Journal of Mechanical Engineering and Automation 2016, 6(5): 110-116
DOI: 10.5923/j.jmea.20160605.03

Approximate Solution of Nonlinear Duffing Oscillator Using Taylor Expansion

A. Okasha El-Nady[1,*], Maha M. A. Lashin[2]

[1]Mechatronics Department, Faculty of Engineering, O6University, Egypt
[2]Mechanical Engineering Department, Shoubra faculty of Engineering, Banha University, Egypt

Abstract The Duffing oscillator is a common model for nonlinear phenomena in science and engineering. Its mathematical model is a second order differential equation with nonlinear spring force used to describe the motion of a damped oscillator with a more complicated potential than in simple harmonic motion. In the present paper, the Duffing oscillator equation is solved using a new simple technique based on Taylor theory. The Duffing oscillator equation is solved with different values of initial conditions and damping. The solution results are compared with Runge–Kutta 4th order numerical solution method to investigate the accuracy and reliability of the suggested technique. Results show an excellent agreement between the proposed technique and the Runge–Kutta method.

Keywords Duffing Oscillator, Nonlinear differential equation, Taylor expansion

1. Introduction

Many physical phenomena are modeled by nonlinear systems of ordinary differential equations. An important problem in the study of nonlinear systems is to find exact solutions and explicitly describe travelling wave behaviours. Motivated by potential applications engineering the damped Duffing equation [1] has received wide interest, it is used for studying the oscillations of a rigid pendulum undergoing with moderately large amplitude motion. It has provided a useful paradigm for studying nonlinear oscillations and chaotic dynamical systems.

The Duffing equation [2] given its characteristic of oscillation and chaotic nature, many scientists are inspired by this nonlinear differential equation given its nature to replicate similar dynamics in our natural world. The Duffing oscillator common model using this oscillator involves an electro-magnetized vibrating beam analyzed as exhibiting cusp catastrophic behaviour for certain parameter values.

Surveying the literature shows that a variety of solution methods have been developed so far to solve the Duffing equation. Some researchers have applied a variety of approximate methods to analyze different types of conservative Duffing equation. The homotopy analysis method [3], harmonic balance method [4], homotopy perturbation method [5-7] frequency–amplitude formulation [8], energy balance method [9-11], max–min approach [12, 13], coupled homotopy-variational approach [14] and modified variational approach [15] have all been employed to solve the conservative Duffing equation. Some researchers in their studies into the Duffing oscillator consider damping [16-20]. When the Duffing oscillator involves damping, the amplitude of oscillation reduces over time and we have a non-conservative system.

Most analytical methods [8-15] are unable to handle non-conservative oscillators. However recently, two new methods, Laplace decomposition [21], and homotopy perturbation transform [22], are introduced for the solution of nonlinear and non-homogeneous differential equations which are capable of solving the non-conservative Duffing oscillator problem including damping.

The differential transform is a semi-analytic and powerful method for solving linear and nonlinear differential equations. This method was first used in the engineering domain by Zhou [23] to analyze electric circuits. The differential transform solution diverges by using a finite numbers of terms. To circumvent this problem the modified differential transform method [24-28] was developed by combining the DTM with the Laplace transform and Padé approximant [29] which can successfully predict the solution of differential equations with finite numbers of terms.

Our motivation in the present study is to obtain the solution of the forced response Duffing oscillator considering different damping effects and with different initial conditions by a simple technique based on Taylor expansion. In this technique the acceleration of Duffing oscillator is obtained by direct substitution of the initial conditions in equilibrium differential equation. The response and its velocity at the next step are obtained using Taylor

* Corresponding author:
rady_nady@yahoo.com (A. Okasha El-Nady)
Published online at http://journal.sapub.org/jmea

expansion. The acceleration of the next point is obtained from the equilibrium equation and so on. Results are compared with that obtained by the fourth order Runge–Kutta method. Results show a good agreement between the proposed technique and the Runge–Kutta method.

2. Basic Fundamentals

In this section the basic fundamentals of Taylor's theorem as well as the forward, backward and central difference approximations of higher order derivative are reviewed.

