z^3 \tag{52}$$

Iterating this process for other helixons $\mathcal{C}_1$, $\mathcal{C}_2$ we get:

$$(\vec{\dot{z}}^1)_\mu = -i\hat{L}_1\, W_\mu^1 z^1 \ , \quad (\vec{\dot{z}}^2)_\mu = -i\hat{L}_2\, W_\mu^2 z^2 \qquad (53)$$

By (30) and (31) and replacing $\hat{L}_1$, $\hat{L}_2$, $\hat{L}_3$ with Pauli matrices σ_1, σ_2, σ_3, assuming definition $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\boldsymbol{W}_\mu = (W_\mu^1, W_\mu^2, W_\mu^3)$ the equation (46) in a compact notation will be read as:

$$V_\mu = -i(\, \sigma_1 W_\mu^1 z^1 + \sigma_2\, W_\mu^2 z^2 + \sigma_3\, W_\mu^3 z^3) \qquad (54)$$

This is rapidity. In helixon model because the three terms in (54) are at the same footing, it is assumed that the modulus and the initial phases of z^1, z^2, z^3 are the same (ω_j are equal). Therefor equation (32) will be simplified as:

$$V_\mu = -i\big(\, \sigma_1 W_\mu^1 + \sigma_2\, W_\mu^2 + \sigma_3\, W_\mu^3\big) z^0 = -i\boldsymbol{\sigma}.\boldsymbol{W}_\mu z^0 \qquad (55)$$

Where $z^0 = z^1 = z^2 = z^3$. Consequently z^0 can be interpreted as Higgs field and $|z^0|$ as the effective radius of Higgs boson r_H . The equivalent operator for total displacement of particle should incorporate the space-time displacement operator ∂_μ and rapidity V_μ of equation (54) and electric charge helixon $\mathcal{C}_0$ (corresponding to vector potential A_μ) i.e.:

$$D_\mu = \partial_\mu - i\boldsymbol{\sigma}.\boldsymbol{W}_\mu + iYA_\mu \qquad (56)$$

We have shown that for each helixon, angular velocity is proportional to *charge to mass ratio* (i.e. coupling constant to mass ratio):

$$\omega \cong {}^{g_2}\!/_{m} \qquad (57)$$

Equation (55) reads as:

$$V_\mu = {}^{g_2}\!/_{m}\,(-i\boldsymbol{\sigma}.\boldsymbol{W}_{\boldsymbol{\mu}}) \qquad (58)$$

And:

$$p_\mu = g_2\,\big(-i\boldsymbol{\sigma}.\boldsymbol{W}_\mu\big) z^0 \qquad (59)$$

The equation (59) indicates an extra-momentum operator i.e. $p_\mu = g_2\,(-i\boldsymbol{\sigma}.\boldsymbol{W}_{\boldsymbol{\mu}})$. This extra momentum term corresponds to either new interactions induced by $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ or second term $\frac{1}{2} i g_2\, \sigma_j W_\mu^j$ in gauge covariant derivative of standard model for electroweak interaction:

$$D_\mu = \partial_\mu - \tfrac{1}{2} i g_2\, \sigma_j W_\mu^j + \tfrac{1}{2} i g_1 Y B_\mu \qquad (60)$$

The term associated with the first helixon $\mathcal{C}_0$ defined for charged particle and interaction with photons and the other term is a consequence of adding helixons $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$. Practically these helixon set is the main structure of *leptons* (e, μ, τ, ν). The first helixon $\mathcal{C}_0$ characterizes the electric charge and would couple with photons. The helixons $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ carry the invariant charges of $su(2)$ symmetry (such as weak hypercharge Y_w and isospin T_3). This is an example of consistency of helixon theory and gauge field theories. The counterpart terms of extra terms in covariant derivatives of gauge theories in helixon model are extra momentums resulted from additional helixons. Therefor for each boson involving in a gauge model there corresponds a helixon with $\vec{z}^j$ as boson field and momentum as corresponding term in gauge covariant derivative. We will examine these properties of helixons in section.11 for electromagnetic field. In equation (59) z^0 stands for a specific boson field that is common between all singular helixons we show that this field is compatible with Higgs boson.

