

Relativistic Invariance of Electromagnetic Fields and Maxwell's Equations in Theory of Electrodynamics

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Abstract We re-examined the four-dimensional spacetime formulation of invariance of electromagnetic fields between two inertial frames under Lorentz transformation, which predicts a pure electric (magnetic) field in one inertial frame is composed the Cartesian components of a pure both electric and magnetic fields in another inertial frame. This contradicts the Lorentz invariance condition which requires that the vector quantities in one inertial frame must have the same form in another inertial frame. In this work, we introduce a three-dimensional quasi-time vector to modify the classical four-dimensional spacetime (3+1) to a new six-dimensional spacetime (3+3) and derive spacetime metric equation and relativistic velocity. We use the classical vector transformation theory to derive expressions for Cartesian components of relativistic velocity and net electromagnetic force vectors. Considering two massive inertial frames form a closed system, we integrated the transformed relativistic velocity with the law of conservation of energy to prove that contrary to the common belief, the electromagnetic field that appears as a purely electric (magnetic) field in one massive inertial frame, it also appears as a pure electric (magnetic) field in another massive inertial frame under Lorentz transformation. As an application of the proposed six-dimensional spacetime theory, we prove Lorentz invariance of Maxwell's equations with and without charge and current source. We also prove the scalar electromagnetic wave equations with and without charge and current source and the conservation laws of the continuity equations of current and densities of electromagnetic energy and linear and angular momentums between two massive inertial frames under Lorentz transformation.

Keywords Six dimensional spacetime, Lorentz transformation, Massive inertial frames, Relativistic velocity transformation, Invariance of electric and magnetic fields, Maxwell equations, Scalar electromagnetic wave equations, Conservation laws for current continuity and electromagnetic energy and momentum

1. Introduction

Maxwell's equations and Lorentz force are the foundations of the electromagnetic theory and describe how the charge and current sources with densities ρ and \vec{J} generate electric and magnetic fields (\vec{E} and \vec{B}), and Lorentz force (\vec{F}) acting on a charge q moving with velocity \vec{v} [1]:

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \rho / \epsilon_0 \quad (\text{Gauss law of electrostatics}), \\ \vec{\nabla} \cdot \vec{B} &= 0 \quad (\text{Gauss law of magnetism}) \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \quad (\text{Faraday's law}), \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \quad (\text{Ampere-Maxwell Law})\end{aligned} \quad (1)$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (\text{Lorentz force}) \quad (3)$$

Maxwell's equations lead to several conservation laws in electrodynamics [1], such as equations of current continuity and laws of conservation of electromagnetic energy and momentum.

Historically, Einstein [2] proved the validity of Maxwell's equations in inertial frames by using Lorentz transformation [3] and proposed two postulates; (i) The laws of physics are invariant in all inertial frames moving with uniform velocities relative to one another. (ii) The speed of light in vacuum is the same in all inertial frames and is independent of the direction of the motion of the emitting body. Einstein combines the rates of change of linear momentum ($d\vec{p}/dt$) and energy (dE/dt) and, after some complicated algebra, finds expressions for the Cartesian components of electric and magnetic fields in inertial frames moving relative to each other along the x-direction [2]

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$$\begin{aligned} E'_x &= E_x, & E'_y &= \gamma(E_y - vB_z), & E'_z &= \gamma(E_z + vB_y), \\ B'_x &= B_x, & B'_y &= \gamma(B_y + (v/c^2)E_z), & B'_z &= \gamma(B_z - (v/c^2)E_y) \end{aligned} \quad (4)$$

$$\begin{aligned} E_x &= E'_x, & E_y &= \gamma(E'_y + vB'_z), & E_z &= \gamma(E'_z - vB'_y), \\ B_x &= B'_x, & B_y &= \gamma(B'_y - (v/c^2)E'_z), & B_z &= \gamma(B'_z + (v/c^2)E'_y) \end{aligned} \quad (5)$$

where $\gamma = 1/(1-v^2/c^2)^{1/2}$ is known as Lorentz factor. Here v is the speed of the inertial frames moving relative to each other and c is the speed of light in vacuum. Equations (4) and (5) state that electric and magnetic fields are Lorentz invariant along the direction of motion (x-axis) while there is a change along the perpendicular directions (y, and z-axes). Close inspection shows that although the scalar product of electric and magnetic fields is invariant, their vector product is not invariant between two inertial frames under Lorentz transformation. This has serious impact on the invariance of Maxwell's equations, electromagnetic wave equations, and conservation laws such as continuity equation of current and electromagnetic energy and momentum between two inertial frames. Therefore, current formulation of the invariance of electromagnetic fields and Maxwell's equations between two inertial frames under Lorentz transformation remains to be one of the century long unsolved problems [4]-[7].

In this article, our aim is to prove the relativistic invariance of electromagnetic fields and Maxwell's equations between two inertial frames under Lorentz transformation. The outline of the presentation is as follows. In section 2 we introduce a three-dimensional quasi-time vector to modify the classical four-dimensional spacetime (3+1) to a new six-dimensional spacetime (3+3) and derive spacetime metric equation and relativistic velocity. In section 3 we use the classical vector transformation method to derive Cartesian components of position, relativistic velocity and electromagnetic force vectors in six dimensional spacetime. In section 4 we use the transformed velocity in the law of energy conservation to prove that contrary to common belief, the electric (magnetic) field in so called a massive inertial frame is composed of electric (magnetic) field in another massive inertial frame. In sections 5 and 6 we prove the invariance of Maxwell equations and electromagnetic wave equations between two massive inertial frames.

2. A six-Dimensional Spacetime Frame

In this section, we introduce time as a three-dimensional quasi-time vector $\vec{t} = (t_x, t_y, t_z)$ and $\vec{t}' = (t'_x, t'_y, t'_z)$, along with three dimensional position vector $\vec{r} = \vec{r}(x, y, z)$ and uniform velocity $\vec{v} = (v_x, v_y, v_z)$, in three-dimensional space to modify the four-dimensional spacetime (3+1) to define a six-dimensional spacetime (3+3). This idea was first proposed by Mignani and Recami [8] and used by others [9]-[16]. They added two extra time coordinates in the primed and unprimed 4-dimensional inertial frames $\Sigma' = \Sigma'(x', y', z', t')$ and $\Sigma = \Sigma(x, y, z, t)$ to interpret the imaginary quantities in the superluminal Lorentz transformations. Time is taken as a vector in the Euclidian 3-dimensional space T^3 , so that an event can be represented in Euclidian 6-dimensional space $M^6 = R^3 \times (icT^3)$ as $P \equiv (x, y, z, cit_x, cit_y, cit_z)$. Cartesian components of position vector do not have any physical meaning for tachyons [8], but the magnitude of time vector $t = (t_x^2 + t_y^2 + t_z^2)^{1/2}$ is observable for bradyons [9]. Pappas [13] later on proposed the time vector as $\vec{t} = (t_x, t_y, t_z)$ in Euclidian 3-dimensional time space T^3 so that an event is represented in a 6-dimensional Euclidian spacetime $M^6 = R^3 \times (cT^3)$ as point $P \equiv (x, y, z, ct_x, ct_y, ct_z)$ described by the set of linear equations [13]-[16]

$$x'_i = \gamma_i(x_i - v_i t_i); \quad t'_i = \gamma_i(t_i - \frac{v_i}{c^2} x_i); \quad x_i = \gamma_i(x'_i + v_i t'_i); \quad t_i = \gamma_i(t'_i + \frac{v_i}{c^2} x'_i) \quad (6)$$

where $\gamma_i = 1/(1-v_i^2/c^2)^{1/2}$ is Lorentz factor, which is anisotropic along the $x'_i = x', y', z'$ and $x_i = x, y, z$ axes in the so called six-dimensional massive inertial frames $\bar{\Sigma}' = \bar{\Sigma}'(x', y', z', t'_x, t'_y, t'_z)$ and $\bar{\Sigma} = \bar{\Sigma}(x, y, z, t_x, t_y, t_z)$ rather than inertial frames. A laboratory or an observatory in which a free body is observed to retain its motion is considered as examples of massive inertial frames.

In this work, we extend our recent work on special relativity [17], [18] to study the relativistic invariance of electromagnetic fields. The theory is based on a six-dimensional spacetime in which two massive inertial frames $\bar{\Sigma}' = \bar{\Sigma}'(x', y', z', t'_x, t'_y, t'_z)$ and $\bar{\Sigma} = \bar{\Sigma}(x, y, z, t_x, t_y, t_z)$ initially coincide with an absolutely stationary inertial frame

$\Sigma_0 = \Sigma_0(x_0, y_0, z_0, t_0)$ at time $t' = t = t_0 = 0$. We assume that Einstein's two postulates are also valid in the 6-dimensional spacetime in which we allow time (space) change in all three Cartesian coordinate axes. We assume that the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ move relative to each other with a three-dimensional uniform velocity $\vec{v} = (v_x, v_y, v_z)$. Time is taken as three-dimensional quasi-vector $\vec{t} = (t_x, t_y, t_z)$. Here $v_x = v \cos \phi \sin \theta$, $v_y = v \sin \phi \sin \theta$, $v_z = v \cos \theta$ and $t_x = t \cos \phi \sin \theta$, $t_y = t \sin \phi \sin \theta$, $t_z = t \cos \theta$ in spherical polar coordinates. The magnitude of quasi-time vectors in the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ ($t = |\vec{t}| = (t_x^2 + t_y^2 + t_z^2)^{1/2}$ and $t' = |\vec{t}'| = (t_x'^2 + t_y'^2 + t_z'^2)^{1/2}$) is measurable and Cartesian components (t_x, t_y, t_z) and (t_x', t_y', t_z') are treated just as mathematical tools in the formulation [9]-[16].

We adopt Einstein's four-dimensional spacetime formulation of the special theory of relativity [2]. We consider an event of sending a light signal from point $P_1(x_1, y_1, z_1, t_{x_1}, t_{y_1}, t_{z_1})$ and second event of the light signal arrival at point $P_2(x_2, y_2, z_2, t_{x_2}, t_{y_2}, t_{z_2})$ in the six-dimensional massive inertial frame $\bar{\Sigma}$. The coordinates of the events are related to each other by the following relation

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_{x_2} - t_{x_1})^2 - c^2(t_{y_2} - t_{y_1})^2 - c^2(t_{z_2} - t_{z_1})^2 = 0 \quad (7a)$$

We can write the following relation for the same two events at points $P'_1(x'_1, y'_1, z'_1, t'_{x_1}, t'_{y_1}, t'_{z_1})$ and $P'_2(x'_2, y'_2, z'_2, t'_{x_2}, t'_{y_2}, t'_{z_2})$ taking place in the second six-dimensional massive inertial frame $\bar{\Sigma}'$

$$(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2 - c'^2(t'_{x_2} - t'_{x_1})^2 - c'^2(t'_{y_2} - t'_{y_1})^2 - c'^2(t'_{z_2} - t'_{z_1})^2 = 0 \quad (7b)$$

where $c = c'$ according to Einstein's second postulate. Defining the coordinates of two events as $(x_1, y_1, z_1, t_{x_1}, t_{y_1}, t_{z_1})$ and $(x_2, y_2, z_2, t_{x_2}, t_{y_2}, t_{z_2})$ in $\bar{\Sigma}$ and $(x'_1, y'_1, z'_1, t'_{x_1}, t'_{y_1}, t'_{z_1})$ and $(x'_2, y'_2, z'_2, t'_{x_2}, t'_{y_2}, t'_{z_2})$ in $\bar{\Sigma}'$, we can write the following six-dimensional spacetime intervals

$$\Delta s = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_{x_2} - t_{x_1})^2 - c^2(t_{y_2} - t_{y_1})^2 - c^2(t_{z_2} - t_{z_1})^2 \right]^{1/2} \quad (8a)$$

$$\Delta s' = \left[(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2 - c'^2(t'_{x_2} - t'_{x_1})^2 - c'^2(t'_{y_2} - t'_{y_1})^2 - c'^2(t'_{z_2} - t'_{z_1})^2 \right]^{1/2} \quad (8b)$$

as the intervals between the events taking place in the six-dimensional massive inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$.

Just as in the case of four dimensional spacetime theory of special relativity [2], the intervals describing the motion of the simultaneous events in six-dimensional massive inertial frames can be positive (space like separation), negative (time like separation), or zero (null separation). A pair of events with null separation can be connected by a signal at the speed of light. The intervals of two event infinitely close to each other in the massive frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ are written as

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2(dt_x^2 + dt_y^2 + dt_z^2) \quad (9a)$$

$$ds'^2 = dx'^2 + dy'^2 + dz'^2 - c'^2(dt_x'^2 + dt_y'^2 + dt_z'^2) \quad (9b)$$

where the differential spacetime along the coordinate axes are defined as [17], [18]

$$\begin{aligned} dx' &= dx - v_x dt_x, & dy' &= dy - v_y dt_y, & dz' &= dz - v_z dt_z \\ dt_x' &= dt_x - (v_x / c^2) dx, & dt_y' &= dt_y - (v_y / c^2) dy, & dt_z' &= dt_z - (v_z / c^2) dz \end{aligned} \quad (10a)$$

$$\begin{aligned} dx &= dx' + v_x dt_x', & dy &= dy' + v_y dt_y', & dz &= dz' + v_z dt_z'; \\ dt_x &= dt_x' + (v_x / c^2) dx', & dt_y &= dt_y' + (v_y / c^2) dy', & dt_z &= dt_z' + (v_z / c^2) dz' \end{aligned} \quad (10b)$$

The intervals in Eqs. (9a) and (9b) are related to each other by a metric equation [17], [18]

$$\begin{aligned} dx^2 + dy^2 + dz^2 - c^2(dt_x^2 + dt_y^2 + dt_z^2) &= \gamma_{\mu\nu}^2(dx'^2 + dy'^2 + dz'^2) - \gamma_{\mu\nu}^2 c'^2(dt_x'^2 + dt_y'^2 + dt_z'^2) \\ &= a_{xx} dx'^2 + a_{yy} dy'^2 + a_{zz} dz'^2 - c'^2 \left(a_{t_x t_x} dt_x'^2 + a_{t_y t_y} dt_y'^2 + a_{t_z t_z} dt_z'^2 \right) \end{aligned} \quad (11)$$

where $a_{\mu\nu}$ are speed dependent coefficients and are given by the following coupled equations

$$a_{xx} = \left(\gamma_{xx}^2 - \gamma_{t_x t_x}^2 \beta_x^2 \right), \quad a_{yy} = \left(\gamma_{yy}^2 - \gamma_{t_y t_y}^2 \beta_y^2 \right), \quad a_{zz} = \left(\gamma_{zz}^2 - \gamma_{t_z t_z}^2 \beta_z^2 \right) \tag{12a}$$

$$a_{t_x t_x} = \left(\gamma_{t_x t_x}^2 - \gamma_{xx}^2 \beta_x^2 \right), \quad a_{t_y t_y} = \left(\gamma_{t_y t_y}^2 - \gamma_{yy}^2 \beta_y^2 \right), \quad a_{t_z t_z} = \left(\gamma_{t_z t_z}^2 - \gamma_{zz}^2 \beta_z^2 \right) \tag{12b}$$