Taylor's Theorem: If f is a function continuous and n times differentiable in an interval $[x, x + h]$, then there exists some point in this interval, denoted by $x + \lambda h$ for some $\lambda \in [0, 1]$, such that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{h^{n-1}}{n!} f^n(x+\lambda h) \quad (1)$$

If f is a so-called analytic function of which the derivatives of all orders exist, then one may consider increasing the value of n indefinitely. Thus, if the condition holds that

$$\lim_{n\to\infty} \frac{h^n}{n!} f^n(x) = 0 \quad (2)$$

which is to say that the terms of the series converge to zero as their order increases, then an infinite-order Taylor-series expansion is available in the form of

$$f(x+h) = \sum_{j=0}^{\infty} \frac{h^j}{j!} f^j(x) \quad (3)$$

This is obtained simply by extending indefinitely the expression from Taylor's Theorem. In interpreting the summary notation for the expansion, one must be aware of the convention that 0! = 1.

Forward Difference:

The first derivative of a function $f(x)$ can be approximated using forward difference:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + o(h) \quad (4)$$

Backward Difference:

The first derivative of a function $f(x)$ can be approximated using backward difference:

$$f'(x) = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h} + o(h) \quad (5)$$

Central Difference:

The first derivative of a function $f(x)$ can be approximated using central difference:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{h} + o(h^2) \quad (6)$$

3. The Problem of Duffing Oscillator

The equation of motion of Duffing oscillator is normally written as

$$\ddot{x} + \alpha\dot{x} + \beta x + \gamma x^3 = F_0 \cos \omega t \quad (7)$$

With initial conditions

$$x(0) = A \quad (8)$$

$$\dot{x}(0) = B \quad (9)$$

Equation (7) is a simple model that can show different types of oscillations such as chaos and limit cycles. The terms associated with this system represent:

$\ddot{x} + \beta x$ Simple harmonic oscillator with angular frequency $\sqrt{\beta}$

$\alpha\dot{x}$ Small damping

γx^3 Small nonlinearity

$F_0 \cos \omega t$ Small periodic forcing term with angular frequency ω

This is a forced oscillator with a nonlinear spring with a restoring force of $F = -\beta x - \alpha x^3$. Different values of α can create either a hardening spring (where $\alpha > 0$) or a softening spring (where $\alpha < 0$). Different values of β can also change the dynamics of the system. For values of β less than zero, the Duffing oscillator displays chaotic motion.

3.1. Methodology of the Proposed Technique

In this technique, the differential equation (7) is rearranged as follows:

$$\ddot{x}(t) = -\alpha\dot{x}(t) - \beta x(t) - \gamma x(t)^3 + F\cos(\omega\, t) \quad (10)$$

By direct substitution of the initial conditions given in (8) and (9) the acceleration at starting point can be written as:

$$\ddot{x}(0) = -\alpha\dot{x}(0) - \beta x(0) - \gamma x(0)^3 + F\cos(\omega \times 0) \quad (11)$$

The approximate displacement function at time $t + \Delta t$ is obtained using Taylor expansion (1) up to the third term:

$$x(t+\Delta t) = x(t) + \dot{x}(t)\Delta t + \frac{1}{2}\ddot{x}(t)\Delta t^2 \quad (12)$$

The approximate velocity function at time $\boldsymbol{t + \Delta t}$ is obtained using the backward difference approximation of the first derivative (5):

$$\dot{x}(t+\Delta t) = \frac{x(t+\Delta t) - x(t)}{\Delta t} \quad (13)$$

Then, the approximate acceleration function at time $\boldsymbol{t + \Delta t}$ is obtained using equation (10) as:

$$\ddot{x}(t+\Delta t) = -\alpha\dot{x}(t+\Delta t) - \beta x(t+\Delta t) - \gamma x(t+\Delta t)^3 + F\,cos(\omega \times (t+\Delta t)) \quad (14)$$

So, the first iteration is obtained from equations (8) through (14) as:

$$x(\Delta t) = x(0) + \dot{x}(0)\Delta t + \frac{1}{2}\ddot{x}(0)\Delta t^2 \quad (15)$$

$$\dot{x}(\Delta t) = (x(\Delta t) - x(0))/\Delta t \quad (16)$$

$$\ddot{x}(\Delta t) = -\alpha\dot{x}(\Delta t) - \beta x(\Delta t) - \gamma x(\Delta t)^3 + F\,cos(\omega\Delta t) \quad (17)$$