11.6.2. Quaternions and Octonions

We show the strict relation between Helixons and hyper-complex numbers i.e. Quaternions and Octonions. As described in last section, first helixon identified as a circular motion of a complex number z^j defined by (8). Here the complex factor $\boldsymbol{i}$ specifies the x^4 coordinate of first helixon $\mathcal{C}_1$. To construct the second helixon $\mathcal{C}_2$ on the world-line which traced out by $\mathcal{C}_1$ we need the other plane $x'^j - x'^4$ perpendicular to the world-line with a circle on this plane. The new local extra spatial coordinate x'^4 could be specified by the new imaginary factor $\boldsymbol{j}$. For an observer on the center of circle generating $\mathcal{C}_2$, the complex number for $\mathcal{C}_2$ i.e. $z'^j = x'^j + \boldsymbol{j}\, x'^4$ represented by new imaginary factor $\boldsymbol{j}$ and new extra momentum proportional to x'^4. Iterating the process for third helixon $\mathcal{C}_3$ gives $z''^j = x''^j + \boldsymbol{k}\, x''^4$. Adding up these three hyper-complex numbers results in a Quaternion number in the general form:

$$Q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \qquad (61)$$

With $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ as Quaternionic basis and $= x^j + x'^j + x''^j$, $b = x^4, c = x'^4, d = x''^4$. One of its isomorphic algebra is $su(2)$ and its generators Pauli matrices [19]:

$$\boldsymbol{i} = -i\sigma_1 \qquad \boldsymbol{j} = -i\sigma_2 \qquad \boldsymbol{k} = -i\sigma_3 \qquad (62)$$

The commutation and anti-commutation relations of Pauli matrices coincides the Quaternion basis relations. Moreover any induced 3-dimensional rotation of vectors by $su(2)$ algebra could be derived by similar transformations of Quaternions. Any member of $su(2)$ algebra that spanned by Pauli matrices corresponds to a rotation in 3-dimensional space and a Quaternion number. So any Quaternion corresponds to a 3-dimensional rotation. This implies that the triple helixon $\mathcal{C}_3$ corresponds to a 3-dimensional rotation of C.O.M and corresponding Quaternion Q. We will show in next sections that these complex helixons identify the fermions and each single helixon corresponds to a boson. This could be understood through the commutation and anti-commutation relations governing the pure imaginary quaternions Q (as creation-annihilation operators of fermions) and $\hat{z}^j, \hat{\bar{z}}^j$ operators (as creation-annihilation of bosons).

11.6.3. Fermion and Bosons

As mentioned in previous sections we achieved definitions for fermions and bosons in the sense of helixon model. $\hat{z}^j, \hat{\bar{z}}^j$ operators would be considered as annihilation and creation operators of related boson. Respect to equations in section (7.1) the commutation relation of these operators is in the form of bosons:

$$[\hat{z}^j, \hat{\bar{z}}^j] = \mathbf{1}\hbar \qquad (63)$$

The corresponding structures for leptons (e, μ, τ, ν) are complex helixons $\mathcal{C}_0 + \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$ these are the

Quaternions defined in previous section.

It is known that for pure quaternions $\boldsymbol{Q} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ with a convenient choice for adjoint conjugate i.e.

$$\boldsymbol{Q}^{\dagger} = -a_1\boldsymbol{i} - a_2\boldsymbol{j} - a_3\boldsymbol{k}$$