Setting $a_{\mu\nu} = 1$ transforms covariant equation (11) into the following invariant form

$$dx^2 + dy^2 + dz^2 - c^2(dt_x^2 + dt_y^2 + dt_z^2) = dx'^2 + dy'^2 + dz'^2 - c^2(dt_x'^2 + dt_y'^2 + dt_z'^2) \tag{13}$$

which states that, just as in the three dimensions, the relativistic quantities (such as velocity and electromagnetic field) should be invariant between two massive inertial frames under Lorentz transformation. Solving the coupled equations (12a) and (12b) for $a_{\mu\nu} = 1$, one then finds

$$\gamma_{xx} = \gamma_{t_x t_x} = 1/\sqrt{1-\beta_x^2}, \quad \gamma_{yy} = \gamma_{t_y t_y} = 1/\sqrt{1-\beta_y^2}, \quad \gamma_{zz} = \gamma_{t_z t_z} = 1/\sqrt{1-\beta_z^2} \tag{14}$$

which are 6-dimensional analogue of classical Lorentz scaling factor. Square of Lorentz scaling factor $\gamma_{\mu\nu}^2$ forms a (6x6) orthogonal boost matrix and metric Eq. (11) can be written as

$$ds^2 = \gamma_{\mu\nu}^2 ds'^2 = \gamma_{\mu\nu}^2 dx'^{\mu} dx'^{\nu} = \begin{pmatrix} \gamma_{xx}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_{yy}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_{zz}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_{t_x t_x}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_{t_y t_y}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_{t_z t_z}^2 \end{pmatrix} dx'^{\mu} dx'^{\nu} \tag{15}$$

We then write the following matrix equation for line elements in the massive inertial frame $\bar{\Sigma}'$

$$\begin{pmatrix} dx' \\ dy' \\ dz' \\ dt'_x \\ dt'_y \\ dt'_z \end{pmatrix} = \begin{pmatrix} \gamma_{xx} & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_{yy} & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_{zz} & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_{t_x t_x} & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_{t_y t_y} & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_{t_z t_z} \end{pmatrix} \begin{pmatrix} dx - v_x dt_x \\ dy - v_y dt_y \\ dz - v_z dt_z \\ dt_x - (v_x/c^2) dx \\ dt_y - (v_y/c^2) dy \\ dt_z - (v_z/c^2) dz \end{pmatrix} \tag{16}$$

In frame $\bar{\Sigma}'$, one replaces v_{x_i} with $-v_{x_i}$ and primed and unprimed subscripts in Eq. (16). One then writes

$$dx' = \gamma_{xx}(dx - v_x dt_x), \quad dy' = \gamma_{yy}(dy - v_y dt_y), \quad dz' = \gamma_{zz}(dz - v_z dt_z) \tag{17a}$$

$$dt'_x = \gamma_{t_x t_x}(dt_x - \beta_x dx/c), \quad dt'_y = \gamma_{t_y t_y}(dt_y - \beta_y dy/c), \quad dt'_z = \gamma_{t_z t_z}(dt_z - \beta_z dz/c)$$

$$dx = \gamma_{xx}(dx' + v_x dt'_x), \quad dy = \gamma_{yy}(dy' + v_y dt'_y), \quad dz = \gamma_{zz}(dz' + v_z dt'_z); \tag{17b}$$

$$dt_x = \gamma_{t_x t_x}(dt'_x + \beta_x dx'/c), \quad dt_y = \gamma_{t_y t_y}(dt'_y + \beta_y dy'/c), \quad dt_z = \gamma_{t_z t_z}(dt'_z + \beta_z dz'/c)$$

Figure 1 shows the variation of Lorentz scaling factor components γ_{xx} and γ_{yy} with azimuthal angle ϕ for polar angle $\theta = \pi/2$ and speed ratio $\beta = v/c$. Figure 1 suggests the replacement of classical Lorentz scaling factor γ in four-dimensions with its analogue $\gamma_{\mu\nu}$ in six-dimensions.

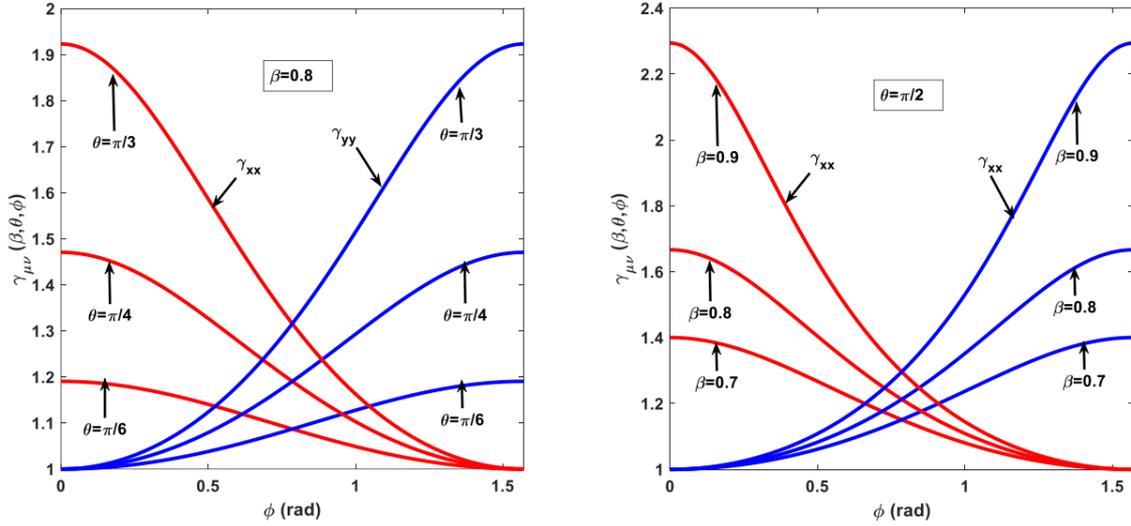


Figure 1. Angle variation of Lorentz scaling factors γ_{xx} (red line) and γ_{yy} (blue line) for (left) $\theta = \pi/6, \pi/4, \pi/3, \pi/2$ and (right) $\beta = v/c$ ratio=0.60, 0.70, 0.80 and 0.90 and $\theta = \pi/2$ in six-dimensional spherical spacetime coordinates as $\bar{\Sigma}'$ frame moves relative to $\bar{\Sigma}$ frame

Using Eqs. (17a) and (17b) with $\gamma_{xx} = \gamma_{t_x t_x}$, $\gamma_{yy} = \gamma_{t_y t_y}$ and $\gamma_{zz} = \gamma_{t_z t_z}$, we can write the following expressions for Cartesian components of the relativistic velocities \bar{u}' and \bar{u} of an event taking place in massive inertial frame $\bar{\Sigma}$ and observed in another massive inertial frame $\bar{\Sigma}'$ [17], [18]

$$u'_x = \frac{dx'}{dt'_x} = (u_x - v_x) \frac{dt_x}{dt'_x} = \frac{(u_x - v_x)}{1 - u_x v_x / c^2}, \quad u_x = \frac{dx}{dt_x} = (u'_x + v_x) \frac{dt'_x}{dt_x} = \frac{(u'_x + v_x)}{1 + u'_x v_x / c^2} \quad (18a)$$

$$u'_y = \frac{dy'}{dt'_y} = (u_y - v_y) \frac{dt_y}{dt'_y} = \frac{u_y - v_y}{1 - u_y v_y / c^2}, \quad u_y = \frac{dy}{dt_y} = (u'_y + v_y) \frac{dt'_y}{dt_y} = \frac{(u'_y + v_y)}{1 + u'_y v_y / c^2} \quad (18b)$$

$$u'_z = \frac{dz'}{dt'_z} = (u_z - v_z) \frac{dt_z}{dt'_z} = \frac{u_z - v_z}{1 - u_z v_z / c^2}, \quad u_z = \frac{dz}{dt_z} = (u'_z + v_z) \frac{dt'_z}{dt_z} = \frac{(u'_z + v_z)}{1 + u'_z v_z / c^2} \quad (18c)$$

When $\bar{\Sigma}'$ moves parallel to x axis of $\bar{\Sigma}$ at the speed of light, Eqs. (18a) - (18c) give $u'_x = -c$ and $u_x = c$ ($u'_y = -c$ and $u_y = c$, $u'_z = -c$ and $u_z = c$), in agreement with 4-dimensional theory [2].

In order to extend the range of the validity of Eqs. (18a), (18b) and (18c) to any relative speed we combine Eqs. (17a) and (17b) and write x, y and z of \bar{u}' and \bar{u} in frames $\bar{\Sigma}'$ and $\bar{\Sigma}$, respectively

$$u'_{x_i} = \frac{dx'_i}{dt'_{x_i}} = \gamma_{x_i x_i} \left(1 - \frac{v_{x_i}^2}{c^2} \right) \frac{dx_i}{dt'_{x_i}} - \gamma_{t_{x_i} t_{x_i}} v_{x_i} = \gamma_{x_i x_i} \left(1 - \frac{v_{x_i}^2}{c^2} \right) (u'_{x_i} + v_{x_i}) - \gamma_{t_{x_i} t_{x_i}} v_{x_i} \quad (19a)$$

$$u_{x_i} = \frac{dx}{dt_{x_i}} = \gamma_{x_i x_i} \left(1 - \frac{v_{x_i}^2}{c^2} \right) \frac{dx'_i}{dt_{x_i}} + \gamma_{t_{x_i} t_{x_i}} v_{x_i} = \gamma_{x_i x_i} \left(1 - \frac{v_{x_i}^2}{c^2} \right) (u_{x_i} - v_{x_i}) + \gamma_{t_{x_i} t_{x_i}} v_{x_i} \quad (19b)$$

which gives the velocity of an event taking place in frame $\bar{\Sigma}$ ($\bar{\Sigma}'$) and observed in frame $\bar{\Sigma}'$ ($\bar{\Sigma}$)

$$\left(1 - \left(1 - \frac{v_{x_i}^2}{c^2} \right) \right) u'_{x_i} = - \left(1 - \left(1 - \frac{v_{x_i}^2}{c^2} \right) \right) v_{x_i} \quad \rightarrow \quad u'_{x_i} = -v_{x_i} \quad (20a)$$

$$\left(1 - \left(1 - \frac{v_{x_i}^2}{c^2} \right) \right) u_{x_i} = \left(1 - \left(1 - \frac{v_{x_i}^2}{c^2} \right) \right) v_{x_i} \quad \rightarrow \quad u_{x_i} = v_{x_i} \quad (20b)$$

The negative sign means that an event is taking place along + direction in the in frame $\bar{\Sigma}$ and observed in -x direction in frame $\bar{\Sigma}'$. Equations (20a) and (20b) can explicitly written as

$$u'_x = -v_x = -v \cos \phi \sin \theta, \quad u'_y = -v_y = -v \sin \phi \sin \theta, \quad u'_z = -v_z = -v \cos \theta \tag{21a}$$

$$u_x = v_x = v \cos \phi \sin \theta, \quad u_y = v_y = v \sin \phi \sin \theta, \quad u_z = v_z = v \cos \theta \tag{21b}$$

Figure 2 shows plot of Eqs. (20c) and (20d) for x and y components of \bar{u}' and \bar{u} at speed of light. As the polar angle increases from $\theta = \pi/6$ to $\pi/2$, the magnitude of components of \bar{u}' and \bar{u} velocity increase and become unity at $\theta = \pi/2$. The overlapping values of the x and y Cartesian components of \bar{u}' and \bar{u} suggest an interaction effect between the events. Equations (20c) and (20d) suggest that u'_x, u'_y, u'_z components of \bar{u}' and u_x, u_y, u_z components of \bar{u} can be determined by using the relative speed of two frames, without requiring one of the unknowns to be known.

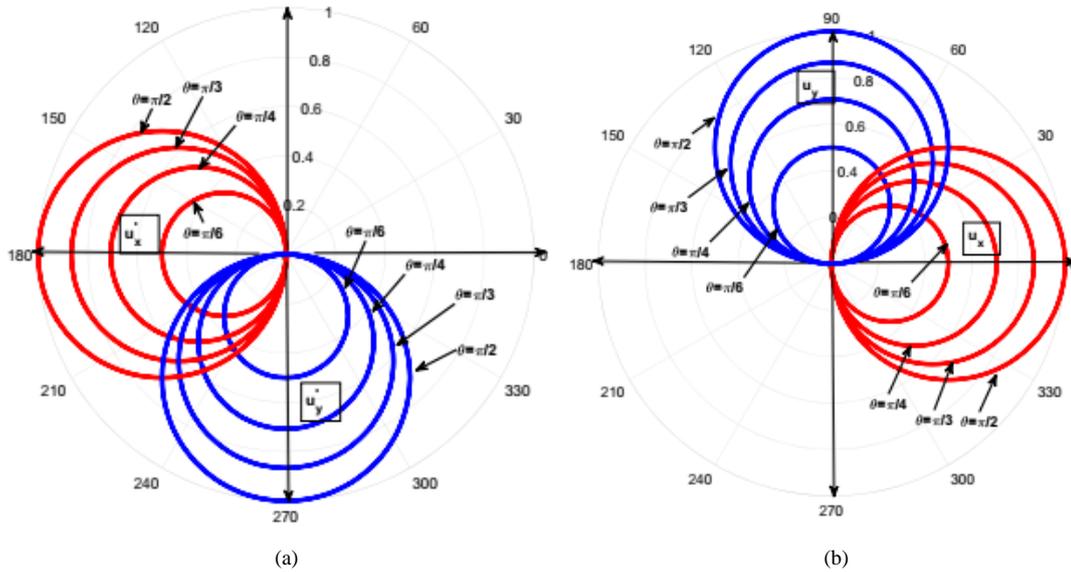


Figure 2. The plot of x- and y- components of \bar{u}' (Fig. 2a) and \bar{u} (Fig. 2b) of an event taking place in massive inertial frame $\bar{\Sigma}$ ($\bar{\Sigma}'$) and observed in massive inertial frame $\bar{\Sigma}'$ ($\bar{\Sigma}$) as a function of azimuthal angle ϕ for polar angle $\theta = \pi/6, \pi/4, \pi/3, \pi/2$