The recurrence formula can be written as:

$$x_n = x_{n-1} + \dot{x}_{n-1}\Delta t + \frac{1}{2}\ddot{x}_{n-1}\Delta t^2 \quad (18)$$

$$\dot{x}_n = (x_n - x_{n-1})/\Delta t \quad (19)$$

$$\ddot{x}_n = -\alpha\dot{x}_n - \beta x_n - \gamma x_n^3 + F\,cos\big(\omega((n-1)\Delta t)\big) \quad (20)$$

3.2. Modified Technique

It was noted that the results obtained by the previous iteration formulae has a big difference with that obtained by Runge–Kutta 4th order method. This because that the error accompanied to the velocity is of order Δt. The approximate velocity formula is modified to be obtained using the central difference approximation of the first derivative (6) as:

$$\dot{x}(\Delta t) = (x(2\Delta t) - x(0))/(2\Delta t) \quad (21)$$

The first iteration of the modified technique is written as:

$$x(\Delta t) = x(0) + \dot{x}(0)\Delta t + \frac{1}{2}\ddot{x}(0)\Delta t^2 \quad (22)$$

$$x(2\Delta t) = x(0) + 2\dot{x}(0)\Delta t + 2\ddot{x}(0)\Delta t^2 \quad (23)$$

$$\dot{x}(\Delta t) = (x(2\Delta t) - x(0))/(2\Delta t) \quad (24)$$

$$\ddot{x}(\Delta t) = -\alpha\dot{x}(\Delta t) - \beta x(\Delta t) - \gamma x(\Delta t)^3 + cos(\omega\Delta t) \quad (25)$$

The recurrence formula of the modified technique can be written as:

$$x_n = x_{n-1} + \dot{x}_{n-1}\Delta t + \frac{1}{2}\ddot{x}_{n-1}\Delta t^2 \quad (26)$$

$$x_{n+1} = x_{n-1} + 2\dot{x}_{n-1}\Delta t + 2\ddot{x}_{n-1}\Delta t^2 \quad (27)$$

$$\dot{x}_n = (x_{n+1} - x_{n-1})/(2\Delta t) \quad (28)$$

$$\ddot{x}_n = -\alpha\dot{x}_n - \beta x_n - \gamma x_n^3 + F\,cos\big(\omega((n-1)\Delta t)\big) \quad (29)$$

4. Results and Discussion

4.1. Free Vibration

The recursive relations in sections 3.1 and 3.2 are applied to the duffing oscillator problem with non exciting force. Three case studies [29] are resolved using the present technique. The first case study includes low damping (periodic behavior), strong nonlinearity and initial displacement. The second case study includes critical damping, strong nonlinearity and initial displacement and the third case study is a combination of initial displacement and velocity with periodic behavior. They are considered as follows:

Example 1

$\alpha = 0.5$, $\beta = \gamma = 25$, $A = 0.1$, $B = 0$, $F_0=0$ (30)

For the values given in Eq. (30), *x(t)* is obtained using the recurrence formulae of the present technique (version 1) and its modification (version 2).

Figure 1 shows the comparison between the results obtained using the present techniques (version 1) and its modification (version 2) and the fourth-order Runge–Kutta numerical method. It is clear that the results using the modified technique have good agreement with the results obtained using the fourth-order Runge–Kutta numerical method. The version 1 of the present technique has a big difference with that obtained by the fourth-order Runge–Kutta numerical method since the error in version 1 is of order (Δt) while in version 2 is of order (Δt^2).

Example 2

$\alpha = 2$, $\beta = 1$, $\gamma = 25$, $A = 0.1$, $B = 0$, $F_0=0$ (31)

For the values given in Eq. (31), *x(t)* is obtained using the recurrence formulae of the present technique (version 1) and its modification (version 2).

