The Transformation of Spacetime Coördinates from Resting to Moving Systems

Phillip Frank, Hermann Rohte

University of Vienna, Austria

Abstract For all equations of transformation which correspond to one-term linear groups, there are three types for which the amount of contraction does not depend upon the direction of motion. Furthermore, only one type has a contraction of lengths as a consequence; namely the Lorentz transformation [Equation (1)]. The two other types, the Galilei and Doppler transformations [Equation (2) -- (129)] the lengths are left unchanged. For the Lorentz transformation the velocity of light in all moving systems has the same value, c. For the Doppler transformation only for propagation in one particular direction occurs when the velocity of light is infinite.

Keywords Lorentz Transformation, Gallilei Transformation, Special Relativity

1. Introduction

The transformation equations which link the spacetime coördinates (x, y, z, t) of a system at rest with those of a system (x', y', z', t') moving with velocity $\vec{q}$ (with constant magnitude and direction) have attained great importance in current physics. Therefore it is worthwhile to examine carefully the physical or other reasonable assumptions that are actually required to obtain the structure of the equations of transformation. According to the theory of relativity they are given by the Lorentz transformation. In familiar notation, if one represents the velocity of light in vacuo by the symbol c and chooses the coördinate systems such that at time $t = 0$ they coincide, and then one of them moves in the x-direction:

$$t' = \frac{1}{\sqrt{1 - q^2/c^2}}(t - q\,x/c^2)$$
$$x' = \frac{1}{\sqrt{1 - q^2/c^2}}(x - q\,t) \tag{1}$$

In the limit $c \to \infty$ [** with $q\,x \ll c^2\,t$ **] we obtain the familiar Galilean transformation:

$$t' = t \qquad x' = x - q\,t$$

The usual standard derivation of equation (1) originates from work of A. Einstein [1], which rests essentially upon the following assumptions:

- α If c is the value of the velocity of light with respect to a system at rest, it must also follow that the velocity of light has that same value with respect to

all systems moving in uniform straight line motion with respect to thee rest system. This assumption is equivalent to the mathematical requirement that the relations (2) follow directly from the equations of transformation.

$$x^2 + y^2 = z^2 - c^2\,t^2 = 0 \;\cdots and\cdots$$
$$x'^2 + y'^2 = z'^2 - c^2\,t'^2 = 0 \tag{2}$$

- β The equations of transformation should be linear and homogeneous in the coördinates; the corresponding coefficients can then depend only on the relative speed q.

- γ If $\vec{q} \to -\vec{q}$, the transform should be mapped into its inverse (that is, Eq. (1) with (x, y, z, t) expressed in terms of (x', y', z', t')).

- δ The contraction factor, $\sqrt{1 - q^2/c^2}$ in Eq. (1), which alters lengths depending on the motion, should depend only on $|\vec{q}|$.

We wish to show that the number of requirements can be considerably reduced, and in particular the one that is presumably the most important assumption. That is, we eliminate α, the requirement that the velocity of light is the same in resting and moving systems.

We can obtain Eq.(1) with only the following assumptions:

- A. The transformation equations labeled by q considered as a variable parameter form a linear, homogenous group, $G(q)$.

- B. The length contraction factor should depend only on the absolute value of q.

The group property of the transformation equations in A must necessarily be required if these equations are to be

valid for all velocities q. If these equations do not form a group, then the composition of the two transformations, that is the transformation to a moving system with the aid of an intermediate system, would lead transformation equations with a structure quite different from that of the original system.

First we will determine the most general transformation consistent with requirement A. By specializing we obtain everything that also satisfies condition B. We obtain either the simple form of the equations which contain no contraction factor, or else the Lorentz transformation (1). The first case is to be identified with a new type of transformation (we call it the Doppler transformation), and obtain the Galilei transformation (2) as a special case.

We have already done a portion of the analysis of these assumptions in the paper [2] "Über eine Allgemeinerung des Relivätsprinzips und die dazu gehörige Mechanik". For example we obain the most general form of the formula for addition of velocities under condition A. [3]

W. v. Ignatowsky [4] has already made an attempt to reduce the Einstein assumptions to a smaller number, If one takes into account explicitly all the tacit assumptions that one has made, one can render the content as follows: he avoids assumption (α) (constancy of the light velocity), retains our assumptions, and also keeps Einstein's assumption (γ). By combining at once all the propositions, he does not obtain the most general transformations satisfying A, and therefore obscures the special position of the Lorentz transformation in this general framework.