3. Relativistic Vector Transformation

In this section, we discuss the classical vector transformation [19] to lay down the groundwork for the study of the invariance of electromagnetic fields between two massive inertial frames. For now, we momentarily set aside the relativity and focus on the transformation of ordinary vectors (i.e., space position, velocity, and force) in three dimensions. The massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ coincide with a rest (an absolutely stationary) inertial frame Σ_0 , at $t' = t = 0$ and have common z-axis ($z = z'$, in Fig. 3). We define unit vectors $(\hat{i}', \hat{j}', \hat{k}')$ in the massive inertial frame $\bar{\Sigma}'$ in terms of unit vectors $(\hat{i}, \hat{j}, \hat{k})$ in the massive inertial frame $\bar{\Sigma}$ by using the following linear transformation matrix equation

$$\begin{pmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{pmatrix} = \begin{pmatrix} \cos(\hat{i}' \cdot \hat{i}) & \cos(\hat{i}' \cdot \hat{j}) & \cos(\hat{i}' \cdot \hat{k}) \\ \cos(\hat{j}' \cdot \hat{i}) & \cos(\hat{j}' \cdot \hat{j}) & \cos(\hat{j}' \cdot \hat{k}) \\ \cos(\hat{k}' \cdot \hat{i}) & \cos(\hat{k}' \cdot \hat{j}) & \cos(\hat{k}' \cdot \hat{k}) \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} \tag{22a}$$

Setting $\phi = -\phi$ in Eq. (21a) we can relate $(\hat{i}, \hat{j}, \hat{k})$ in frame $\bar{\Sigma}$ to $(\hat{i}', \hat{j}', \hat{k}')$ in frame $\bar{\Sigma}'$

$$\begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} \cos(\hat{i}' \cdot \hat{i}) & \cos(\hat{i}' \cdot \hat{j}) & \cos(\hat{i}' \cdot \hat{k}) \\ \cos(\hat{j}' \cdot \hat{i}) & \cos(\hat{j}' \cdot \hat{j}) & \cos(\hat{j}' \cdot \hat{k}) \\ \cos(\hat{k}' \cdot \hat{i}) & \cos(\hat{k}' \cdot \hat{j}) & \cos(\hat{k}' \cdot \hat{k}) \end{pmatrix} \begin{pmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{pmatrix} \tag{22b}$$

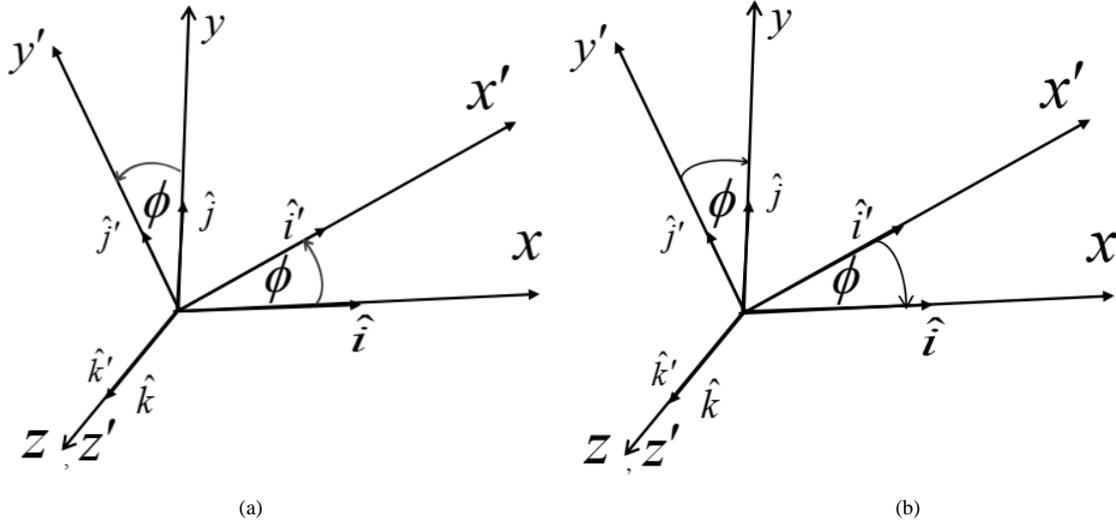


Figure 3. The schematic diagram of unit vectors in a rotation through azimuthal angle ϕ in counterclockwise of (x, y) plane into (x', y') plane (a) and in clockwise of (x', y') into (x, y) plane for $0 \leq \phi \leq 2\pi$ and $\theta = \pi/2$. In both cases z or z' axes are kept the same

We now consider the transformation of Cartesian components of position, velocity and force vectors. The translation of a position vector does not affect its Cartesian components [19], which transform under rotation according to Eqs. (22a) and (22b). We define three-dimensional stationary position vectors $\vec{r}' = (x', y', z')$ and $\vec{r} = (x, y, z)$ in the inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ relative to unit vectors $(\hat{i}', \hat{j}', \hat{k}')$ and $(\hat{i}, \hat{j}, \hat{k})$, as shown in Fig. 4, by the following equations

$$\vec{r}' = x'\hat{i}' + y'\hat{j}' + z'\hat{k}' = (x \cos \phi + y \sin \phi)\hat{i} + (-x \sin \phi + y \cos \phi)\hat{j} + z\hat{k} \quad (23a)$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = (x' \cos \phi - y' \sin \phi)\hat{i}' + (x' \sin \phi + y' \cos \phi)\hat{j}' + z'\hat{k}' \quad (23b)$$

which can be written in the form of transformation matrix equations for Cartesian components

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (24a)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad (24b)$$

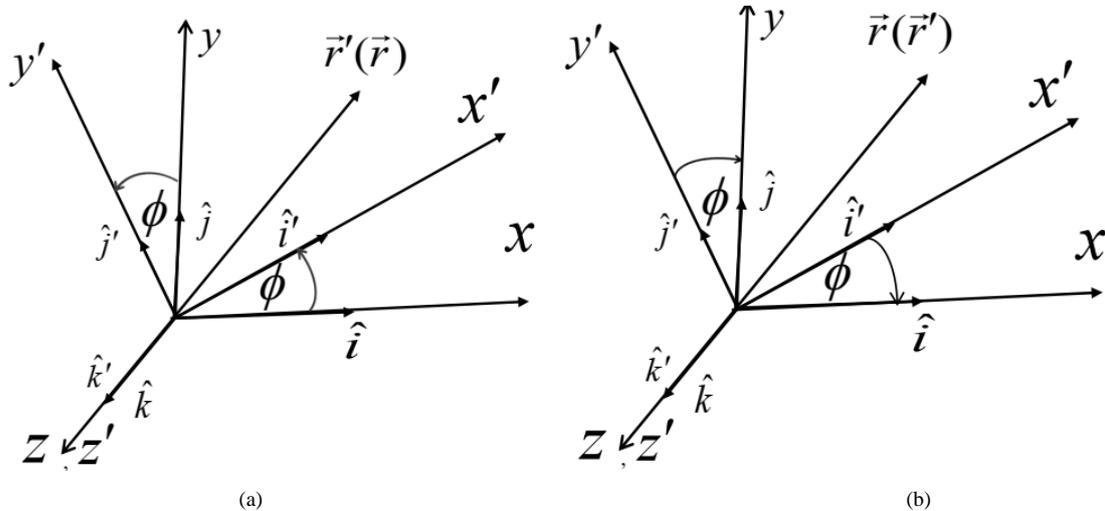


Figure 4. The schematic diagram of \vec{r}' and \vec{r} in terms of unit vectors in a rotation through angle ϕ in counterclockwise of (x, y) plane into (x', y') plane (a) and in clockwise of (x', y') into (x, y) plane for $0 \leq \phi \leq 2\pi$ and $\theta = \pi/2$. In both cases z or z' axes are kept the same

The scalar product of \vec{r}' with itself in $\bar{\Sigma}'$ (or \vec{r} in $\bar{\Sigma}$) frame leads to $\vec{r}' \cdot \vec{r}' = \vec{r} \cdot \vec{r}$, which states that the magnitude of the position vectors \vec{r}' and \vec{r} Lorentz scalar and have the same length $|\vec{r}'| = |\vec{r}|$ from the origin. The spacetime metric equation is also invariant relative to the rotation of coordinates about third axis ($ds'^2 = ds^2$) for any coordinate rotations of reference system [19].

Likewise, one can write the following transformation matrix equations for Cartesian components of relativistic velocity vectors \vec{u}' and \vec{u} in the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$, respectively

$$\begin{pmatrix} u'_x \\ u'_y \\ u'_z \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \tag{25a}$$

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u'_x \\ u'_y \\ u'_z \end{pmatrix} \tag{25b}$$

from which the relativistic velocity vectors in the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ are written as

$$\vec{u}' = u'_x \hat{i}' + u'_y \hat{j}' + u'_z \hat{k}' = (u_x \cos \phi + u_y \sin \phi) \hat{i} + (-u_x \sin \phi + u_y \cos \phi) \hat{j} + u_z \hat{k} \tag{26a}$$

$$\vec{u} = u_x \hat{i} + u_y \hat{j} + u_z \hat{k} = (u'_x \cos \phi - u'_y \sin \phi) \hat{i}' + (u'_x \sin \phi + u'_y \cos \phi) \hat{j}' + u'_z \hat{k}' \tag{26b}$$

Figure 5 compares the Cartesian components of the transformed relativistic velocity of an event taking place in frame $\bar{\Sigma}$ ($\bar{\Sigma}'$) and observed in frame $\bar{\Sigma}'$ ($\bar{\Sigma}$). As the light source at the origin of a frame is flashed on and off rapidly, observers in both frames see a spherical shell of radiation which expands outward from the origin in all directions [2].

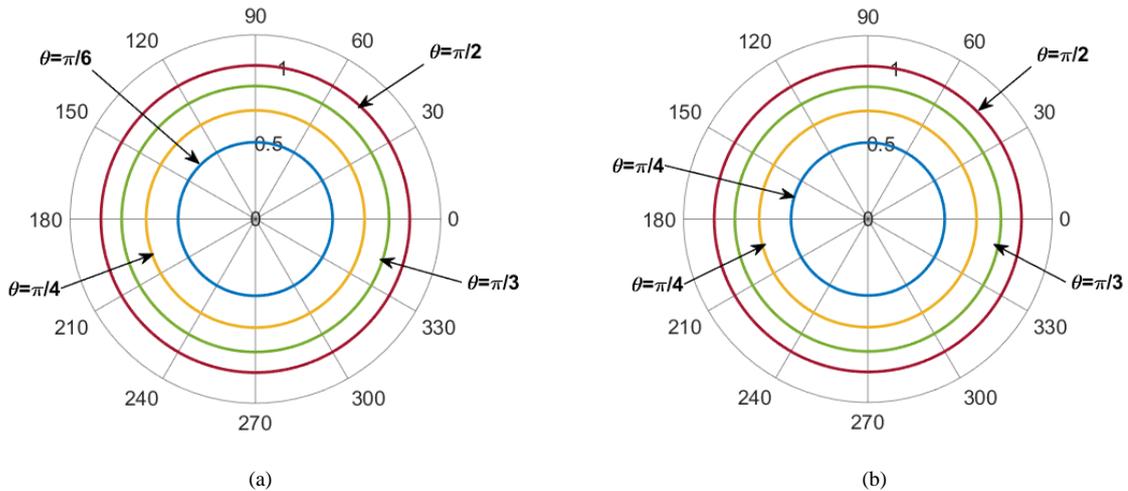


Figure 5. Polar plot of u'_x and u'_y components of \vec{u}' (a) and u_x , u_y components of \vec{u} (b) plotted as a function of azimuthal angle ϕ for $\theta = \pi / 6, \pi / 4, \pi / 3, \pi / 2$, respectively [17], [18]

The upper limit for the relative speed in both frames is equal to speed of light in vacuum

$$\frac{ds'^2}{dt'^2} = u_x'^2 + u_y'^2 + u_z'^2 - c'^2 = |\vec{u}'|^2 - c'^2 = 0, \quad \Rightarrow \quad |\vec{u}'| = c' \tag{27}$$

$$\frac{ds^2}{dt^2} = u_x^2 + u_y^2 + u_z^2 - c^2 = |\vec{u}|^2 - c^2 = 0, \quad \Rightarrow \quad |\vec{u}| = c$$

which proves that the speed of light is Lorentz scalar ($c' = c$) between massive inertial frames.

Furthermore, similar to defining stationary position vectors \vec{r}' and \vec{r} in Fig. 4, we define stationary net force vectors $\vec{F}' = (F'_x, F'_y, F'_z)$ and $\vec{F} = (F_x, F_y, F_z)$ relative to unit vectors $(\hat{i}', \hat{j}', \hat{k}')$ and $(\hat{i}, \hat{j}, \hat{k})$ in the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$, respectively. We then write the following matrix equations

$$\begin{pmatrix} F'_x \\ F'_y \\ F'_z \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \quad (28a)$$

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F'_x \\ F'_y \\ F'_z \end{pmatrix} \quad (28b)$$

from which, similar to writing Eqs. (21a) and (21b), we can write the net force vectors as

$$\vec{F}' = F'_x \hat{i}' + F'_y \hat{j}' + F'_z \hat{k}' = (F_x \cos \phi + F_y \sin \phi) \hat{i} + (-F_x \sin \phi + F_y \cos \phi) \hat{j} + F_z \hat{k} \quad (29a)$$

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} = (F'_x \cos \phi - F'_y \sin \phi) \hat{i}' + (F'_x \sin \phi + F'_y \cos \phi) \hat{j}' + F'_z \hat{k}' \quad (29b)$$

The scalar product of \vec{F}' with itself in the massive inertial frame $\bar{\Sigma}'$ leads to the scalar product of \vec{F} with itself in the massive inertial frame $\bar{\Sigma}$: $\vec{F}' \cdot \vec{F}' = \vec{F} \cdot \vec{F}$, which states that net forces have the same length ($|\vec{F}'| = |\vec{F}|$) from the origin, so vector transformation is identified as a rotation if it causes no change in vector's magnitude [19].

4. Relativistic Invariance of Electromagnetic Fields

Considering the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ form a closed system in the six-dimensional spacetime, the rate at which work is done by the net force on a particle are written as [17], [18]

$$\frac{dE}{dt_{x_i}} = \vec{F}_{x_i} \cdot \vec{u}_{x_i} = F_x u_x + F_y u_y + F_z u_z \quad (30a)$$

$$\frac{dE'}{dt'_{x_i}} = \vec{F}'_{x_i} \cdot \vec{u}'_{x_i} = F'_x u'_x + F'_y u'_y + F'_z u'_z \quad (30b)$$

where \vec{F}'_{x_i} and \vec{F}_{x_i} are the Cartesian components of Lorentz force massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$

$$F'_x = q(E'_x + u'_y B'_z - u'_z B'_y), \quad F'_y = q(E'_y + u'_z B'_x - u'_x B'_z), \quad F'_z = q(E'_z + u'_x B'_y - u'_y B'_x) \quad (31a)$$

$$F_x = q(E_x + u_y B_z - u_z B_y), \quad F_y = q(E_y + u_z B_x - u_x B_z), \quad F_z = q(E_z + u_x B_y - u_y B_x) \quad (31b)$$

where u'_j (u'_k) and u_j (u_k) are x, y and z components of \vec{u}' and \vec{u} , given by Eqs. (25a) and (25b).