Figure 2 shows the comparison between the results obtained using the present techniques (version 1) and its modification (version 2) and the fourth-order Runge–Kutta numerical method. It is clear that the results using the modified technique have good agreement with the results obtained using the fourth-order Runge–Kutta numerical method. The version 1 of the present technique has a big difference with that obtained by the fourth-order Runge–Kutta numerical method since the error in version 1 is of order (Δt) while in version 2 is of order (Δt^2).

Example 3

$\alpha = 1$, $\beta = 20$, $\gamma = 2$, $A = -0.2$, $B=2$, $F_0=0$ (32)

For the values given in Eq. (32), *x(t)* is obtained using the recurrence formulae of the present technique (version 1) and its modification (version 2).

Figure 3 shows the comparison between the results obtained using the present techniques (version 1) and its modification (version 2) and the fourth-order Runge–Kutta numerical method. It is clear that the results using the modified technique have good agreement with the results obtained using the fourth-order Runge–Kutta numerical method. The version 1 of the present technique has a big difference with that obtained by the fourth-order Runge–Kutta numerical method since the error in version 1 is of order (Δt) while in version 2 is of order (Δt^2).

As shown in Figures 1, 2, and 3 the results of the solution of the Duffing equation by the proposed technique has excellent agreement with that obtained by the Runge-Kutta 4th order (RK4) method.

4.2. Forced Vibration

The responses of the forced duffing oscillators given in the previous examples are obtained using the modified proposed technique. The amplitude of the exciting force F_0 =1 and the exciting frequency ω =0.5.

For the same data given in Example 1, Example 2 and Example 3 with F_0=1 and ω =0.5, the responses *x(t)* of the forced duffing oscillator are obtained using the recurrence formulae of the present technique (version 2). The results are compared with that obtained using the fourth-order Runge–Kutta method and shown in figures 4, 5 and 6. The comparison shows an excellent agreement between the present technique and Runge–Kutta 4th order.