This work proceeds as follows. At the outset we make the restrictions

$$y' = y, \quad z' = z$$

because the analysis of the equations is fairly easy, and avoids needless difficulties that might otherwise arise. Therefore we consider only the transformations of x and t.

In Chapter I we briefly review the concepts and rules of transformation groups. [5] In Chapters II and III we apply these results to the transformations specified by condition A. In Chapter IV we introduce a parameter q [** the group parameter **] which has the characteristic of a velocity, and leads immediately to the addition theorem for parallel velocities. In Chapter V we give an example of these developments. In Chapter VI we finally calculate the form of the most general transformation satisfying condition A., and in particular the contraction factor as a function of the velocities considered in IV. Finally, in Chapter VII we impose condition B. and obtain all systems that satisfy our requirements.

2. Chapter I

1. Let t, x, p be three variables, where (t, x) signifies the rectilinear coördinate of a point P in the $t - x$ plane, with p a parameter. Also let

$$\phi(t, p, x) \text{ and } \psi(t, x, p) \tag{3}$$

be two single-valued continuous and differentiable functions of the three arguments [6] for which the Jacobian (functional determinant)

$$\frac{\partial(\phi, \psi)}{\partial(t, x)} \equiv \begin{vmatrix} \dfrac{\partial \phi}{\partial t} & \dfrac{\partial \psi}{\partial t} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \psi}{\partial x} \end{vmatrix} \tag{4}$$

does not vanish identically, nor do the factors

$$\frac{\partial \phi}{\partial p} \equiv 0; \quad \frac{\partial \psi}{\partial p} \equiv 0 \tag{5}$$

vanish identically. From these two equations

$$t' = \phi(t, z, p), \quad x' = \psi(t, x, p) \tag{6}$$

if the parameter p has a fixed value, each ordered pair t, x corresponds to a second pair t', x'. This correspondence is called a transformation, which is represented by T_p, a pointwise mapping of the t, x plane onto the t', x' plane.

In effect, we map a point P: (t, x) into the point P': (t', x'), using the same underlying coördinate system for each of them.

2. The parameter p may pass through all numerical values, or vary over a restricted interval, so that we have an infinite set S of transformations T_p, for which each has a particular value corresponding to the parameter p. This is called a one parameter continuous family of transformations.

Let $T_{p'}$ denote a second transformation in the set S, which belongs to parameter p' and transforms t', x' into t'', x'', so that

$$t'' = \phi(t', x', p') \quad x'' = \psi(t', x', p') \tag{7}$$

We eliminate t', x' from equations (6) and (7), to obtain the equations

$$\begin{aligned} t'' &= \phi(\phi(t, z, p), \psi(t, z, p), p') \\ x'' &= \psi(\phi(t, z, p), \psi(t, z, p), p') \end{aligned} \tag{8}$$

which represent the transformation T which takes t, x directly into t'', x'' and is the product of the two transformations T_p and $T_{p'}$. That is,

$$T = T_p T_{p'} \tag{9}$$

where the ordering of the factors in the product corresponds to the order in which the transforms should be performed. In general,

$$T_{p'} T_p \neq T_p T_{p'} \tag{10}$$

That is, the combination of transformations T does not satisfy the commutative rule

3. The product of two transformations in the set S as represented by q will, in general, not lie in the set S. However, if the product of two arbitrary transformations in

S also lies in S, then the transfomations is S are said to possess the group property. In that case:

$$\left[T_p, T_{p'}\right] T_{p''} = T_p \left[T_{p'}, T_{p''}\right] \qquad (11)$$

That is, the product of three (and arbitrarily many) factors obeys the associative law.

If S is a group; that is, if T belongs to S, Eq. (8) must have the form

$$t'' = \phi(t, x, p''), \quad x'' = \psi(t, x, p'') \qquad (12)$$

where

$$p'' = \pi(p, p') \qquad (13)$$

is a function of p and p' alone,

The transformations in S form a group G when the following conditions are met:

- A. The transformations of G obey the group composition property.
- B. There is a particular parameter $p = p_0$, for which

$$\phi(t, x, p_0) \equiv t, \quad \psi(t, x, p_0) \equiv x \qquad (14)$$

The transformation T_p corresponding to this parameter p_0, which satisfies the equations

$$t' = t, \quad x' = x \qquad (15)$$

leaves the pair t, x unchanged, and is called the identity transformation.