Since the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ form a closed system in six-dimensional spacetime, we can write the following relation for Lorentz invariance of the conservation of relativistic power law (rate of relativistic energy change) between the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ [17], [18]

$$\frac{dE'}{dt'_{x_i}} = \frac{dE}{dt_{x_i}} \Rightarrow F'_x u'_x + F'_y u'_y + F'_z u'_z = F_x u_x + F_y u_y + F_z u_z \quad (32)$$

Combining the velocity transformation matrix equations (25b) and (25a) with the conservation of power law in equation (32) we can write the following explicit algebraic equations

$$\begin{aligned} F'_x u'_x + F'_y u'_y + F'_z u'_z &= F_x u_x + F_y u_y + F_z u_z \\ &= (F_x \cos \phi + F_y \sin \phi) u'_x + (-F_x \sin \phi + F_y \cos \phi) u'_y + F'_z u'_z \end{aligned} \quad (33a)$$

$$\begin{aligned} F_x u_x + F_y u_y + F_z u_z &= F'_x u'_x + F'_y u'_y + F'_z u'_z \\ &= (F'_x \cos \phi - F'_y \sin \phi) u_x + (F'_x \sin \phi + F'_y \cos \phi) u_y + F'_z u_z \end{aligned} \quad (33b)$$

Component by component matching both sides of Eq. (33a) and then (33b), respectively, one obtains the transformation matrix equations (28a) and (28b) for the net electromagnetic forces.

Substituting Eqs. (31a) and (31b) into Eq. (33a) we write the following set of equations

$$(E'_x + u'_y B'_z - u'_z B'_y) = (E_x + u_y B_z - u_z B_y) \cos \phi + (E_y + u_z B_x - u_x B_z) \sin \phi \quad (34a)$$

$$(E'_y + u'_z B'_x - u'_x B'_z) = -(E_x + u_y B_z - u_z B_y) \sin \phi + (E_y + u_z B_x - u_x B_z) \cos \phi \quad (34b)$$

$$E'_z + u'_x B'_y - u'_y B'_x = E_z + u_x B_y - u_y B_x \quad (34c)$$

Using Eq. (25a) for u'_x , u'_y , and u'_z in Eqs. (34a) - (34c), we write the transformation equations

$$\begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \quad (35a)$$

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \quad (35b)$$

which allows us to write the following expressions for electric and magnetic fields in frame $\bar{\Sigma}'$

$$\vec{E}' = E'_x \hat{i}' + E'_y \hat{j}' + E'_z \hat{k}' = (E_x \cos \phi + E_y \sin \phi) \hat{i} + (-E_x \sin \phi + E_y \cos \phi) \hat{j} + E_z \hat{k} \quad (36a)$$

$$\vec{B}' = B'_x \hat{i}' + B'_y \hat{j}' + B'_z \hat{k}' = (B_x \cos \phi + B_y \sin \phi) \hat{i} + (-B_x \sin \phi + B_y \cos \phi) \hat{j} + B_z \hat{k} \quad (36b)$$

Likewise, using Eq. (25b) for u_x , u_y , and u_z in Eq. (33b) and following the steps in writing Eqs. (35a) and (35b), we then write the following transformation equations

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} \quad (37a)$$

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} \quad (37b)$$

from which the electric and magnetic field vectors in the inertial frame $\bar{\Sigma}$ are written as

$$\vec{E} = E_x \hat{i} + E_y \hat{j} + E_z \hat{k} = (E'_x \cos \phi - E'_y \sin \phi) \hat{i}' + (E'_x \sin \phi + E'_y \cos \phi) \hat{j}' + E'_z \hat{k}' \quad (38b)$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k} = (B'_x \cos \phi + B'_y \sin \phi) \hat{i}' + (-B'_x \sin \phi + B'_y \cos \phi) \hat{j}' + B'_z \hat{k}' \quad (38b)$$

Equations (35)-(38) are in full agreement with the first postulate of special relativity: Just as with three dimensional quantities, the relativistic vector quantities such as electric and magnetic fields should not change between two massive inertial frames under Lorentz transformation.

As verification, let us consider a static point charge Q that is at rest at the origin of the massive inertial frame $\bar{\Sigma}$ which is assumed to coincide with the absolutely stationary inertial frame Σ_0 and second massive frame $\bar{\Sigma}'$ moves relative to first massive frame with uniform velocity $\vec{u} = (u_x, u_y, u_z)$. The produced electric fields in the massive frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ are written as

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} = \frac{Q}{4\pi\epsilon_0 r'^{3/2}} (x\hat{i} + y\hat{j} + z\hat{k}) = E_x \hat{i} + E_y \hat{j} + E_z \hat{k} \quad (39a)$$

$$\vec{E}' = \frac{Q}{4\pi\epsilon_0} \frac{\hat{r}'}{r'^2} = \frac{Q}{4\pi\epsilon_0 r'^{3/2}} [x'\hat{i}' + y'\hat{j}' + z'\hat{k}'] = E'_x \hat{i}' + E'_y \hat{j}' + E'_z \hat{k}' \quad (39b)$$

where $r'^2 = r^2$. Using the unit vectors $(\hat{i}', \hat{j}', \hat{k}')$ described in Eq. (22a) we can re-write Eq. (39b) as

$$\vec{E}' = \frac{Q}{4\pi\epsilon_0 r'^2} \hat{r}' = \frac{Q}{4\pi\epsilon_0 r'^{3/2}} [x\hat{i} + y\hat{j} + z\hat{k}] = E_x\hat{i} + E_y\hat{j} + E_z\hat{k} = \vec{E} \quad (39c)$$

which states a pure electric field in one frame appears as a pure electric field in another frame.

Since observer in frame $\bar{\Sigma}'$ sees the charge Q in frame $\bar{\Sigma}$ as moving with uniform velocity $\vec{u} = (u_x, u_y, u_z)$, the magnetic field produced by charge Q in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ can be written as

$$\vec{B} = \frac{\mu_0 Q (\vec{u} \times \hat{r})}{4\pi r^2} = \frac{\mu_0 Q}{4\pi r^{3/2}} (u_x\hat{i} + u_y\hat{j} + u_z\hat{k}) \times (x\hat{i} + y\hat{j} + z\hat{k}) = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} \quad (40a)$$

$$\vec{B}' = \frac{\mu_0 Q (\vec{u}' \times \hat{r}')}{4\pi r'^2} = \frac{\mu_0 Q}{4\pi r'^{3/2}} [(u'_y z' - u'_z y')\hat{i}' + (-u'_x z' + u'_z x')\hat{j}' + (u'_x y' - u'_y x')\hat{k}'] \quad (40b)$$

Substituting unit vectors $(\hat{i}', \hat{j}', \hat{k}')$ from Eq. (22a) into Eq. (40b), one can write

$$\begin{aligned} \vec{B}' &= \frac{\mu_0 Q}{4\pi r'^{3/2}} [(u'_y z' - u'_z y')(\cos\phi\hat{i} + \sin\phi\hat{j}) + (-u'_x z' + u'_z x')(-\sin\phi\hat{i} + \cos\phi\hat{j}) + (u'_x y' - u'_y x')\hat{k}] \\ &= (B'_x \cos\phi - B'_y \sin\phi)\hat{i} + (B'_x \sin\phi + B'_y \cos\phi)\hat{j} + B'_z\hat{k} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} = \vec{B} \end{aligned} \quad (40c)$$

which states a pure magnetic field in one frame appears as a pure magnetic field in another one.

One can easily show that scalar and vector products of electric and magnetic fields are Lorentz invariant between the massive inertial frames. The scalar products of \vec{E}' and \vec{B}' in the massive inertial frame $\bar{\Sigma}'$ (\vec{E} and \vec{B} in the inertial frame $\bar{\Sigma}$) are written as

$$\begin{aligned} \vec{E}' \cdot \vec{B}' &= E'_x B'_x + E'_y B'_y + E'_z B'_z = (E_x \cos\phi + E_y \sin\phi)(B_x \cos\phi + B_y \sin\phi) + \\ &+ (-E_x \sin\phi + E_y \cos\phi)(-B_x \sin\phi + B_y \cos\phi) + (E_z B_z) = E_x B_x + E_y B_y + E_z B_z = \vec{E} \cdot \vec{B} \end{aligned} \quad (41a)$$

$$\begin{aligned} \vec{E} \cdot \vec{B} &= E_x B_x + E_y B_y + E_z B_z = (E'_x \cos\phi - E'_y \sin\phi)(B'_x \cos\phi - B'_y \sin\phi) + \\ &+ (E'_x \sin\phi + E'_y \cos\phi)(B'_x \sin\phi + B'_y \cos\phi) + (E'_z B'_z) = E'_x B'_x + E'_y B'_y + E'_z B'_z = \vec{E}' \cdot \vec{B}' \end{aligned} \quad (41b)$$

which are Lorentz scalars between two inertial frames. Likewise, the scalar product of electric field \vec{E}' (\vec{E}) with itself and magnetic field \vec{B}' (\vec{B}) with itself in the massive inertial frame $\bar{\Sigma}'$ ($\bar{\Sigma}$) yields

$$\begin{aligned} \vec{E}' \cdot \vec{E}' &= E'^2_x + E'^2_y + E'^2_z = (E_x \cos\phi + E_y \sin\phi)^2 + (-E_x \sin\phi + E_y \cos\phi)^2 + (E_z)^2 \\ &= E^2_x + E^2_y + E^2_z = (E'_x \cos\phi - E'_y \sin\phi)^2 + (E'_x \sin\phi + E'_y \cos\phi)^2 + (E'_z)^2 = \vec{E} \cdot \vec{E} \end{aligned} \quad (42a)$$

$$\begin{aligned} \vec{B}' \cdot \vec{B}' &= B'^2_x + B'^2_y + B'^2_z = (B_x \cos\phi + B_y \sin\phi)^2 + (-B_x \sin\phi + B_y \cos\phi)^2 + (B_z)^2 \\ &= B^2_x + B^2_y + B^2_z = (B'_x \cos\phi - B'_y \sin\phi)^2 + (B'_x \sin\phi + B'_y \cos\phi)^2 + (B'_z)^2 = \vec{B} \cdot \vec{B} \end{aligned} \quad (42b)$$

which state that $E' = E$ and $B' = B$ (Lorentz scalar invariants). We can then write

$$\frac{1}{2c^2} (\vec{E}^2 + c^2 \vec{B}^2) = \frac{1}{2c^2} (\sum E_{x_i}^2 + c^2 \sum B_{x_i}^2) = \frac{1}{2c^2} (\sum E'^2_{x_i} + c'^2 \sum B'^2_{x_i}) = \frac{1}{2c'^2} (\vec{E}'^2 + c'^2 \vec{B}'^2) \quad (43)$$

where $x_i = x, y, z$. Equation (43) is important for determining electromagnetic energy density.

The vector product of electric and magnetic fields can be determined by using Eqs. (36a) and (36b) for \vec{E}' and \vec{B}' and Eq. (38a) and (38b) for \vec{E} and \vec{B} , which yield the following equations

$$\begin{aligned} \vec{E}' \times \vec{B}' &= (E'_y B'_z - B'_y E'_z)\hat{i}' + (B'_x E'_z - E'_x B'_z)\hat{j}' + (E'_x B'_y - B'_x E'_y)\hat{k}' \\ &= ((E_y B_z - E_z B_y)\cos\phi + (E_z B_x - E_x B_z)\sin\phi)\hat{i}' \\ &+ ((E_z B_x - E_x B_z)\cos\phi + (E_z B_y - E_y B_z)\sin\phi)\hat{j}' + (E_x B_y - E_y B_x)\hat{k}' = \vec{E} \times \vec{B} \end{aligned} \quad (44a)$$

$$\begin{aligned} \vec{E} \times \vec{B} &= (E_y B_z - E_z B_y) \hat{i} + (E_z B_x - E_x B_z) \hat{j} + (E_x B_y - E_y B_x) \hat{k} \\ &= \left((E'_y B'_z - E'_z B'_y) \cos \phi + (E'_x B'_z - E'_z B'_x) \sin \phi \right) \hat{i} \\ &\quad + \left((E'_z B'_x - E'_x B'_z) \cos \phi - (E'_y B'_y - E'_y B'_z) \sin \phi \right) \hat{j} + (E_x B_y - E_y B_x) \hat{k} = \vec{E}' \times \vec{B}' \end{aligned} \tag{44b}$$

which states that the vector product of electric and magnetic fields is Lorentz invariant between two massive inertial frames Σ' and Σ ($\vec{E}' \times \vec{B}' = \vec{E} \times \vec{B}$), which is essential in proving the Lorentz invariance of Poynting vector and conservation laws for charge (current) continuity equation and electromagnetic energy and momentum. This eliminates the non-invariance of electric and magnetic fields ($\vec{E}' \times \vec{B}' \neq \vec{E} \times \vec{B}$) according to classical four-dimensional spacetime theory [2].

5. Relativistic Invariance of Maxwell's Equations

Maxwell's equation of electrodynamics must take the same form (invariant) in every inertial frame. And it is quite tedious to demonstrate this invariance explicitly by using the transformation rules in the classical four-dimensional theory of special relativity [2]. This motivated us to demonstrate the Lorentz invariance of Maxwell's equations between massive inertial frames by using the vector transformation rules described in the previous section. In doing so, we begin with the differential forms of Maxwell's equations in the six-dimensional massive inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$, written as

$$\bar{\nabla}_i \cdot \vec{E} = \rho / \epsilon_0, \quad \bar{\nabla}_i \cdot \vec{B} = 0, \quad \bar{\nabla}_i \times \vec{E} = -\frac{\partial \vec{B}}{\partial t_{x_i}}, \quad \bar{\nabla}_i \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t_{x_i}} \tag{45a}$$

$$\bar{\nabla}'_i \cdot \vec{E}' = \rho' / \epsilon'_0, \quad \bar{\nabla}'_i \cdot \vec{B}' = 0, \quad \bar{\nabla}'_i \times \vec{E}' = -\frac{\partial \vec{B}'}{\partial t'_{x_i}}, \quad \bar{\nabla}'_i \times \vec{B}' = \mu'_0 \vec{J}' + \frac{1}{c'^2} \frac{\partial \vec{E}'}{\partial t'_{x_i}} \tag{45b}$$

Recall that Gauss' theorem relates the flux of a vector field $\vec{V}(\vec{r})$ through a closed surface to the volume integral of its divergence inside the surface [20] in massive inertial frames are written as

$$\Phi = \oint_S \vec{V} \cdot d\vec{A} = \oint_V \bar{\nabla} \cdot \vec{V} dv = \frac{1}{\epsilon_0} \oint_V \rho dv, \quad \Phi' = \oint_{S'} \vec{V}' \cdot d\vec{A}' = \oint_{V'} \bar{\nabla}' \cdot \vec{V}' dv' = \frac{1}{\epsilon'_0} \oint_{V'} \rho' dv' \tag{46}$$