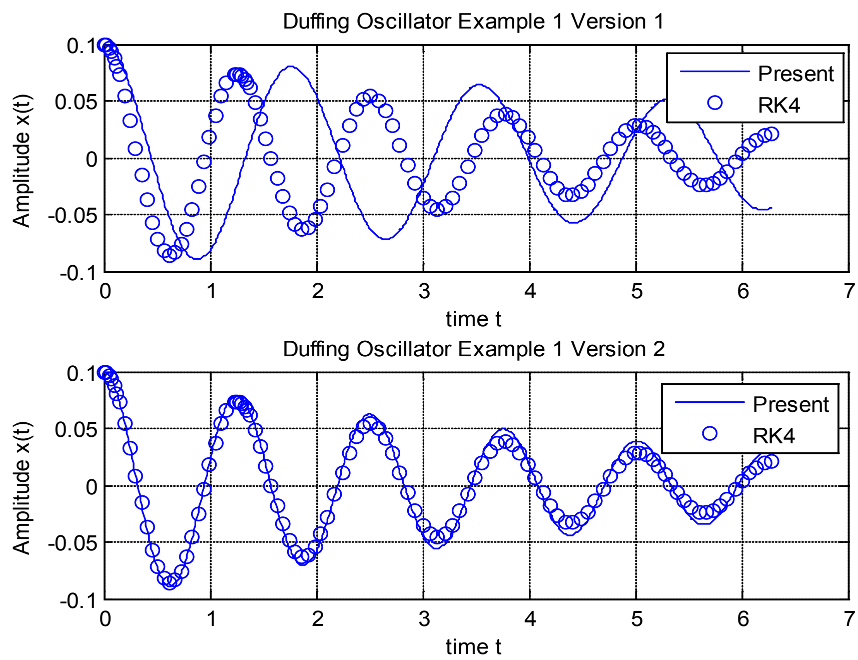

Figure 1. Solution of Duffing Oscillator of Example 1

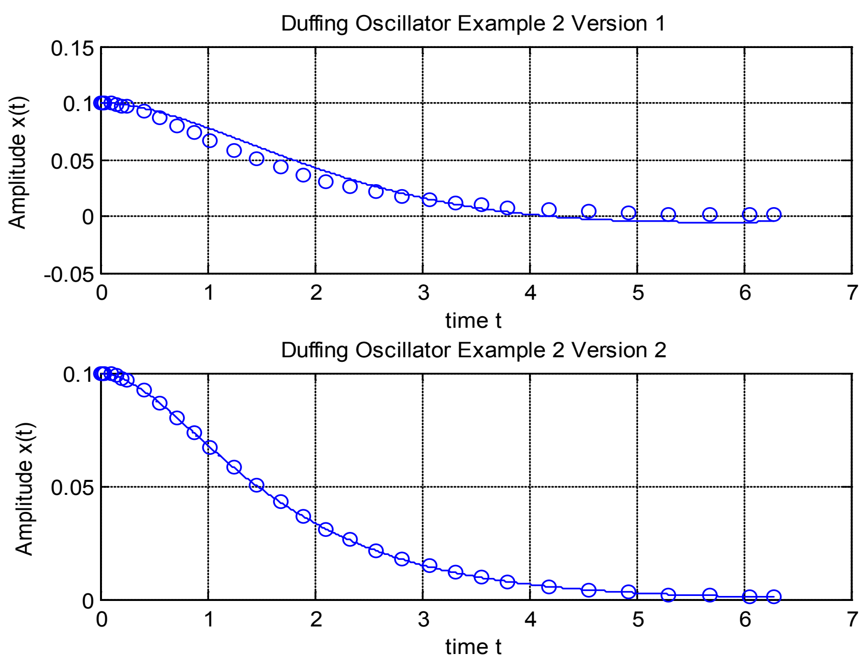

Figure 2. Solution of Duffing Oscillator of Example 2

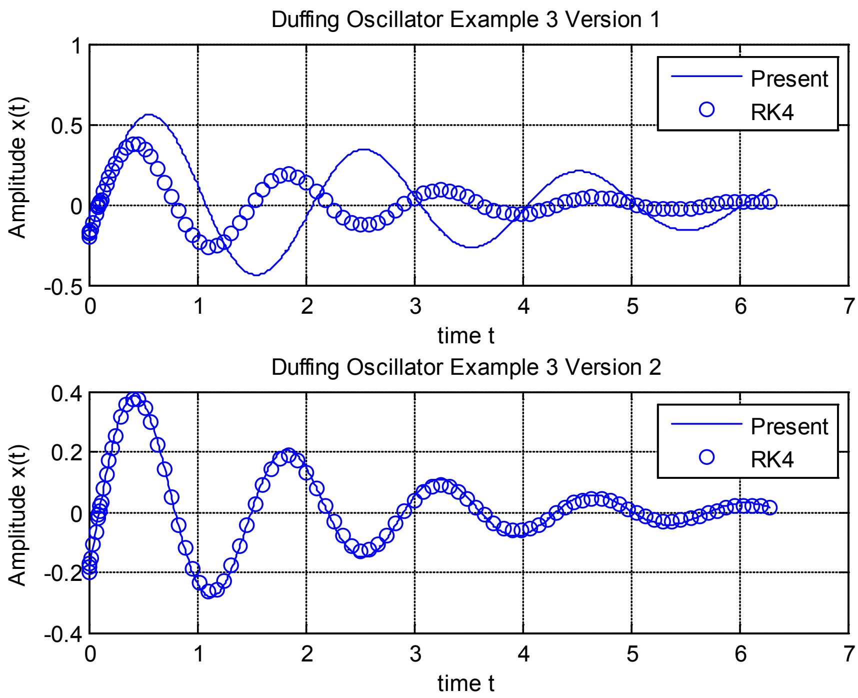

Figure 3. Solution of Duffing Oscillator of Example 3

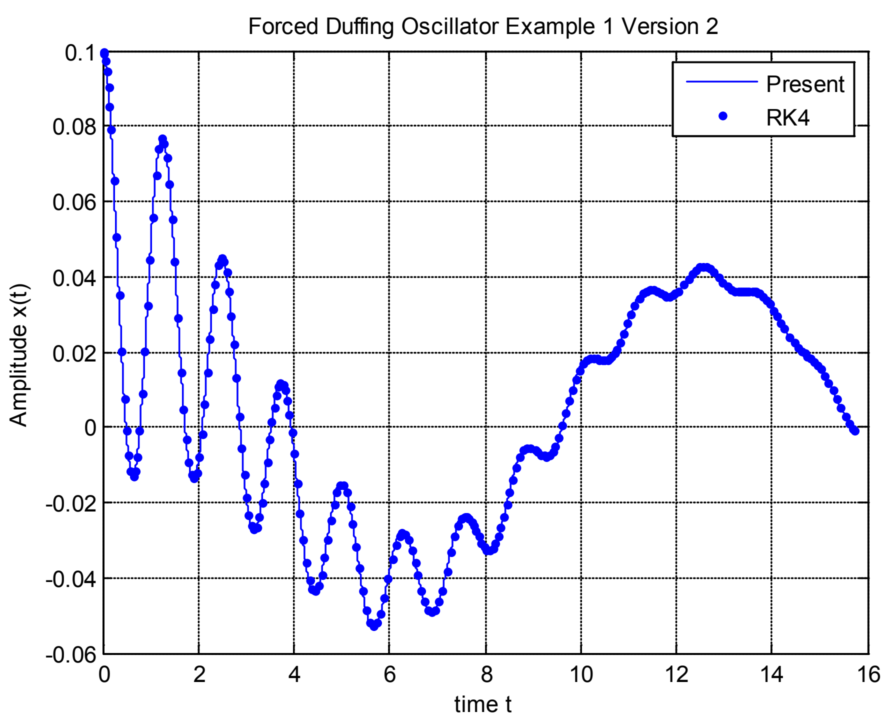

Figure 4. Solution of Force Duffing Oscillator of Example 1

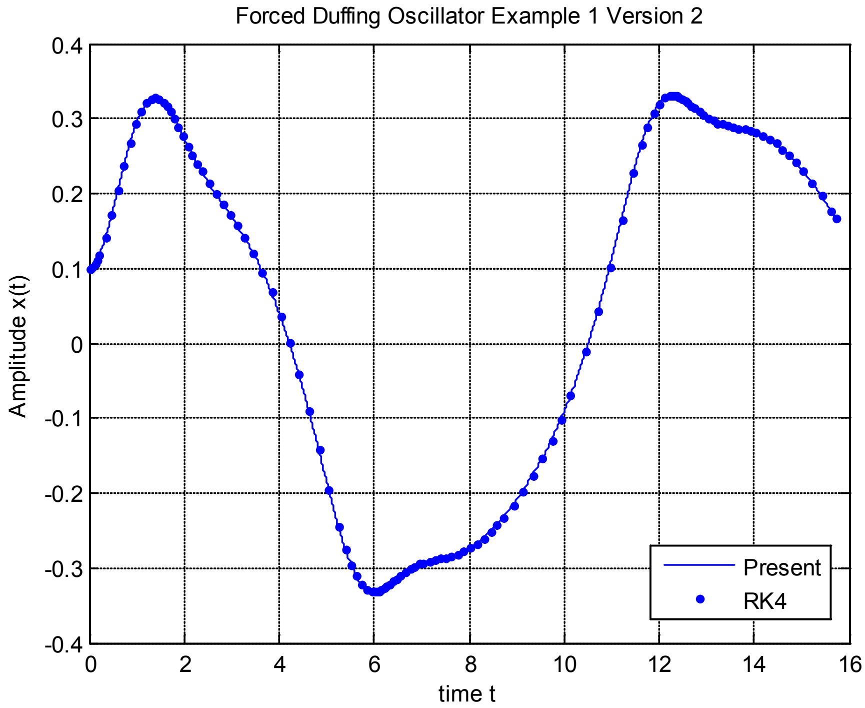

Figure 5. Solution of Force Duffing Oscillator of Example 2

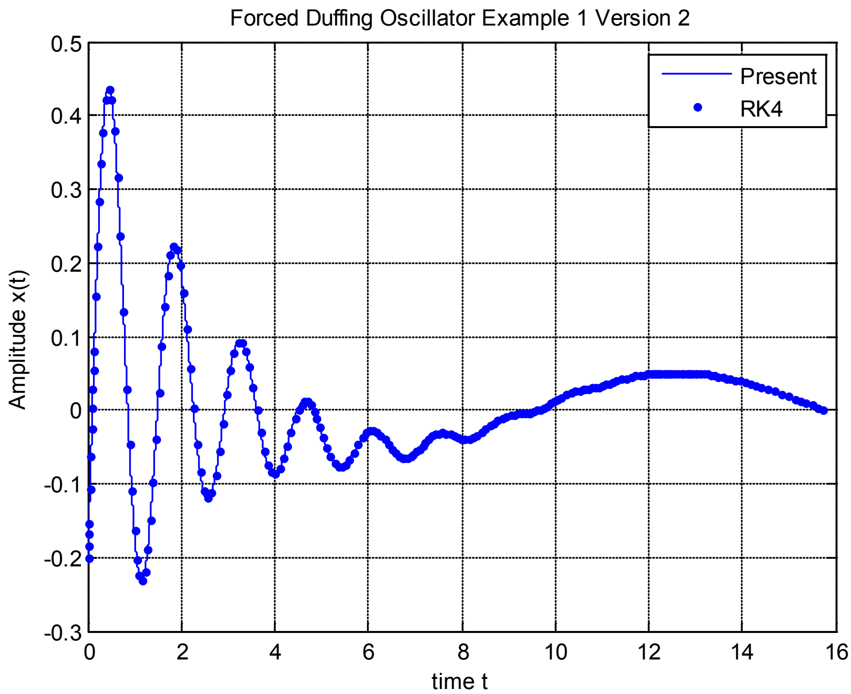

Figure 6. Solution of Force Duffing Oscillator of Example 3

5. Conclusions

In the present study, a simple technique based on the Taylor expansion was applied to determine an approximate solution for a nonlinear Duffing oscillator with damping effect under different initial conditions. A comparison of results with fourth-order Runge–Kutta method indicates excellent accuracy of the solution. We