- C. For each transformation T_p there is a second member of S for which in either order of combination there results the identity transformation T_{p0}. These two transformations are called inverse transformations, and we call T_p^{-1} the inverse transformation to T_p, so that

$$T_p T_p^{-1} = T_p^{-1} T_p = T_{p0} \qquad (16)$$

The inverse transformation to T_p is obtained if one solves Eq. (6) for x, t, which is always possible, since the functional determinant never vanishes identically. If the inverse transformation T_p^{-1} to T_p corresponds to the parameter $\bar{p}$, which depends only on p, then we make the identification

$$T_p^{-1} = T_{\bar{p}} \qquad (17)$$

According to (13) the two values p and $\bar{p}$ thereby satisfy the equation.

$$\pi(p, \bar{p}) = p_0 \qquad (18)$$

Then G is a one-parameter group, which consists of an infinite number of transformations.

4. If we consider p in Eq. (18) as a variable that is to be transformed and p' as a parameter (or inversely) then eqn (18) defines a one parameter family of transformations,

labeled by the transformed value p'', which constitute a one-parameter subgroup of G.

5. If δp is an infinitesimal parameter, which corresponds to the parameter

$$p = p_0 + \delta p \qquad (19)$$

it yields a change which is infinitesimally close to the identity transformation.

It is called an infinitesimal transformation of the group which transforms the point $P = (t, x)$ into an infinitely near point P', with these coördinates.

$$t' = t + \delta t, \quad x' = x + \delta x \qquad (20)$$

where

$$\delta t' = \tau(t, x)\delta p, \quad \delta x = \xi(t, x)\delta p \qquad (21)$$

if we set

$$\phi'_p(t, x, p_0) \equiv \tau(t, x), \quad \psi'_p(t, x, p_0) \equiv \xi(t, x) \qquad (22)$$

In the equations (21), which define the infinitesimal transformations, it is permissible to substitute $\kappa \delta p$ for δp, where κ denotes a non-vanishing constant, without in any essential way altering the structure of the group G. If one regards two infinitesimal transformations connected in this way as identical then every one parameter group contains only a single infinitesimal transformation. Inversely, each arbitrary infinitesimal transformation (21) induces a definite one-parameter group. The finite equations (6) can be obtained by integration of the simultaneous system

$$\frac{dt'}{\tau(t', x')} = \frac{dx'}{\xi(t', x')} = dp \qquad (23)$$

with the initial conditions

$$t' = t, \quad x' = x \quad \text{for} \quad p = p_0 \qquad (24)$$

6. Let us now think of x as a function of t

$$x = f(t) \qquad (25)$$

and thereby obtain a curve Γ in the $t - x$ plane. It transforms under (6) into a curve Γ'. that satisfies the relation

$$x' = f_1(t') \qquad (26)$$

If we set

$$w = \frac{dx}{dt} = f'(t), \quad w' = \frac{dx'}{dt'} = f_1'(t) \qquad (27)$$

then

$$w' = \frac{\psi'_t(t, x, p) + \psi'_x(t, x, p)w}{\phi'_t(t, x, p) + \phi'_x(t, x, p)w} \qquad (28)$$

We introduce the abbreviated notation

$$w' = \chi(t, x, w, p) \qquad (29)$$

for the transformation in eq. (6) related to w. For $p = p_0$ we use (14) to obtain

$$\phi'_t(t, x, p_0) = 1, \qquad \phi'_x(t, x, p_0) = 0$$
$$\psi'_t(t, x, p_0) = 0, \qquad \psi'_x(t, x, p_0) = 1 \qquad (30)$$

Therefore

$$\chi(t, x, w, p_0) = w \qquad (31)$$

that is,

$$w' = w \qquad (32)$$

For $p = p_0 + \delta p$ we obtain

$$w' = w_0 + \delta w \qquad (33)$$

and make an infinitesimal transformation of w

$$\delta w = [\xi'_t + (\xi'_x - \tau'_t)w - \tau'_x w^2]\delta p \qquad (34)$$

or, briefly,

$$\delta w = \eta(t, x, w)\delta p \qquad (35)$$

Eqns. (6) and (28) together form a one-parameter group G_1 of transformations, which convert the variables t, x, w into t', x', w'. This group G_1 is called the first extended group. Its infinitesimal transformation is given through Eqns. (21) and (34), and one gets the finite equations (6) and (28) by integration of this system:

$$\frac{dt'}{\tau(t', x')} = \frac{dx'}{\xi(t', x')} = \frac{dw'}{\eta(t', x', w')} = dp \qquad (36)$$

with the initial conditions

$$t' = t, \; x' = x, \; w' = w, \; \text{for} \; p = p_0 \qquad (37)$$

3. Chapter II

7. Let us choose a coördinate system S containing a fixed direction, the x axis, and with a fixed origin O. On the x axis we place observers with a clock at each point.