Since any scalar function, such as electric and magnetic fields, is continuous at any point in space in both frames ($\varphi = \varphi'$) [21], we write the following chain rules for differential operators

$$\frac{\partial \varphi}{\partial x_i} = \frac{\partial \varphi'}{\partial x_i} = \frac{\partial x'_i}{\partial x_i} \frac{\partial \varphi'}{\partial x'_i} + \frac{\partial t'_{x_i}}{\partial x_i} \frac{\partial \varphi'}{\partial t'_{x_i}} = \frac{\partial x'_i}{\partial x_i} \frac{\partial \varphi'}{\partial x'_i} + \frac{\partial t'_{x_i}}{\partial x_i} \frac{\partial x_i}{\partial t'_{x_i}} \frac{\partial x'_i}{\partial x_i} \frac{\partial \varphi'}{\partial x'_i} = a_{x_i x_i} \frac{\partial \varphi'}{\partial x'_i} \tag{47a}$$

$$\frac{\partial \varphi}{\partial t_{x_i}} = \frac{\partial \varphi'}{\partial t_{x_i}} = \frac{\partial t'_{x_i}}{\partial t_{x_i}} \frac{\partial \varphi'}{\partial t'_{x_i}} + \frac{\partial x'_i}{\partial t_{x_i}} \frac{\partial \varphi'}{\partial x'_i} = \frac{\partial \varphi'}{\partial t'_{x_i}} \frac{\partial t'_{x_i}}{\partial t_{x_i}} + \frac{\partial \varphi'}{\partial x'_i} \frac{\partial t'_{x_i}}{\partial x_i} \frac{\partial x_i}{\partial t'_{x_i}} \frac{\partial t'_{x_i}}{\partial t_{x_i}} = a_{t_{x_i} t_{x_i}} \frac{\partial \varphi'}{\partial t'_{x_i}} \tag{47b}$$

$$\frac{\partial^2 \varphi}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left(\frac{\partial \varphi'}{\partial x_i} \right) \frac{\partial x'_i}{\partial x_i} + \frac{\partial}{\partial t'_{x_i}} \left(\frac{\partial \varphi'}{\partial x_i} \right) \frac{\partial t'_{x_i}}{\partial x_i}, \quad \frac{\partial^2 \varphi'}{\partial x_i^2} = \frac{\partial}{\partial x'_i} \left(\frac{\partial \varphi'}{\partial x'_i} \right) \frac{\partial x'_i}{\partial x_i} + \frac{\partial}{\partial t'_{x_i}} \left(\frac{\partial \varphi'}{\partial x'_i} \right) \frac{\partial t'_{x_i}}{\partial x_i} \tag{47c}$$

where $a_{x_i x_i} = \gamma^2_{x_i x_i} - \gamma^2_{t_{x_i} t_{x_i}} \beta^2_{x_i}$ and $a_{t_{x_i} t_{x_i}} = \gamma^2_{t_{x_i} t_{x_i}} - \gamma^2_{x_i x_i} \beta^2_{x_i}$ are given in Eqs. (12a) and (12b) with $\gamma^2_{x_i x_i} = \gamma^2_{t_{x_i} t_{x_i}}$ given by Eq. (14). In the following we will use Stokes theorem and chain rules to prove the Lorentz invariance of Maxwell's equations between two massive inertial frames.

5.1. Gauss Law of Electrostatics

Since the electric field wave function is continuous at any point in space in both frames, taking $\varphi = E$ and $\varphi' = E'$ and applying the chain rule in Eq. (47a) we write

$$\rho = \epsilon_0 \bar{\nabla} \cdot \vec{E} = \epsilon_0 \gamma^2_{xx} (1 - \beta_x^2) \frac{\partial E'}{\partial x'} + \epsilon_0 \gamma^2_{yy} (1 - \beta_y^2) \frac{\partial E'}{\partial y'} + \epsilon_0 \gamma^2_{zz} (1 - \beta_z^2) \frac{\partial E'}{\partial z'} = \epsilon_0 \gamma^2_{x_i x_i} (1 - \beta^2_{x_i}) \bar{\nabla}'_i \cdot \vec{E}' \tag{48}$$

where $\gamma_{x_i x_i}^2 (1 - \beta_{x_i}^2) = 1$ and $\vec{\nabla}' \cdot \vec{E}' = \rho' / \varepsilon'_0$. Matching both sides of Eq. (48) yields $\varepsilon'_0 \rho = \varepsilon_0 \rho'$. Since $\varepsilon'_0 = \varepsilon_0$, we have $\rho' = \rho$. Consequently, covariant Eq. (44) is transformed into invariant form

$$\vec{\nabla} \cdot \vec{E} - \frac{\rho}{\varepsilon_0} = \vec{\nabla}' \cdot \vec{E}' - \frac{\rho'}{\varepsilon'_0} \quad (49)$$

which is the Lorentz invariant Gauss law of electrostatics between the six-dimensional massive inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ (or between four-dimensional inertial frames Σ and Σ'). Here we should keep in mind that $\rho = \rho_0 / \sqrt{1 - \beta^2}$ and $\rho' = \rho_0 / \sqrt{1 - \beta'^2}$ defined with respect to the rest charge density ρ_0 in the absolutely steady inertial frame Σ_0 , in static equilibrium.

5.2. Gauss law of Magnetism

Since magnetic field wave function is continuous at any point in space [21], taking $\varphi = B$ and $\varphi' = B'$ and using the chain rule in Eq. (47a) we write

$$\vec{\nabla} \cdot \vec{B} = \gamma_{xx}^2 (1 - \beta_x^2) \frac{\partial B'}{\partial x'} + \gamma_{yy}^2 (1 - \beta_y^2) \frac{\partial B'}{\partial y'} + \gamma_{zz}^2 (1 - \beta_z^2) \frac{\partial B'}{\partial z'} = \gamma_{x_i x_i}^2 (1 - \beta_{x_i}^2) \vec{\nabla}' \cdot \vec{B}' \quad (50)$$

Matching both sides of Eq. (47), one finds $\gamma_{x_i x_i} = 1 / (1 - \beta_{x_i}^2)^{1/2}$, which transforms Eq. (46) to

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla}' \cdot \vec{B}' \quad (51)$$

which is Lorentz invariant Gauss law of magnetism between the six-dimensional massive inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ (or between four-dimensional inertial frames Σ and Σ').

5.3. Faraday's Law of Induction

Let us write the differential form of Faraday's law of induction in Eq. (45a) in x directions of system of Cartesian polar coordinates

$$(\vec{\nabla} \times \vec{E})_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t_x}, \quad (\vec{\nabla} \times \vec{E})_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t_y}, \quad (\vec{\nabla} \times \vec{E})_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t_z} \quad (52)$$

Applying the chain rule for differential operators in Eq. (47a) and (47b) to x, y, and z components in Eq. (52) for the differential form of Faraday's law of induction we write

$$\frac{\partial E_{x_i}}{\partial x_i} = \frac{\partial E'_{x_i}}{\partial x_i} = \frac{\partial E'_{x_i}}{\partial x'_i} \frac{\partial x'_i}{\partial x_i} + \frac{\partial E'_{x_i}}{\partial t'_{x_i}} \frac{\partial t'_{x_i}}{\partial x_i} = \frac{\partial E'_{x_i}}{\partial x'_i} - \frac{v_{x_i}}{c'^2} \frac{\partial E'_{x_i}}{\partial x'_i} \frac{\partial x'_i}{\partial x_i} \frac{\partial x_i}{\partial t'_{x_i}} = (1 - \beta_{x_i}^2) \frac{\partial E'_{x_i}}{\partial x'_i} \quad (53a)$$

$$\frac{\partial B_{x_i}}{\partial t_{x_i}} = \frac{\partial B'_{x_i}}{\partial t_{x_i}} = \frac{\partial B'_{x_i}}{\partial t'_{x_i}} \frac{\partial t'_{x_i}}{\partial t_{x_i}} + \frac{\partial B'_{x_i}}{\partial x'_i} \frac{\partial x'_i}{\partial t_{x_i}} = \frac{\partial B'_{x_i}}{\partial t'_{x_i}} - \frac{\partial x'_i}{\partial t_{x_i}} \frac{\partial t'_{x_i}}{\partial t_{x_i}} \left(\frac{v_{x_i}}{c'^2} \frac{\partial t'_{x_i}}{\partial x'_i} \right) \frac{\partial B'_{x_i}}{\partial t'_{x_i}} = (1 - \beta_{x_i}^2) \frac{\partial B'_{x_i}}{\partial t'_{x_i}} \quad (53b)$$

where $x_i = x, y, z$. Combining Eqs. (49a) and (49b) side by side, we write a covariant equation

$$(\vec{\nabla} \times \vec{E})_{x_i} + \frac{\partial \vec{B}_{x_i}}{\partial t_{x_i}} = \gamma_{x_i x_i}^2 (1 - \beta_{x_i}^2) (\vec{\nabla}' \times \vec{E}')_{x_i} + \gamma_{t_{x_i} t_{x_i}}^2 (1 - \beta_{x_i}^2) \frac{\partial \vec{B}'_{x_i}}{\partial t'_{x_i}} \quad (54)$$

Matching both sides of Eq. (54), one finds $\gamma_{x_i x_i} = \gamma_{t_{x_i} t_{x_i}} = 1 / (1 - v_{x_i}^2 / c'^2)^{1/2}$ in Eq. (14) for Lorentz factor. When $\bar{\Sigma}$ and $\bar{\Sigma}'$ move along +x axis, 6-dimensional spacetime (3+3) becomes 4-dimensional (3+1) with $\gamma_{xx} = \gamma_{tt} = \gamma = 1 / (1 - v^2 / c'^2)^{1/2}$ and Eq. (54) reduces to four-dimensional invariant form

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E}' + \frac{\partial \vec{B}'}{\partial t'} \quad (55)$$

which is Lorentz invariant Faraday's law of induction between four-dimensional inertial frames Σ and Σ' frames in vacuum.

5.4. Ampere-Maxwell Law

We write the following forms of Ampere-Maxwell’s law (extended form of Maxwell law of induction in Eq. (45a) to include the current) in x, y and z-directions

$$(\vec{\nabla} \times \vec{B})_x = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x + \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t_x} = \mu_0 \sigma E_x + \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t_x} \tag{56a}$$

$$(\vec{\nabla} \times \vec{B})_y = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 J_y + \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t_y} = \mu_0 \sigma E_y + \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t_y} \tag{56b}$$

$$(\vec{\nabla} \times \vec{B})_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 J_z + \mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t_z} = \mu_0 \sigma E_z + \mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t_z} \tag{56c}$$

where $\sigma = \sigma_0 / \sqrt{1 - \beta^2}$ and $\sigma' = \sigma_0 / \sqrt{1 - \beta'^2}$ are the conductivities in the massive inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ and σ_0 is the rest conductivity in the absolutely steady inertial frame Σ_0 . Applying chain rule in Eq. (47a) and (47b) to the differential form of Ampere-Faraday’s law in x, y, and z-directions and write first, second and third terms in Eqs. (56a), (56b), and (56c), we write

$$\frac{\partial B_{x_i}}{\partial x_i} = \frac{\partial B'_{x_i}}{\partial x_i} = \frac{\partial B'_{x_i}}{\partial x'_i} \frac{\partial x'_i}{\partial x_i} + \frac{\partial B'_{x_i}}{\partial t'_{x_i}} \frac{\partial t'_{x_i}}{\partial x_i} = \frac{\partial B'_{x_i}}{\partial x'_i} - \frac{v_{x_i}}{c'^2} \frac{\partial B'_{x_i}}{\partial x'_i} \frac{\partial x'_i}{\partial x_i} \frac{\partial x_i}{\partial t'_{x_i}} = \left(1 - \frac{v_{x_i}^2}{c'^2}\right) \frac{\partial B'_{x_i}}{\partial x'_i} \tag{57a}$$

$$\frac{\partial \bar{E}_{x_i}}{\partial t_{x_i}} = \frac{\partial \bar{E}'_{x_i}}{\partial t_{x_i}} = \frac{\partial \bar{E}'_{x_i}}{\partial t'_{x_i}} \frac{\partial t'_{x_i}}{\partial t_{x_i}} + \frac{\partial \bar{E}'_{x_i}}{\partial x'_i} \frac{\partial x'_i}{\partial t_{x_i}} = \frac{\partial \bar{E}'_{x_i}}{\partial t'_{x_i}} - \frac{\partial x'_i}{\partial t_{x_i}} \frac{\partial t'_{x_i}}{\partial t_{x_i}} \left(\frac{v_{x_i}}{c'^2} \frac{\partial t_{x_i}}{\partial x'_i}\right) \frac{\partial \bar{E}'_{x_i}}{\partial t'_{x_i}} = \left(1 - \frac{v_{x_i}^2}{c'^2}\right) \frac{\partial \bar{E}'_{x_i}}{\partial t'_{x_i}} \tag{57b}$$

Combining Eqs. (57a), and (57b) side by side and adding $\mu_0 \sigma \bar{E}_{x_i}$ and $\mu'_0 \sigma' \bar{E}'_{x_i}$ on the left and right sides we write the Ampere-Maxwell law which is covariant between two massive inertial frames

$$(\vec{\nabla} \times \vec{B})_{x_i} + \mu_0 \sigma \bar{E}_{x_i} + \varepsilon_0 \frac{\partial \bar{E}_{x_i}}{\partial t_{x_i}} = \gamma_{x_i x_i}^2 \left(1 - \frac{v_{x_i}^2}{c'^2}\right) (\vec{\nabla}' \times \vec{B}')_{x_i} + \mu'_0 \sigma' \bar{E}'_{x_i} + \gamma_{t_{x_i} t_{x_i}}^2 \varepsilon'_0 \left(1 - \frac{v_{x_i}^2}{c'^2}\right) \frac{\partial \bar{E}'_{x_i}}{\partial t'_{x_i}} \tag{58}$$

Matching both sides of Eq. (58), one finds $\gamma_{x_i x_i} = \gamma_{t_{x_i} t_{x_i}} = 1 / (1 - v_{x_i}^2 / c'^2)^{1/2}$ in Eq. (14) for Lorentz scaling factor, which transforms covariant Eq. (54) into the following invariant form

$$\vec{\nabla}_i \times \vec{B} + \mu_0 \sigma \bar{E} + \varepsilon_0 \frac{\partial \bar{E}}{\partial t_{x_i}} = \nabla'_i \times \vec{B}' + \mu'_0 \sigma' \bar{E}' + \varepsilon'_0 \frac{\partial \bar{E}'}{\partial t'_{x_i}} \tag{59}$$

When frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ move in one dimension (e.g., along x axis), 6-dimensional spacetime (3+3) reduces to 4-dimensional spacetime (3+1). Consequently, the Lorentz scaling factor reduces to $\gamma_{xx} = \gamma_{tt} = \gamma = 1 / (1 - v^2 / c'^2)^{1/2}$ and Eq. (59) then takes the following 4-dimensional invariant form

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E}' + \frac{\partial \vec{B}'}{\partial t'} \tag{60}$$

which is the Lorentz invariant Maxwell law of induction between four-dimensional inertial frames Σ and Σ' in vacuum. Recall that $\vec{E}(\vec{B})$ and $\vec{E}'(\vec{B}')$ are invariant vectors and μ_0, σ and μ'_0, σ' are invariant scalars ($\mu_0 \sigma = \mu'_0 \sigma'$) between two frames.