We obtain:

$$\{\boldsymbol{Q}^{\dagger}, \boldsymbol{Q}\} = \mathbf{1}\hbar \tag{64}$$

If the real coefficients take the value $a_i = \frac{\hbar}{\sqrt{6}}$. Pure quaternions are equivalent algebra for $su(2)$ and regular complex basis i denoted as the generator of $u(1)$. This implies that $u(1) \times su(2)$ as the governing symmetry of leptons in the context of helixon model, could be represented by a tensor product $\mathbb{C} \times \overline{\mathbb{H}}$ where $\mathbb{C}$ stands for complex number and $\overline{\mathbb{H}}$ denotes the pure quaternion numbers. Now we can represent the *annihilation* and *creation* operators via a diagonal matrix with elements of $\boldsymbol{Q}$ and z:

$$\mathbb{a} = \begin{bmatrix} \boldsymbol{Q} & 0 \\ 0 & z \end{bmatrix} \qquad \mathbb{a}^{\dagger} = \begin{bmatrix} \boldsymbol{Q}^{\dagger} & 0 \\ 0 & z^{*} \end{bmatrix}$$

The anti-commutation relation $\{\mathbb{a}, \mathbb{a}^{\dagger}\} = \mathbf{1}\hbar$ of these operators leads to:

$$2zz^{*} = 2|z|^2 = \hbar$$

This results in an estimate for $|z|$ about $10^{-18}m$ which is compatible with effective radius of Higgs bosons [20]:

$$|z| \sim r_H$$

Equations (63) and (64) indicate the bosonic and fermionic properties of $\hat{z}^j$ and $\boldsymbol{Q}$ respectively.

11.6.4. Higgs Field as the Potential Well of Helixons

One of the main cornerstones of the helixon theory is the potential well whereby the C.O.M is restricted to trace out the helixon geodesics. Minimum of this potential well traces out the circular trajectory on single helixon where the *radius* determines the vacuum expectation value (vev). For massless bosons the related helixons are perpendicular to time axis. However for massive bosons there is some inclination that results in a *longitudinal polarization* which be presented as an oscillation along the time axis and assumed to be the origin of mass for massive bosons. So the Higgs field in helixon theory is the amplitude of this precession.

11.6.5. The $U(1) \times SU(2) \times SU(3)$ Symmetry

It is well known that $su(N)$ algebras are the generators of symmetry groups of the degenerate states of N-dimensional isotropic harmonic oscillators [2]. Thus symmetry groups $SU(N)$ leaves the related Hamiltonian invariant. Helixon model incorporates these group symmetries by combining the different frequency modes in a single geodesic motion of C.O.M as described in previous sections. This geodesic motion identifies a N-dimensional harmonic oscillator. Let C.O.M passes the geodesics on 3-torus in M^4 with 3 frequencies denoted by $\omega_1, \omega_2, \omega_3$. As we showed through previous sections ω_1 and ω_2 represents electroweak interactions. Adding an extra helixon $\mathcal{C}_3$ with ω_3 associated with 3-dimensional isotropic harmonic oscillator, results in a larger symmetry $SU(3)$ with algebra $su(3)$ with new sets of particle multiplets and associated invariant charges. Taking into account the $|z^j|$ as an invariant (constant), respect to constraints (13) and (17), we conclude that the acceptable charges ω_j of a multiplet of particles should be located on the simplex determined by equation (13). These simplexes for dimensions 2 and 3 represented by a line element and triangle respectively. We show that the hadronic isospin multiplet representation coincides the discrete points on these simplexes corresponding to the states (particles) with approximately the same mass (energy) when the charges ω_j (or a linear combination of charges) plotted as the coordinates. This conclusion predicts a linear relations of charges ω_j for particles included in a multiplet with approximately the same mass which may be replaced by $\mathcal{Q}_j$ (invariant charges) as an example this results in formulas such as the Gell-Mann–Nishijima formula:

$$Q = I_3 + \frac{1}{2}Y \tag{65}$$

Where *hypercharge* denoted by Y equals the sum of B (baryonic number) and S (strangeness number). Therefor for $SU(3)$ symmetry we have three independent charges i.e. I_3, B, S which correspond to three helixon with independent frequencies ω_1, ω_2, ω_3. For more charges and helixon numbers extra charges should be added to satisfy the higher order symmetries $SU(4)$ and so on. This reveals a connection between the frequencies ω_j and the weights of adjoint representation Lie algebras of related group symmetries that generally are considered as unitary symmetries $SU(N)$.