conclude that the modified technique is an accurate tool in handling a nonlinear oscillator with a high level of accuracy. Using the suggested technique, there is no need to transform the higher order differential equations to state space.

REFERENCES

[1] Zhaosheng Feng, Goong Chen, Sze-Bi Hsu, "A Qualitative Study of the Damped Duffing Equation and Applications", University of Texas-Pan American, Edinburg, TX, 2005.

[2] Tamas Kalmar, Balakumar Balachandran, "Forced Harmonic Vibration of a Duffing Oscillator with Linear Viscous Damping-Nonlinear Oscillators and their Behavior", First Edition, Published John Wiley & Sons, Ltd. ISBN: 978-0-470-71549-9, 2011.

[3] Liao, S.J. and Cheung, A.T. ''Application of homotopy analysis method in nonlinear oscillations'', ASME Journal of Applied Mechanics, 65, pp. 914–922 (1998).

[4] Mickens, R.E. ''Mathematical and numerical study of the Duffing-harmonic oscillator'', Journal of Sound and Vibration, 244, pp. 563–567 (2001).

[5] He, J.H. ''Homotopy perturbation technique'', Computer Methods in Applied Mechanics and Engineering, 178, pp. 257–262 (1999).

[6] Younesian, D., Askari, H., Saadatnia, Z. and Yazdi, M.K. ''Free vibration analysis of strongly nonlinear generalized Duffing oscillators using He's variational approach & homotopy perturbation method'', Nonlinear Science Letters A, 2, pp. 11–16 (2011).

[7] Ganji, D.D., Alipour, M.M., Fereidoon, A.H. and Rostamiyan, Y. ''Analytic approach to investigation of fluctuation and frequency of the oscillators with odd and even nonlinearities'', IJE Transactions A, 23, pp. 41–56 (2010).

[8] He, J.H. ''Comment on He's frequency formulation for nonlinear oscillators'', European Journal of Physics, 29, pp. 19–22 (2008).

[9] He, J.H. ''Preliminary report on the energy balance for nonlinear oscillations'', Mechanics Research Communications, 29, pp. 107–111 (2002).