6. Relativistic Invariance of Electromagnetic Wave Equations

One of the consequences of Maxwell’s equations are the scalar wave equations [20]. Using the formula $\vec{\nabla} \times \vec{\nabla} \times = \vec{\nabla} \vec{\nabla} \cdot - \vec{\nabla}^2$ [20], we write Faraday’s law of induction in a charge free medium

$$\vec{\nabla}_i(\vec{\nabla}_i \cdot \vec{E}) - \vec{\nabla}_i^2 \vec{E} = -\sigma\mu \frac{\partial \vec{E}}{\partial t_{x_i}} - \frac{\partial}{\partial t_{x_i}}(\vec{\nabla}_i \times \vec{B}) = -\sigma\mu \frac{\partial \vec{E}}{\partial t_{x_i}} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t_{x_i}^2} \quad (61a)$$

$$\vec{\nabla}'_i(\vec{\nabla}'_i \cdot \vec{E}') - \vec{\nabla}'_i{}^2 \vec{E}' = -\sigma'\mu' \frac{\partial \vec{E}'}{\partial t'_{x_i}} - \frac{\partial}{\partial t'_{x_i}}(\vec{\nabla}'_i \times \vec{B}') = -\sigma'\mu' \frac{\partial \vec{E}'}{\partial t'_{x_i}} - \frac{1}{c'^2} \frac{\partial^2 \vec{E}'}{\partial t'_{x_i}{}^2} \quad (61b)$$

where $\vec{\nabla}_i(\vec{\nabla}_i \cdot \vec{E}) = 0$ and $\vec{\nabla}'_i(\vec{\nabla}'_i \cdot \vec{E}') = 0$. Using the chain rule in Eqs. (47a), (47b), and (47c) for differential operators one then writes each component of Eq. (61a) and (61b) as

$$\nabla_i^2 \vec{E} = \frac{\partial}{\partial x'_i} \left(\frac{\partial \vec{E}'}{\partial x_i} \right) \frac{\partial x'_i}{\partial x_i} + \frac{\partial}{\partial t'_{x_i}} \left(\frac{\partial \vec{E}'}{\partial x_i} \right) \frac{\partial t'_{x_i}}{\partial x_i} = \gamma_{x_i x_i}^2 (1 - \beta_{x_i}^2) \frac{\partial^2 \vec{E}'}{\partial x_i'^2} \quad (62a)$$

$$\frac{\partial \vec{E}}{\partial t_{x_i}} = \frac{\partial \vec{E}'}{\partial t_{x_i}} = \frac{\partial \vec{E}'}{\partial t'_{x_i}} \frac{\partial t'_{x_i}}{\partial t_{x_i}} + \frac{\partial \vec{E}'}{\partial x'_i} \frac{\partial x'_i}{\partial t_{x_i}} = \frac{\partial \vec{E}'}{\partial t'_{x_i}} - \frac{\partial x'_i}{\partial t_{x_i}} \frac{\partial t'_{x_i}}{\partial t_{x_i}} \left(\frac{v_{x_i}}{c^2} \frac{\partial t_{x_i}}{\partial x'_i} \right) \frac{\partial \vec{E}'}{\partial t'_{x_i}} = \gamma_{x_i x_i}^2 (1 - \beta_{x_i}^2) \frac{\partial \vec{E}'}{\partial t'_{x_i}} \quad (62b)$$

$$-\frac{1}{c^2} \left(\frac{\partial^2 \vec{E}}{\partial t_{x_i}^2} \right) = -\frac{1}{c'^2} \frac{\partial}{\partial x'_i} \left(\frac{\partial \vec{E}'}{\partial t_{x_i}} \right) \frac{\partial x'_i}{\partial t_{x_i}} + \frac{\partial}{\partial t'_{x_i}} \left(\frac{\partial \vec{E}'}{\partial t_{x_i}} \right) \frac{\partial t'_{x_i}}{\partial t_{x_i}} = -\frac{1}{c'^2} \gamma_{t_{x_i} t_{x_i}}^2 (1 - \beta_{x_i}^2) \frac{\partial^2 \vec{E}'}{\partial t_{x_i}'^2} \quad (62c)$$

In a charge free medium, covariant form of Faraday's law between two frames is then written as

$$\nabla_i^2 \vec{E} - \sigma\mu \frac{\partial \vec{E}}{\partial t_{x_i}} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t_{x_i}^2} = \gamma_{x_i x_i}^2 (1 - \beta_{x_i}^2) \frac{\partial^2 \vec{E}'}{\partial x_i'^2} - \sigma'\mu' \gamma_{x_i x_i}^2 (1 - \beta_{x_i}^2) \frac{\partial \vec{E}'}{\partial t'_{x_i}} - \frac{1}{c'^2} \gamma_{t_{x_i} t_{x_i}}^2 (1 - \beta_{x_i}^2) \frac{\partial^2 \vec{E}'}{\partial t_{x_i}'^2} \quad (63)$$

Matching both sides of Eq. (63) one finds $\gamma_{x_i x_i} = \gamma_{t_{x_i} t_{x_i}} = 1/(1 - \beta_{x_i}^2)^{1/2}$ in Eq. (14) for the components of Lorentz scaling factor and Eq. (63) becomes Lorentz invariant since \vec{E} and \vec{E}' are invariant vectors ($\vec{E} = \vec{E}'$) and μ, σ and μ', σ' are invariant scalars ($\mu\sigma = \mu'\sigma'$). Maxwell's scalar wave equation (63) for the electric field in the six-dimensional spacetime can then be written as

$$\begin{aligned} \nabla_i^2 \vec{E} - \frac{1}{c^2} \left(\frac{\partial^2}{\partial t_x^2} + \frac{\partial^2}{\partial t_y^2} + \frac{\partial^2}{\partial t_z^2} \right) \vec{E} &= \left(\gamma_{xx}^2 - \gamma_{t_x t_x}^2 \beta_x^2 \right) \frac{\partial^2 \vec{E}}{\partial x'^2} + \left(\gamma_{yy}^2 - \gamma_{t_y t_y}^2 \beta_y^2 \right) \frac{\partial^2 \vec{E}}{\partial y'^2} + \left(\gamma_{zz}^2 - \gamma_{t_z t_z}^2 \beta_z^2 \right) \frac{\partial^2 \vec{E}}{\partial z'^2} \\ &- \frac{1}{c^2} \left(\gamma_{t_x t_x}^2 - \gamma_{xx}^2 \beta_x^2 \right) \frac{\partial^2 \vec{E}}{\partial t_x'^2} - \frac{1}{c^2} \left(\gamma_{t_y t_y}^2 - \gamma_{yy}^2 \beta_y^2 \right) \frac{\partial^2 \vec{E}}{\partial t_y'^2} - \frac{1}{c^2} \left(\gamma_{t_z t_z}^2 - \gamma_{zz}^2 \beta_z^2 \right) \frac{\partial^2 \vec{E}}{\partial t_z'^2} \end{aligned} \quad (64)$$

Component by component matching both sides of Eq. (64) then yield the components of Lorentz scaling factor given by Eq. (14). When the massive inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ move along x axis, the 6-dimensional spacetime (3+3) reduces to the 4-dimensional spacetime (3+1). Lorentz scaling factor reduces to $\gamma_{xx} = \gamma_{tt} = \gamma = 1/(1 - v^2/c^2)^{1/2}$ and Eq. (64) reduces to the following four-dimensional invariant form

$$\nabla_i^2 \vec{E} - \frac{1}{c^2} \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right) = \nabla_i'^2 \vec{E}' - \frac{1}{c'^2} \left(\frac{\partial^2 \vec{E}'}{\partial t'^2} \right) \quad (65)$$

which is the Lorentz invariant Maxwell wave equation between four-dimensional inertial frames Σ and Σ' in vacuum.

Likewise, using the formula from vector analysis $\vec{\nabla} \times \vec{\nabla} \times = \vec{\nabla} \vec{\nabla} \cdot - \vec{\nabla}^2$ [20] one writes the differential equations for Ampere-Maxwell law in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ in charge free medium as

$$\vec{\nabla}_i(\vec{\nabla}_i \cdot \vec{B}) - \vec{\nabla}_i^2 \vec{B} = -\sigma\mu \frac{\partial \vec{B}}{\partial t_{x_i}} - \frac{\partial}{\partial t_{x_i}}(\vec{\nabla}_i \times \vec{B}) = -\sigma\mu \frac{\partial \vec{B}}{\partial t_{x_i}} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t_{x_i}^2} \quad (66a)$$

$$\vec{\nabla}'_i(\vec{\nabla}'_i \cdot \vec{B}') - \vec{\nabla}'_i{}^2 \vec{B}' = -\sigma'\mu' \frac{\partial \vec{B}'}{\partial t'_{x_i}} - \frac{\partial}{\partial t'_{x_i}}(\vec{\nabla}'_i \times \vec{B}') = -\sigma'\mu' \frac{\partial \vec{B}'}{\partial t'_{x_i}} - \frac{1}{c'^2} \frac{\partial^2 \vec{B}'}{\partial t'_{x_i}{}^2} \quad (66b)$$

where $\vec{\nabla}_i(\vec{\nabla}_i \cdot \vec{B}) = 0$, and $\vec{\nabla}'_i(\vec{\nabla}'_i \cdot \vec{B}') = 0$. Following the steps in writing Eq. (63), we write

$$\nabla_i^2 \bar{B} = \frac{\partial}{\partial x'_i} \left(\frac{\partial \bar{B}'}{\partial x_i} \right) \frac{\partial x'_i}{\partial x_i} + \frac{\partial}{\partial t'_i} \left(\frac{\partial \bar{B}'}{\partial x_i} \right) \frac{\partial t'_i}{\partial x_i} = \gamma_{x_i x_i}^2 (1 - \beta_{x_i}^2) \frac{\partial^2 \bar{B}'}{\partial x_{x_i}^2} \tag{67a}$$

$$\frac{\partial \bar{B}}{\partial t_{x_i}} = \frac{\partial \bar{B}'}{\partial t_{x_i}} = \frac{\partial \bar{B}'}{\partial t'_{x_i}} \frac{\partial t'_{x_i}}{\partial t_{x_i}} + \frac{\partial \bar{B}'}{\partial x'_i} \frac{\partial x'_i}{\partial t_{x_i}} = \frac{\partial \bar{B}'}{\partial t'_{x_i}} - \frac{\partial x'_i}{\partial t_{x_i}} \frac{\partial t'_{x_i}}{\partial t_{x_i}} \left(\frac{v_{x_i}}{c^2} \frac{\partial t_{x_i}}{\partial x'_i} \right) \frac{\partial \bar{B}'}{\partial t'_{x_i}} = \gamma_{x_i x_i}^2 (1 - \beta_{x_i}^2) \frac{\partial \bar{B}'}{\partial t'_{x_i}} \tag{67b}$$

$$-\frac{1}{c^2} \left(\frac{\partial^2 \bar{E}}{\partial t_{x_i}^2} \right) = -\frac{1}{c'^2} \frac{\partial}{\partial x'_i} \left(\frac{\partial \bar{E}'}{\partial t_{x_i}} \right) \frac{\partial x'_i}{\partial t_{x_i}} + \frac{\partial}{\partial t'_{x_i}} \left(\frac{\partial \bar{E}'}{\partial t_{x_i}} \right) \frac{\partial t'_{x_i}}{\partial t_{x_i}} = -\frac{1}{c'^2} \gamma_{t_{x_i} t_{x_i}}^2 (1 - \beta_{x_i}^2) \frac{\partial^2 \bar{E}'}{\partial t_{x_i}^2} \tag{67c}$$

In a charge free medium, combining Eqs. (67a), (67b) and (67c) covariant form of Maxwell’s wave equation in six dimensional spacetime between two massive inertial frames is written as

$$\nabla_i^2 \bar{B} - \sigma \mu \frac{\partial \bar{B}}{\partial t_{x_i}} - \frac{1}{c^2} \left(\frac{\partial^2 \bar{B}}{\partial t_{x_i}^2} \right) = \gamma_{x_i x_i}^2 (1 - \beta_{x_i}^2) \frac{\partial^2 \bar{B}'}{\partial x_{x_i}^2} + \sigma' \mu' \gamma_{x_i x_i}^2 (1 - \beta_{x_i}^2) \frac{\partial \bar{B}'_{x_i}}{\partial t'_{x_i}} - \frac{1}{c'^2} \gamma_{t_{x_i} t_{x_i}}^2 (1 - \beta_{x_i}^2) \frac{\partial^2 \bar{B}'}{\partial t_{x_i}^2} \tag{68}$$

Matching both sides of Eq. (68), one finds $\gamma_{x_i x_i} = \gamma_{t_{x_i} t_{x_i}} = 1 / (1 - v^2 / c'^2)^{1/2}$ in Eq. (14) for Lorentz scaling factor and Eq. (68) becomes invariant. Since \bar{B} and \bar{B}' are invariant vectors ($\bar{B} = \bar{B}'$) and μ, σ and μ', σ' are invariant scalars ($\mu \sigma = \mu' \sigma'$), wave equation (68) is then written as

$$\begin{aligned} \nabla_i^2 \bar{B} - \frac{1}{c^2} \left(\frac{\partial^2}{\partial t_x^2} + \frac{\partial^2}{\partial t_y^2} + \frac{\partial^2}{\partial t_z^2} \right) \bar{B} &= \left(\gamma_{xx}^2 - \gamma_{t_x t_x}^2 \beta_x^2 \right) \frac{\partial^2 \bar{B}'}{\partial x^2} + \left(\gamma_{yy}^2 - \gamma_{t_y t_y}^2 \beta_y^2 \right) \frac{\partial^2 \bar{B}'}{\partial y^2} + \left(\gamma_{zz}^2 - \gamma_{t_z t_z}^2 \beta_z^2 \right) \frac{\partial^2 \bar{B}'}{\partial z^2} \\ - \frac{1}{c^2} \left(\gamma_{t_x t_x}^2 - \gamma_{xx}^2 \beta_x^2 \right) \frac{\partial^2 \bar{B}'}{\partial t_x^2} - \frac{1}{c^2} \left(\gamma_{t_y t_y}^2 - \gamma_{yy}^2 \beta_y^2 \right) \frac{\partial^2 \bar{B}'}{\partial t_y^2} - \frac{1}{c^2} \left(\gamma_{t_z t_z}^2 - \gamma_{zz}^2 \beta_z^2 \right) \frac{\partial^2 \bar{B}'}{\partial t_z^2} \end{aligned} \tag{69}$$

Component by component matching both sides of Eq. (68) yield the components of Lorentz scaling factor in six dimensions given by Eq. (14). When $\bar{\Sigma}$ and $\bar{\Sigma}'$ move along x axis, the 6-dimensional spacetime (3+3) reduces to the 4-dimensional spacetime (3+1) and Lorentz factor reduces to $\gamma_{xx} = \gamma_{tt} = \gamma = 1 / (1 - v^2 / c'^2)^{1/2}$ and Eq. (69) reduces to the following four-dimensional invariant form

$$\nabla_i^2 \bar{B} - \frac{1}{c^2} \left(\frac{\partial^2 \bar{B}}{\partial t^2} \right) = \nabla_i^2 \bar{B}' - \frac{1}{c'^2} \left(\frac{\partial^2 \bar{B}'}{\partial t'^2} \right) \tag{70}$$

which is the Lorentz invariant Maxwell wave equation between four-dimensional inertial frames Σ and Σ' in vacuum.