11.6.6. Mass Formula for Particles

Equation (60) reveals that $g_1 Y$ and $g_2\,\sigma_j$ correspond to ω_1 and ω_j proportional to $\mathcal{Q}_1$ and $\mathcal{Q}_j$:

$$\omega_1 = \frac{\mathcal{Q}_1}{m} \cong g_1 Y \;;\; \omega_j = \frac{\mathcal{Q}_j}{m} \cong g_2\,\sigma_j \tag{66}$$

For a helixon with radius $|z^j|$ and angular velocity ω_j, the linear velocity of C.O.M on geodesic should be invariant under small fluctuation of $|z^j|$ (i.e. small deviation of geodesic) as the result of conservation of velocity magnitude of C.O.M along the motion on geodesic. Then we get:

$$V = \omega_j |z^j| = \frac{\mathcal{Q}_j}{m}|z^j| = cte \tag{67}$$

Let v denotes the vacuum expectation value of Higgs field. Thus $v/2$ corresponds to $|z^j|$ and we could obtain:

$$\frac{\mathcal{Q}_j}{m} \cong \frac{2}{v} \rightarrow \frac{g_2}{m} \cong \frac{1}{v} \rightarrow m \cong \frac{1}{2}\,g_2\,v \tag{68}$$

As an example, for W bosons m stands for the mass of related gauge boson i.e. m_W and g_2 is weak coupling constant g_W. These results of the model are compatible with the outcomes of Higgs mechanism where the mass of gauge bosons after symmetry breaking is proportional to coupling constant and vacuum expectation value of Higgs bosons. The Z boson mass will be calculated after introduction of Weinberg angle concept in helixon theory.

11.6.7. Weinberg Angle

The consistency of helixon and standard model can be verified by the interpretation of mixing (Weinberg) angle in helixon theory. The exact relation of g, g', e and Weinberg angle will be the aim of next articles.

12. Other Results

12.1. Montgomery Conjecture

Helixon model proves the relation of heavy nucleus energy level distribution and distribution of non-trivial zeros of Riemann zeta as conjectured by Montgomery. As we have shown, $\mathbf{T}^n$ is a compact Kahler manifold and hence one can calculate the number of closed geodesics with a length less than L lying on this manifold. It is noteworthy to mention that this enumeration has been fulfilled via the formula:

$$\mathcal{N} \sim \frac{e^L}{L} \tag{69}$$

By definition, small perturbations around stable geodesics, returns to geodesics very close to the original geodesics. Hence it will be rational to choose normally the stable geodesics as permitted orbits (geodesics) for particles and conclude that the possible energy levels restricted to these orbits and their energy levels. The inequality (31) shows that the maximum length of $L(\gamma)$ for a certain $S(\gamma)$ is proportional to this action. This means that there is a one to one correspondence between possible energy levels and closed geodesics.

Recall that for a combination of normal modes of vibrations in a system containing the vibrating particles with fixed frequencies (e.g. molecules), superposition of these modes result in a periodic overall motion of system with a time period much larger than the time periods of individual modes [6]. Hence for complex systems of particles whose time periods is a long period in scale of Planck time), energy levels could be approximated by length and number of closed geodesics on $\mathbf{T}^n$ and number and spacing of energy levels converges to distribution of these closed geodesics and consequently prime number distribution. This proves the Montgomery conjecture about heavy nucleus energy levels and its relation to non-trivial Riemann zeta zeros.