Let us consider the motion of a material point M along the x-axis, with corresponding readings of the clocks lying along the x-axis, forming the ordered pair t, x, over a particular set of observers. This motion is specified by eqn. (25) and the velocity is given by the first equation in (27).

If we interpret the quantities t, x as coördinates of a point in the t, x plane, for each point in M there is a point P in space-time, which lies in a system S of space-time coördinates. The complete motion of M is represented by a continuous path of space-time coördinates; that is through the curve Γ, eqn. (25), which is called the world-line for this motion. The velocity w at time t is equal in value to the tangent to the world-line at point P. The corresponding motion of M lies along a straight line.

8. Along with the system S we consider with the same straight line another simple infinite set of systems S' (that is, of length and time measurements), each of which has a definite value of the parameter p, such that different values of p correspond to different S' systems. An arbitrary point P in the system S with space-time coördinates t, x has coördinates t', x' in each system S', which is dependent only on t, x and p. That is, the space-time coördinates of P which are, respectively, t, x and t', x' and are connected by equations of the form (6). We call the quantities t', x' the space-time coördinates of P as measured in S'. The infinite number of values of P, lead to an infinite number of values of t', x', which arise from t, x from the one-parameter set S of transformations. [7]

If we compose two transformations of the set S using the transformations (6), we go from system S to a second system S' and then again through equation (7) to a third system S'', so that the transformation (8), which goes directly from S to S'', should also lie in the set S. That is, the set S should possess the group composition property.

Furthermore, we assume that the original system S, which also lies among the systems S', corresponds to the parameter value p_0, so that for $p = p_0$ equation (6) must become equation (15); thus the set S contains the identity transformation.

Finally we require that each transformation also has an inverse in S, so that for each p there is a parameter $\bar{p}$ such that equation (1) is satisfied. Thus the transformations form a one-parameter group G, which we describe as follows:

The transformations (6), which take the original space-time coördinates t, x in the system S into coördinates t', x' in the system S', form a one-parameter group with the parameter p.

9. In order to specify the group G more precisely, we make the following additional assumptions.

A. Each motion which is uniform with respect to a system S at rest is also uniform with respect to a system S' in motion. Therefore when the world line of a curve Γ in S is a straight line, the world line curve Γ' must also be a straight line. That is, the transformations of the group G must generate straight lines from straight lines. The only transformations of this sort are projections, [8] that is, those whose equations (6) are of the following special form:

$$t' = \frac{a_{11}(p)t + a_{12}(p)x + a_{13}(p)}{a_{31}(p)t + a_{32}(p)x + a_{33}(p)}$$
$$x' = \frac{a_{21}(p)t + a_{22}(p)x + a_{23}(p)}{a_{31}(p)t + a_{32}(p)x + a_{33}(p)} \qquad (38)$$

The group G is then called a one-term projective group. [** It is at this point that the explicit expressions which ultimately lead to the Lorentz transformations enter the development of the theory **]

B. Each space-time point which has finite coördinates t, x must also have finite coördinates t', x' in the system S'. It follows [9] that in equation (38)

$$a_{31}(p) \equiv 0 . \quad a_{32}(p) \equiv 0 \qquad (39)$$

If we make the replacements

$$\frac{a_{ik}(p)}{a_{33}(p)} \to a_{ik} \qquad (40)$$

for $i = 1, 2$ and $k = 1, 2, 3$, (38) can be rewritten in the form

$$t' = a_{11}(p)t + a_{12}(p)x + a_{13}(p)$$
$$x' = a_{21}(p)t + a_{22}(p)x + a_{13}(p) \qquad (41)$$

The transformations (41) which leave infinite regions of the t, x plane invariant, are called affine; the group G is then called affine or general linear.