7. Results and Discussions

Throughout previous sections, we demonstrated that use of classical vector transformation allows one first to derive expressions for Cartesian components of relativistic invariant vector quantities, having the same length from the origin, so that a vector transformation is identified as a rotation if it causes no change in the magnitude of a vector [19]. With transformed velocity components used in the law of conservation of electromagnetic energy in a closed system, we proved that the electromagnetic fields and Maxwell’s equations and scalar wave equations are Lorentz invariant between two massive inertial frames. In this section we will give a summary discussion about the applications of the proposed theory in deriving the expressions for the relativistic invariance of the electromagnetic wave equations in the materials medium and the conservation laws of charge (current) continuity equation and continuity of electromagnetic energy and linear and angular Momentums between two massive inertial frames under Lorentz transformation.

7.1. Relativistic Invariance of Electromagnetic Wave Equations in Materials Medium

One can easily extend the six-dimensional spacetime theory to material medium wherein $\vec{D} = \epsilon \vec{E}$ and $\vec{H} = \vec{B} / \mu$ by replacing the speed of light in vacuum with that in material medium as $c'_m = 1 / \sqrt{\mu'_m \epsilon'_m}$ and $c_m = 1 / \sqrt{\mu_m \epsilon_m}$ in the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$. As an example, the covariant Maxwell’s scalar wave equations for the electric and magnetic fields in material medium between two massive inertial frames under Lorentz transformation can then be written as

$$\begin{aligned}
\nabla_i^2 \bar{D} - \frac{1}{c_m^2} \left(\frac{\partial^2}{\partial t_x^2} + \frac{\partial^2}{\partial t_y^2} + \frac{\partial^2}{\partial t_z^2} \right) \bar{D} &= \nabla_i'^2 \bar{D}' - \frac{1}{c_m'^2} \left(\frac{\partial^2}{\partial t_x'^2} + \frac{\partial^2}{\partial t_y'^2} + \frac{\partial^2}{\partial t_z'^2} \right) \bar{D}' \\
&= \left(\gamma_{m,xx}' - \gamma_{m,t_x t_x}^2 \beta_x^2 \right) \frac{\partial^2 \bar{D}'}{\partial x'^2} + \left(\gamma_{m,yy}' - \gamma_{m,t_y t_y}^2 \beta_y^2 \right) \frac{\partial^2 \bar{D}'}{\partial y'^2} + \left(\gamma_{m,zz}' - \gamma_{m,t_z t_z}^2 \beta_z^2 \right) \frac{\partial^2 \bar{D}'}{\partial z'^2} \\
&\quad - \frac{1}{c_m'^2} \left(\gamma_{m,t_x t_x}^2 - \gamma_{m,xx}^2 \beta_x^2 \right) \frac{\partial^2 \bar{D}'}{\partial t_x'^2} - \frac{1}{c_m'^2} \left(\gamma_{m,t_y t_y}^2 - \gamma_{m,yy}^2 \beta_y^2 \right) \frac{\partial^2 \bar{D}'}{\partial t_y'^2} - \frac{1}{c_m'^2} \left(\gamma_{m,t_z t_z}^2 - \gamma_{m,zz}^2 \beta_z^2 \right) \frac{\partial^2 \bar{D}'}{\partial t_z'^2}
\end{aligned} \tag{71a}$$

$$\begin{aligned}
\nabla_i^2 \bar{H} - \frac{1}{c_m^2} \left(\frac{\partial^2}{\partial t_x^2} + \frac{\partial^2}{\partial t_y^2} + \frac{\partial^2}{\partial t_z^2} \right) \bar{H} &= \nabla_i'^2 \bar{H}' - \frac{1}{c_m'^2} \left(\frac{\partial^2}{\partial t_x'^2} + \frac{\partial^2}{\partial t_y'^2} + \frac{\partial^2}{\partial t_z'^2} \right) \bar{H}' \\
&= \left(\gamma_{m,xx}' - \gamma_{m,t_x t_x}^2 \beta_x^2 \right) \frac{\partial^2 \bar{H}'}{\partial x'^2} + \left(\gamma_{m,yy}' - \gamma_{m,t_y t_y}^2 \beta_y^2 \right) \frac{\partial^2 \bar{H}'}{\partial y'^2} + \left(\gamma_{m,zz}' - \gamma_{m,t_z t_z}^2 \beta_z^2 \right) \frac{\partial^2 \bar{H}'}{\partial z'^2} \\
&\quad - \frac{1}{c_m'^2} \left(\gamma_{m,t_x t_x}^2 - \gamma_{m,xx}^2 \beta_x^2 \right) \frac{\partial^2 \bar{H}'}{\partial t_x'^2} - \frac{1}{c_m'^2} \left(\gamma_{m,t_y t_y}^2 - \gamma_{m,yy}^2 \beta_y^2 \right) \frac{\partial^2 \bar{H}'}{\partial t_y'^2} - \frac{1}{c_m'^2} \left(\gamma_{m,t_z t_z}^2 - \gamma_{m,zz}^2 \beta_z^2 \right) \frac{\partial^2 \bar{H}'}{\partial t_z'^2}
\end{aligned} \tag{71b}$$

where $\varepsilon_m(\varepsilon_m')$ and $\mu_m(\mu_m')$ are the Lorentz scalar dielectric constant and magnetic permittivity of a material medium in massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$. Here $\gamma_{m,\alpha\beta}$ are the Lorentz scaling factors

$$\gamma_{m,xx} = \gamma_{m,t_x t_x} = 1/\sqrt{1-\beta_{mx}^2}; \quad \gamma_{m,yy} = \gamma_{m,t_y t_y} = 1/\sqrt{1-\beta_{my}^2}; \quad \gamma_{m,zz} = \gamma_{m,t_z t_z} = 1/\sqrt{1-\beta_{mz}^2} \tag{72}$$

where $\beta_{mx} = v_x/c_m$, $\beta_{my} = v_y/c_m$, $\beta_{mz} = v_z/c_m$ are the normalized x, y, and z components of relative velocity of the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ in a material medium.

We can extend the derivations carried out above to the cases in which charge and current densities are not zero. Using $\nabla_i \cdot \bar{E} = \rho_m / \varepsilon_m$, $\nabla_i' \cdot \bar{E}' = \rho_m' / \varepsilon_m'$ with $\bar{B} = \nabla_i \times \bar{A}$ and $\bar{B}' = \nabla_i' \times \bar{A}'$, with \bar{E} and \bar{E}' which are defined in the massive inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ according to

$$\bar{E} = -\nabla_i \Phi - \partial \bar{A} / \partial t_i; \quad \bar{E}' = -\nabla_i' \Phi' - \partial \bar{A}' / \partial t_i' \tag{73a}$$

$$\nabla_i \cdot \bar{A} + \frac{1}{c_m^2} \frac{\partial \Phi}{\partial t_i} = 0, \quad \nabla_i' \cdot \bar{A}' + \frac{1}{c_m'^2} \frac{\partial \Phi'}{\partial t_i'} = 0 \quad (\text{Lorentz gauge}) \tag{73b}$$

$$\nabla_i \cdot \bar{J}_{mi} + \frac{\partial \rho_{mi}}{\partial t_i} = 0, \quad \nabla_i' \cdot \bar{J}'_{mi} + \frac{\partial \rho'_{mi}}{\partial t_i'} = 0 \quad (\text{Continuity equation}) \tag{73c}$$

with $i, j, k = x, y, z$, respectively. Here Φ (Φ') and \bar{A} (\bar{A}') are the scalar and vector potentials. $\bar{J}_{mi} = \rho_{mi} \bar{u}_{mi}$ and $\bar{J}'_{mi} = \rho'_{mi} \bar{u}'_{mi}$ are current densities with charge densities $\rho_{mi} = \rho_0 / \sqrt{1-\beta_{mi}^2}$ and $\rho'_{mi} = \rho_0 / \sqrt{1-\beta_{mi}'^2}$ in a material medium in the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ with ρ_0 the charge density in the absolutely stationary inertial (rest) frame. We can then write the following equations

$$\left(\nabla_i^2 - \frac{1}{c_m^2} \frac{\partial^2}{\partial t_i^2} \right) \Phi_i + \rho_i / \varepsilon_m = \left(\nabla_i'^2 - \frac{1}{c_m'^2} \frac{\partial^2}{\partial t_i'^2} \right) \Phi_i' + \rho_{mi}' / \varepsilon_m' \tag{74a}$$

$$\left(\nabla_i^2 - \frac{1}{c_m^2} \frac{\partial^2}{\partial t_i^2} \right) \bar{A}_i + \mu_m \bar{J}_{mi} = \left(\nabla_i'^2 - \frac{1}{c_m'^2} \frac{\partial^2}{\partial t_i'^2} \right) \bar{A}_i' + \mu_m' \bar{J}'_{mi} \tag{74b}$$

which are invariant between the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$. Setting $\rho'_{mi} = \rho_{mi}$ in Eqs. (68a) and (68b) one finds $u'_{mx} = u_{mx}$, $u'_{my} = u_{my}$ and $u'_{mz} = u_{mz}$, respectively, for the velocity components.

7.2. Relativistic Invariance of Conservation Laws in Electrodynamics

Since electromagnetic waves are associated with propagation of energy and momentum in space [19], the proof of the

relativistic invariance of the following relations is essential

$$\nabla_i \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0, \quad \nabla'_i \cdot \vec{J}' + \frac{\partial \rho'}{\partial t'} = 0, \text{ Electric charge continuity equations} \tag{75a}$$

$$\vec{\nabla}' \cdot \vec{S}' + \frac{\partial u'_{em}}{\partial t'} + \vec{J}' \cdot \vec{E}' = 0, \quad \vec{\nabla} \cdot \vec{S} + \frac{\partial u_{em}}{\partial t} + \vec{J} \cdot \vec{E} = 0, \text{ Energy continuity equations} \tag{75b}$$

$$\frac{\partial \vec{g}'}{\partial t'} - \vec{\nabla}' \cdot \vec{T}' + \vec{f}' = 0, \quad \frac{\partial \vec{g}}{\partial t} - \vec{\nabla} \cdot \vec{T} + \vec{f} = 0, \text{ Maxwell energy-momentum stress tensor} \tag{75c}$$

where $\vec{S}' = (\vec{E}' \times \vec{B}') / \mu'_o$ and $\vec{S} = (\vec{E} \times \vec{B}) / \mu_o$ are the energy fluxes, known as Poynting vectors, and $u'_{em} = (\vec{E}'^2 + c'^2 \vec{B}'^2) / 2$ and $u_{em} = (\vec{E}^2 + c^2 \vec{B}^2) / 2$ are the electromagnetic energy densities, $\vec{g}' = \epsilon_o (\vec{E}' \times \vec{B}') = \vec{S}' / c'^2$ and $\vec{g} = \epsilon_o (\vec{E} \times \vec{B}) = \vec{S} / c^2$ are the electromagnetic linear momentum densities and $\vec{f}' = \rho' \vec{E}' + \vec{J}' \times \vec{B}'$ and $\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B}$ are the Lorentz force per unit volume exerted by the fields on the electric charge. Finally, \vec{T}' and \vec{T} are Maxwell energy-stress tensors, written as

$$T'_{ij} = \epsilon'_o (E'_i E'_j + c'^2 B'_i B'_j) - \frac{1}{2} \delta_{ij} (E'^2 + c'^2 B'^2), \quad T_{ij} = \epsilon_o (E_i E_j + c^2 B_i B_j) - \frac{1}{2} \delta_{ij} (E^2 + c^2 B^2) \tag{75d}$$

where the indices i and j refer to the coordinates x, y and z . δ_{ij} is the Kronecker delta which is unity if the indices are the same ($\delta_{xx} = \delta_{yy} = \delta_{zz}$) and zero otherwise ($\delta_{xy} = \delta_{yz} = \delta_{zx} = 0$). Maxwell energy-stress tensor \vec{T} is the force (stress) per unit area acting on a surface in both inertial frames with diagonal elements representing pressure and off diagonal elements are shears. In the following sub-sections we will discuss the invariance electromagnetic energy and momentum densities between two inertial frames under Lorentz transformation.

7.2.1. Current Continuity Equation

We start with employing the chain rule to differentiation in Eqs. (47a) and (47b) to write the following differential relations

$$\frac{\partial \rho}{\partial t_{x_i}} = \gamma^2 \frac{\partial \rho'}{\partial t'_{x_i}} + \gamma^2 \left(-\frac{v_x}{c^2} \frac{\partial \rho'}{\partial x'} - \frac{v_y}{c^2} \frac{\partial \rho'}{\partial y'} - \frac{v_z}{c^2} \frac{\partial \rho'}{\partial z'} \right) = \gamma^2 \left(1 - \beta_{x_i}^2 \right) \frac{\partial \rho'}{\partial t'_{x_i}} \tag{76a}$$

$$\vec{\nabla} \cdot \vec{J} = \gamma_{xx}^2 \left(1 - \beta_x'^2 \right) \frac{\partial J'_x}{\partial x'} + \gamma_{yy}^2 \left(1 - \beta_y'^2 \right) \frac{\partial J'_y}{\partial y'} + \gamma_{zz}^2 \left(1 - \beta_z'^2 \right) \frac{\partial J'_z}{\partial z'} \tag{76b}$$

Side by side additions of Eqs. (76a) and (76b) allows us to write the following equation

$$\vec{\nabla}_i \cdot \vec{J} + \frac{\partial \rho}{\partial t_{x_i}} = \gamma_{x_i x_i}^2 \left(1 - \frac{v^2}{c^2} \right) \vec{\nabla}'_i \cdot \vec{J}' + \gamma^2 \left(1 - \beta_{x_i}^2 \right) \frac{\partial \rho'}{\partial t'_{x_i}} \tag{77}$$

where $x_i = x, y, z$ and c' is the Lorentz scalar speed of light ($c' = c$). Matching both sides of Eq. (76) yields $\gamma_{x_i x_i} = \gamma_{t_{x_i} t_{x_i}} = 1 / (1 - \beta_{x_i}^2)^{1/2}$ in Eq. (14). When the massive inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ move in one dimension (e.g., along x axis), the six-dimensional spacetime (3+3) reduces to the classical four-dimensional spacetime (3+1). Lorentz scaling factor reduces to $\gamma_{xx} = \gamma_{tt} = 1 / (1 - v^2 / c^2)^{1/2}$ and covariant Eq. (77) is then transformed into the following four-dimensional invariant form

$$\vec{\nabla}_i \cdot \vec{J} + \frac{\partial \rho}{\partial t} = \vec{\nabla}'_i \cdot \vec{J}' + \frac{\partial \rho'}{\partial t'} \tag{78}$$

which is the Lorentz invariant current continuity equation between four-dimensional inertial frames Σ and Σ' in vacuum.