[10] Ganji, D.D., Gorji, M., Soleimani, S. and Esmaeil pour, M. ''Solution of nonlinear cubic-quintic Duffing oscillators using He's energy balance method'', Journal of Zhejiang University: Science A, 10, pp. 1263–1268 (2009).

[11] Durmaz, S., Demirbag, S.A. and Kaya, M.O. ''High order He's energy balance method based on collocation method'', International Journal of Nonlinear Sciences and Numerical Simulation, 11, pp. 1–5 (2010).

[12] He, J.H. ''Max-min approach to nonlinear oscillators'', International Journal of Nonlinear Sciences and Numerical Simulation, 9, pp. 207–210 (2008).

[13] Yazdi, M.K., Ahmadian, H., Mirzabeigy, A. and Yildirim, A. ''Dynamic analysis of vibrating systems with nonlinearities'', Communications in Theoretical Physics, 57, pp. 183–187 (2012).

[14] Khan, Y., Akbarzade, M. and Kargar, A. ''Coupling of homotopy and the variational approach for a conservative oscillator with strong odd nonlinearity'', Scientia Iranica, 19, pp. 417–422 (2012).

[15] Yazdi, M.K., Mirzabeigy, A. and Abdollahi, H. ''Nonlinear oscillators with non-polynomial and discontinuous elastic restoring forces'', Nonlinear Science Letters A, 3, pp. 48–53 (2012).

[16] Wu, B.S. and Sun, W.P. ''Construction of approximate analytical solutions to strongly nonlinear damped oscillators'', Archive of Applied Mechanics, 81, pp. 1017–1030 (2011).

[17] Cveticanin, L. ''Oscillators with nonlinear elastic and damping forces'', Computers & Mathematics with Applications, 62, pp. 1745–1757 (2011).

[18] Leung, A.Y.T., Guo, Z. and Yang, H.X. ''Residue harmonic balance analysis for the damped Duffing resonator driven by a van der Pol oscillator'', International Journal of Mechanical Sciences, 63, pp. 59–65 (2012).

[19] Turkyilmazoglu, M. ''An effective approach for approximate analytical solutions of the damped Duffing equation'', PhysicaScripta, (2012), http://dx.doi.org/10.1088/0031-8949/86/01/015301.

[20] Elias-Zungia, A. ''Analytical solution of the damped Helmholtz–Duffing equation'', Applied Mathematics Letters, 25, pp. 2349–2353 (2012).

[21] Khan, Y. and Austin, F. ''Application of the Laplace decomposition method to nonlinear homogeneous and non-homogenous advection equations'', Zeitschriftfür Naturforschung A, 65a, pp. 849–853 (2010).

[22] Khan, Y. and Wu, Q. ''Homotopy perturbation transform method for nonlinear equations using He's polynomials'', Computers & Mathematics with Applications, 61, pp. 1963–1967 (2011).

[23] Zhou, J.K., Differential Transformation and its Application for Electrical Circuits, Huazhong University Press, Wuhan, China (in Chinese) (1986).

[24] El-Shahed, M. ''Application of differential transform method to non-linear oscillatory systems'', Communications in Nonlinear Science and Numerical Simulation, 13, pp. 1714–1720 (2008).

[25] Momani, S. and Ertürk, V.S. ''Solutions of non-linear oscillators by the modified differential transform method'', Computers & Mathematics with Applications, 55, pp. 833–842 (2008).

[26] Yildirim, A., Gokdogan, A. and Merdan, M. ''Chaotic systems via multistep differential transformation method'', Canadian Journal of Physics, 90, pp. 391–406 (2012).

[27] Mirzabeigy, A. and Yildirim, A. ''Approximate periodic solution for nonlinear jerk equation as a third-order nonlinear equation via modified differential transform method'', Engineering Computations, (2013).

[28] Ertürk, V.S., Yildirim, A., Momani, S. and Khan, Y. ''The differential transform method and Padé approximants for a fractional population growth model'', International Journal of Numerical Methods for Heat & Fluid Flow, 22, pp. 791–802 (2012).

[29] Nourazar, S., and A. Mirzabeigy. "Approximate solution for nonlinear Duffing oscillator with damping effect using the modified differential transform method." *Scientia Iranica* 20.2 (2013): 364-368.