C. Finally the specification of the origin must be the same in all systems, so that if

$$t = 0, \quad x = 0$$

it should follow that

$$t' = 0, \quad x' = 0$$

Therefore

$$a_{13}(p) = 0, \quad a_{23}(p) = 0 \qquad (42)$$

so that equation (41) becomes the following:

$$t' = a_{11}(p)t + a_{12}(p)x, \quad x' = a_{21}(p)t + a_{22}(p)x \qquad (43)$$

These transformations (6) for t', x' are linear homogeneous functions of t, x with coefficients that depend only on the parameter p. The group G is called a one-parameter linear homogeneous group. Its transformations map infinite straight lines in the $t.x$ plane into straight lines and leave the origin invariant. [10]

It is hardly necessary to remark that the coefficients a_{ik} cannot be chosen arbitrarily, but must satisfy certain conditions for the transformations to form a group. Determination of the form of these coefficients is given below.

4. Chapter III

10. We can summarize all of our assumptions concerning the transformations (6) again as follows: The transformations (6) which take the original system S into a system S', form a linear homogeneous group with the single parameter p.

We shall consider Eq. (43) subject to the initial conditions at $p = p_0$, which correspond to the identity transformation:

$$a_{11}(p_0) = 1, \quad a_{12}(p_0) = 0$$
$$a_{21}(p_0) = 0, \quad a_{22}(p_0) = 1 \qquad (44)$$

For $p = p_0 + \delta p$, with δp small, we obtain

$$a_{11}(p_0 + \delta p) = 1 + a'_{11}(p_0)\delta p$$
$$a_{21}(p_0 + \delta p) = a'_{21}(p_0)\delta p$$
$$a_{12}(p_0 + \delta p) = a'_{12}(p_0)\delta p$$
$$a_{22}(p_0 + \delta p) = 1 + a'_{22}(p_0)\delta p \qquad (45)$$

When we define the quantities

$$a'_{11}(p_0) = \alpha_{11}, \quad a'_{12}(p_0) = \alpha_{12}$$
$$a'_{21}(p_0) = \alpha_{21}, \quad a'_{22}(p_0) = \alpha_{22} \qquad (46)$$

Then the equations for an infinitesimal transformation [see eqns. (19)-(22) in 5.] appear in the form

$$\delta t = [\alpha_{11}t + \alpha_{12}x]\delta p \quad \delta x = [\alpha_{21}t + \alpha_{22}x]\delta p \qquad (47)$$

so that for the linear homogeneous group (43) we obtain

$$\tau(t,x) = \alpha_{11}t + \alpha_{12}x \quad \xi(t,x) = \alpha_{21}t + \alpha_{22}x \qquad (48)$$

The coefficients $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ may then be chosen arbitrarily; only their mutual relation is relevant. As a result there are ∞^3 infinitesimal relations (47), and each one generates a particular one-parameter linear homogeneous group (43).

11. If we consider a given transformation in G, with the parameter p held fixed, by differentiation of (43) we obtain the equations

$$dt' = a_{11}(p)dt + a_{12}(p)dx , \quad dx' = a_{21}(p)dt + a_{22}(p)dx \qquad (49)$$

From this it follows that the differentials dt, dx transform just like the finite quantities t, x; thus the two pairs t, x and dt, dx are co-gradient systems.

From Eqn (49) it follows that

$$\frac{dx'}{dt'} = \frac{a_{21}(p)dt + a_{22}(p)dx}{a_{12}(p)dt + a_{12}(p)dx} \qquad (50)$$

so that from (27)

$$w' = \frac{a_{21}(p) + a_{22}w}{a_{11}(p) + a_{12}w} \qquad (51)$$

This equation, which gives the transformation of velocity w into w', now replaces eq. (28) and represents Equation (43), the first extended group G_1. Of special importance is the situation in which the linear group requires that w' is a function only of w and p, and not dependent on t and x. From (34) and (48) we obtain the infinitesimal transformation of velocity as

$$\delta w = -[-\alpha_{21} + (\alpha_{11} - \alpha_{12})w + \alpha_{12}w^2]\delta p \qquad (52)$$

which implies that the function $\eta(t,x,w)$ in (35) is independent of t and x. This infinitesimal transformation changes the velocity so that w goes into $w' = w + \delta w$. It remains unchanged only when

$$\delta w = 0 \qquad (53)$$

This occurs for velocities w which are roots of the quadratic equation

$$-\alpha_{21} + (\alpha_{11} - \alpha_{22})w + \alpha_{12}w^2 = 0 \qquad (54)$$

They remain the same under an infinitesimal transformation, and we will verify directly that [see the conclusion of No. 12.] that each finite transformation also leaves the group G unchanged. In the succeeding we shall call these the exceptional velocities.