12.2. Geodesic Deviation and Golden Ratio

As mentioned before, C.O.M traces out helixons as geodesics on 2-dimensional torus in 4+1 space-time. For stability of these trajectories, the helixons should be closed geodesics for observers co-moving with center of helixon. This requires the *rational* value for angular velocities ratio ω_1/ω_2 and related energies. Rational number distribution investigated by several authors. The well-known distribution is based on Minkowski's number lattice [21]. Due to this analysis the rational number "density" approaches minima at the points determined by the roots of $x^2 - xp - 1$. For $p = 1$ the roots will be read as "golden ratio":

$$x_1 = \frac{1+\sqrt{5}}{2} \qquad x_2 = \frac{-1+\sqrt{5}}{2} \tag{70}$$

The density of geodesics fan with rational ratio ω_1/ω_2 should be minimized at golden ratio and therefor the *geodesics* deviation η should be maximized i.e. $\eta' = 0$. Considering Fermi coordinate while first axis coordinate coincides the geodesic and the other coordinate as $r = \theta$, the Geodesic deviation equations on a Riemannian manifold could be written in regular local coordinate as [23]:

$$\frac{d^2\eta}{dr^2} - R^2_{121}\eta = 0 \tag{71}$$

By supposing small negative constant value for Ricci-Riemann curvature R^2_{121} the general solution reduces to:

$$\eta = k(e^{-\alpha r} + e^{\alpha r}) \tag{72}$$

Where $\alpha = |R^2_{121}|$. The exponentials $e^{-\alpha r}, e^{\alpha r}$ can be considered as exponential maps. For small α with $\alpha^2 \sim 0$ we choose $\eta = \alpha k r$ as specific solution to (71), and thus for complete solution we have:

$$\eta = k(e^{-\alpha r} + e^{\alpha r} + \alpha r) \tag{73}$$

Then for derivative respect to *r* we obtain:

$$\eta' = k(-\alpha e^{-\alpha r} + \alpha e^{\alpha r} + \alpha) \tag{74}$$

By replacing $x = e^{-\alpha r}$ the roots for the solutions of $\eta' = 0$ will read as:

$$-\alpha x + \alpha x^{-1} + \alpha = 0$$

It turns out that the accepted root is golden ratio; $x_1 = \frac{1+\sqrt{5}}{2}$ and implies that the relative probability (intensity) of two modes is golden mean. This result is compatible with experiments [24].

13. Conclusions

Introducing a new model (Helixon Theory) based on the geodetical motion of particles C.O.M on toral manifold embedded in 4+1 Euclidean space results in a set of degree of freedoms of a multidimensional harmonic oscillator which interpreted as the invariant charges of internal symmetries of hadrons and leptons and realizes the $U(1), SU(2)$ and $SU(3)$ symmetries as the symmetries of helixon multidimensional harmonic oscillator and boson and fermions as single and compound helixons. Helixon theory describes the vector potential, Larmor frequency and Higgs vacuum expectation value in a new way and introduces the Quaternion and Octonions as the basic mathematical structures for helixons. This article provides a logical background for Montgomery conjecture in the subject of heavy nucleus energy levels and golden ratio in the context of quantum mechanics. Small deviation of longitudinal relaxation rate at lower Larmor precession is another prediction of helixon theory which is compatible with experiment and conveys a promising prospect for further development and applicability of the model.

Appendix

The units adopted in helixon theory is a geometrical unit system. We use the convention applied in the recently work of Kim.et.al [22]:

$$G = c = \frac{\mu_0}{16\pi} = 16\pi\varepsilon_0 = 1 \tag{75}$$

Hence the numerical values of the terms containing these constants in term of MKS system will be represented in length unit (i.e. meter). For example in the relation of contravariant vector potential and metric tensor of Kaluza-Klein theory:

$$A^{\mu} = Kg^{\mu 4} \tag{76}$$

Where K denoted as $\frac{2}{\tilde{c}}\sqrt{4\pi\tilde{G}\tilde{\varepsilon}_0}$. $\tilde{c}$, $\tilde{G}, \tilde{\varepsilon}_0$ refer to the numerical values of G, c, ε_0 in MKS system. Then K expressed in meters:

$$K = 0{,}57 \times 10^{-18}\ \text{m}$$

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