7.2.2. Electromagnetic Energy Continuity Equation

Since Poynting's theorem [20] states that the power flowing out of the volume and the time rate of increase of energy storage inside the volume equal to the total power delivered by the source to a closed electrical circuit and they can be summarized according to the following relations in the massive inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$

$$\oint_S \mu_0 \vec{S} \cdot d\vec{a} + \int_V \frac{\partial u_{em}}{\partial t_{x_i}} dv = - \int_V (\vec{E} \cdot \vec{J}) dv, \quad \oint_{S'} \mu'_0 \vec{S}' \cdot d\vec{a}' + \int_{V'} \frac{\partial u'_{em}}{\partial t'_{x_i}} dv' = - \int_{V'} (\vec{E}' \cdot \vec{J}') dv \quad (79)$$

Considering \vec{E} and \vec{J} are in the same direction ($\vec{E} \cdot \vec{J} = \sigma E^2$) using Ampere-Maxwell equation, the differential forms of Eq. (79) in the massive inertial frames Σ and Σ' are written as

$$\vec{\nabla}_i \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla}_i \times \vec{E}) - \vec{E} \cdot (\vec{\nabla}_i \times \vec{B}) = -\vec{B} \cdot \frac{\partial \vec{B}}{\partial t_{x_i}} - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t_{x_i}} - \vec{E} \cdot \vec{J} = -\frac{\partial u_{em}}{\partial t_{x_i}} - \sigma E^2 \quad (80a)$$

$$\vec{\nabla}'_i \cdot (\vec{E}' \times \vec{B}') = \vec{B}' \cdot (\vec{\nabla}'_i \times \vec{E}') - \vec{E}' \cdot (\vec{\nabla}'_i \times \vec{B}') = -\vec{B}' \cdot \frac{\partial \vec{B}'}{\partial t'_{x_i}} - \varepsilon'_0 \vec{E}' \cdot \frac{\partial \vec{E}'}{\partial t'_{x_i}} - \vec{E}' \cdot \vec{J}' = -\frac{\partial u'_{em}}{\partial t'_{x_i}} - \sigma' E'^2 \quad (80b)$$

where $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b})$ [19] is used. Since $\vec{E}' \times \vec{B}' = \vec{E} \times \vec{B}$, one can easily show that the Poynting vector is Lorentz invariant between two massive inertial frames

$$\vec{S}' = \frac{1}{\mu'_0} (\vec{E}' \times \vec{B}') = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \vec{S} \quad \Leftrightarrow \quad \vec{S}' = c'^2 \varepsilon'_0 (\vec{E}' \times \vec{B}') = c^2 \varepsilon_0 (\vec{E} \times \vec{B}) = \vec{S} \quad (81)$$

which states that direction of light propagation is independent of massive inertial frames. Since $\vec{E}' \times \vec{B}' = \mu'_0 \vec{S}'$, $\vec{E} \times \vec{B} = \mu_0 \vec{S}$, $u'_{em} = (\vec{E}'^2 + c'^2 \vec{B}'^2) / 2$, and $u_{em} = (\vec{E}^2 + c^2 \vec{B}^2) / 2$, letting $\varphi = S$ or u_{em} and $\varphi' = S'$ or u'_{em} , the first and second terms on the left side of Eq. (80a) in frame Σ are

$$\vec{\nabla} \cdot \vec{S} = \gamma_{xx}^2 (1 - \beta_x'^2) \frac{\partial S'}{\partial x'} + \gamma_{yy}^2 (1 - \beta_y'^2) \frac{\partial S'}{\partial y'} + \gamma_{zz}^2 (1 - \beta_z'^2) \frac{\partial S'}{\partial z'} \quad (82a)$$

$$\frac{\partial u_{em}}{\partial t_{x_i}} = \gamma^2 \frac{\partial u'_{em}}{\partial t'_{x_i}} - \gamma^2 \left(\frac{v_x}{c^2} \frac{\partial t_x}{\partial x'} + \frac{v_y}{c^2} \frac{\partial t_y}{\partial y'} + \frac{v_z}{c^2} \frac{\partial t_z}{\partial z'} \right) \frac{\partial u'_{em}}{\partial t'_{x_i}} = \gamma^2 (1 - \beta_{x_i}'^2) \frac{\partial u'_{em}}{\partial t'_{x_i}} \quad (82b)$$

Side by side addition of Eqs. (82a) and (82b) allows us to write the following equation for the invariance of the continuous electromagnetic energy between two massive inertial frames

$$\vec{\nabla}_i \cdot \vec{S} + \frac{\partial u_{em}}{\partial t_{x_i}} + \sigma E^2 = \gamma_{x_i x_i}^2 (1 - \beta_{x_i}'^2) \vec{\nabla}'_i \cdot \vec{S}' + \gamma_{t_i t_i}^2 (1 - \beta_{x_i}'^2) \frac{\partial u'_{em}}{\partial t'_{x_i}} + \sigma' E'^2 \quad (83)$$

Since $\vec{E}' \cdot \vec{E}' = \vec{E} \cdot \vec{E}$ (or $\sigma E^2 = \sigma' E'^2$) is Lorentz invariant, matching both sides of Eq. (83), one finds $\gamma_{x_i x_i} = \gamma_{t_i t_i} = 1 / (1 - \beta_{x_i}'^2)^{1/2}$ in Eq. (14). When Σ and Σ' move in the x-direction, 6-dimensional spacetime (3+3) reduces to 4-dimensional spacetime (3+1). Lorentz scaling factor reduces to $\gamma_{xx} = \gamma_{tt} = 1 / (1 - v^2 / c^2)^{1/2}$ and Eq. (83) is transformed into 4-dimensional invariant form

$$\vec{\nabla}_i \cdot \vec{S} + \frac{\partial u_{em}}{\partial t} = \vec{\nabla}'_i \cdot \vec{S}' + \frac{\partial u'_{em}}{\partial t'} \quad (84)$$

which is Lorentz invariant between four-dimensional frames Σ and Σ' in vacuum with no charge source.

7.2.3. Electromagnetic Momentum Continuity Equation

The electromagnetic field linear and angular momentums per unit volume in both frames are defined as [1]

$$\vec{g}' = \varepsilon_0 (\vec{E}' \times \vec{B}') = \frac{1}{c^2} \vec{S}', \quad \vec{g} = \varepsilon_0 (\vec{E} \times \vec{B}) = \frac{1}{c^2} \vec{S} \quad (85a)$$

$$\vec{l}' = \vec{r}' \times \vec{g}' = \frac{1}{c^2} (\vec{r}' \times \vec{S}'), \quad \vec{l} = \vec{r} \times \vec{g} = \frac{1}{c^2} (\vec{r} \times \vec{S}) \quad (85b)$$

Since $\vec{S}' = \vec{S}$ and $\vec{r}' \times \vec{S}' = \vec{r} \times \vec{S}$, so are $\vec{g}' = \vec{g}$ and $\vec{l}' = \vec{l}$, both are Lorentz invariant between the Σ and Σ' frames. Employing the chain rule in Eqs. (47a) and (47b) to the components of Maxwell stress tensor equation (75d) allows us to write the following relations in the Σ frame

$$\frac{\partial \bar{g}}{\partial t_{x_i}} = \frac{\partial \bar{g}'}{\partial t_{x_i}} = \frac{\partial \bar{g}'}{\partial t'_{x_i}} \frac{\partial t'_{x_i}}{\partial t_{x_i}} + \frac{\partial \bar{g}}{\partial x'_i} \frac{\partial x'_i}{\partial t_{x_i}} = \frac{\partial \bar{g}'}{\partial t'_{x_i}} - \frac{\partial x'_i}{\partial t_{x_i}} \frac{\partial t'_{x_i}}{\partial t_{x_i}} \left(\frac{v_{x_i}}{c'^2} \frac{\partial t_{x_i}}{\partial x'_i} \right) \frac{\partial \bar{g}'}{\partial t'_{x_i}} = \gamma^2_{t_{x_i} t'_{x_i}} \left(1 - \beta'^2_{x_i} \right) \frac{\partial \bar{g}'}{\partial t'_{x_i}} \quad (86a)$$

$$\bar{\nabla} \cdot \bar{T} = \gamma^2_{xx} \left(1 - \frac{v_x^2}{c'^2} \right) \frac{\partial T'_{ij}}{\partial x'} + \gamma^2_{yy} \left(1 - \frac{v_y^2}{c'^2} \right) \frac{\partial T'_{ij}}{\partial y'} + \gamma^2_{zz} \left(1 - \frac{v_z^2}{c'^2} \right) \frac{\partial T'_{ij}}{\partial z'} \quad (86b)$$

$$\bar{f} = \rho \bar{E} + \bar{J} \times \bar{B} = \rho' \bar{E}' + \sigma' (\bar{E}' \times \bar{B}') = \bar{f}' \quad (86c)$$

Since $\bar{f} = \bar{f}'$, addition of Eqs. (86a), (86b), and (86c) allows us to write the following equation

$$\frac{\partial \bar{g}}{\partial t_{x_i}} - \bar{\nabla} \cdot \bar{T} + \bar{f} = \gamma^2_{t_{x_i} t'_{x_i}} \left(1 - \beta'^2_{x_i} \right) \frac{\partial \bar{g}'}{\partial t'_{x_i}} - \gamma^2_{x_i x_i} \left(1 - \beta'^2_{x_i} \right) \frac{\partial T'_{ij}}{\partial x'_i} + \bar{f}' \quad (87)$$

Matching both sides of Eq. (87), one finds $\gamma_{x_i x_i} = \gamma_{t_{x_i} t'_{x_i}} = 1 / (1 - \beta'^2_{x_i})^{1/2}$ in Eq. (14). When $\bar{\Sigma}$ and $\bar{\Sigma}'$ move in one dimension (e.g., along x axis), the six-dimensional spacetime (3+3) reduces to the classical four-dimensional spacetime (3+1). Lorentz scaling factor reduces to $\gamma_{xx} = \gamma_{tt} = 1 / (1 - v^2 / c'^2)^{1/2}$ and Eq. (87) is transformed into four-dimensional invariant form

$$\frac{\partial \bar{g}}{\partial t} - \bar{\nabla}_i \cdot \bar{T} = \frac{\partial \bar{g}'}{\partial t'} - \bar{\nabla}'_i \cdot \bar{T}' \quad (88)$$

which is Lorentz invariant between the four-dimensional inertial frames Σ and Σ' in vacuum.

8. Conclusions

We introduced a new six-dimensional spacetime frame which allows the space and time influence of each other in the system of spherical coordinates. After satisfying Lorentz invariance of metric equation between two massive inertial frames, we derived expressions for Cartesian components of the relativistic velocity which is valid at any speed. Using the classical vector transformation method, we derived expressions for the Cartesian components of transformed relativistic velocity and electromagnetic force vectors. Considering two massive inertial frames form a closed system, we implemented the transformed relativistic velocity components into the law of conservation of energy to prove that contrary to the common belief, the electromagnetic field that appears as a purely electric (magnetic) field in one massive inertial frame, it also appears as a pure electric (magnetic) field in another massive inertial frame under Lorentz transformation. As applications of the proposed theory, we proved the relativistic invariance of Maxwell equations and scalar wave equations with and without charge and current sources, and conservation laws for the continuity of current and electromagnetic energy and momentum. Since the magnitudes of quasi-time vectors are measurable in both massive inertial frames and their Cartesian components are treated as mathematical tools, we proved that the predictions of the invariance of scalar and vector quantities in the

six-dimensional spacetime frame reduce to those in the four-dimensional spacetime frame.

Data Availability Statement

This manuscript has no associated data or the data will not be deposited.

REFERENCES

- [1] Griffiths, D. J., Resource letter EM-1: Electromagnetic momentum, *Am. J. of Phys.*, 80, 7 (2012).
- [2] Einstein, A., On the Electrodynamics of Moving Bodies, *Annalen der Physik*, 322, 10 891 (1905).
- [3] Lorentz, H. A., *Electromagnetic Phenomena in a System Moving with any Velocity less than that of Light*, Proc. Acad. Sci. Amsterdam 6 (1904).
- [4] Jefminko, O. D., On the relativistic invariance of Maxwell's equations, *Z. Naturforsch*, 54a, 637 (1999).
- [5] Redžić, D. V., Comment on 'Maxwell's equations and Lorentz transformations', *Eur. J. Phys.* 43, 068002 (2022).
- [6] Sheng, X. L., Y Li, S Pu, Q Wang, Lorentz transformation in Maxwell equations for slowly moving media, *Symmetry*, 14, 164 (2022).
- [7] Anghinoni, B. G. A. S. Flizikowski, L. C. Malacarne, M. Partanen, S. E. Bialkowski, and N. G. C. Astrath, On the formulation of the electromagnetic stress-energy tensor, *Annals of Physics*, 432 169004 (2022).
- [8] Recami, E. and R. Mignani, *Riv. Nuovo Cimento*, 4, 209 (1974).
- [9] Demers, P., *Canad. J. Phys.* 53, 1687 (1975).
- [10] Mignani, R. and E. Recami, *Lettre al Nuovo Cimento*, 16, 669 (1976).
- [11] Cole, E. A. B., *Nuovo Cimento*, 40A, 171 (1977).
- [12] Dattoli, G. and R. Mingani, *Lettre al Nuovo Cimento*, 22, 65 (1978).

- [13] Pappas, P. T., *Lettere al Nuovo Cimento*, Vol. 22. 15, p. 601-607 (1978); *ibid.* 25, 14 (1979).
- [14] Teli, M. T., *Physics Letters A*, 122, Issue 9, p. 447-450 (1987)
- [15] Guy, B., *Journal of New Energy*, 6, Issue 3, p. 46-71 (2002).
- [16] Franco, A. and Jorge, A. R., *EJTP* 9, 35-64 (2006).
- [17] Ünlü, H., Invariance of electric and magnetic fields in relativistic electrodynamics, Researchsquare.com (2022); <https://doi.org/10.21203/rs.3.rs-543691/v1>.
- [18] Ünlü, H., Special relativity in six dimensions, *Journal of Asian Scientific Research*, Vol 12 (2022) 188-217.
- [19] Kolenkow. R., *An Introduction to Groups and their Matrices for Science Students*, p 168, Cambridge Univ. Press (2022).
- [20] Reitz, J. R., F., J. Milford and R. W. Christy, *Foundations of Electromagnetic Theory* (3rd Ed.) Addison- Wesley Publishing Co. (1979).
- [21] Freeman, J. C., A new investigation into an unresolved mathematical procedure in the 1905 paper on special relativity, *Physics Essays* 31, 3, 310 (2018).