Note that the case of rest ($w = 0$) is not an exceptional velocity, which means that necessarily

$$\alpha_{21} \neq 0 \qquad (55)$$

Because of condition (55) the case

$$\alpha_{11} = \alpha_{22}, \quad \alpha_{12} = 0 = \alpha_{21}$$

in which (54) is identically satisfied, implies the possibility that every velocity w is an exceptional velocity, is excluded. In every other case we have two exceptional velocities, which we call c_1 and c_2; namely

$$c_1 = \frac{\alpha_{22} - \alpha_{11} + \sqrt{\theta}}{2\alpha_{12}} \qquad c_2 = \frac{\alpha_{22} - \alpha_{11} - \sqrt{\theta}}{2\alpha_{12}} \qquad (56)$$

where

$$\theta = (\alpha_{11} - \alpha_{22})^2 - 4\alpha_{12}\alpha_{21} \qquad (57)$$

The symmetric combinations of the roots c_1 and c_2 are

$$c_1 + c_2 = \frac{\alpha_{22} - \alpha_{11}}{\alpha_{12}} \qquad c_1 c_2 = -\frac{\alpha_{21}}{\alpha_{12}} \qquad (58)$$

and with these relations the infinitesimal transformation (52) can be brought into the form

$$\delta w = \alpha_{21} \left(1 - \frac{w}{c_1}\right) \left(1 - \frac{w}{c_2}\right) \delta p \qquad (59)$$

in which the significance of c_1 and c_2 as the exceptional velocities is immediately apparent.

5. Chapter IV

12. In order to obtain the finite forms (48) and (51) of the extended group G_1, which is generated by the infinitesimal transformations (47) and (52), we proceed to integrate the simultaneous system (36)

$$\frac{dt'}{\alpha_{11}t' + \alpha_{12}x'} = \frac{dx'}{\alpha_{21}t' + \alpha_{22}x'}$$
$$= \frac{-dw'}{-\alpha_{21} + (\alpha_{11} - \alpha_{22})w' + \alpha_{12}w'^2} = dp \qquad (60)$$

with the initial conditions (37). In the interim we shall apply this method to Eqn. (51) for the velocity w in order to determine its transformation. It will be useful to note that w' depends only on w and p but is independent of t and x. This implies that in (60) we can separately consider the equation

$$\frac{dw'}{-[-\alpha_{21} + (\alpha_{11} - \alpha_{22})w' + \alpha_{12}w'^2]} = dp \qquad (61)$$

with the initial conditions

$$w' = w \text{ at } p = p_0 \qquad (62)$$

The direct integration, then has the form

$$\int_w^{w'} \frac{dw'}{-[-\alpha_{21} + (\alpha_{11} - \alpha_{22})w' + \alpha_{12}w'^2]} = \int_{p_0}^p dp \qquad (63)$$

In evaluating the integral on the l.h.s. we take into account that the two exceptional velocities c_1 and c_2 are the zero points of the denominator. The result can be written the form

$$\frac{1}{\sqrt{\theta}} \log d(c_1, c_2, w, w') = p - p_0 \qquad (64)$$

where $d(c_1, c_2, w, w')$ represents the double relation of the four values of c_1, c_2, w, w'; i.e the expression

$$d(c_1, c_2, w, w') = \frac{(c_1 - w)(c_2 - w')}{(c_1 - w')(c_2 - w)} \qquad (65)$$

According to (61) we finally obtain

$$d(c_1, c_2, w, w') = e^{\sqrt{\theta}(p - p_0)} \qquad (66)$$

Let us solve for w' to obtain

$$w' = \frac{c_1 c_2 [1 - e^{\sqrt{\theta}(p - p_0)}] - [c_2 - c_1 e^{\sqrt{\theta}(p - p_0)}]w}{[c_1 - c_2 e^{\sqrt{\theta}(p - p_0)}] - [1 - e^{\sqrt{\theta}(p - p_0)}]w} \qquad (67)$$

which is the finite equation for the transformation of the velocity [compare (25) and (51)]. Furthermore we can set c_1 and c_2 equal to their values in (56). [11]

Observe that in equation (67) the two exceptional values c_1 and c_2 are indeed unchanged, and thus have the same value in every system obtained by the finite group transformation. If we replace $w = [c_1 \text{ or } c_2]$, respectively, it follows that $w' = [c_1 \text{ or } c_2] = w$ for every value of the parameter p.

13. The systems S', which we introduced in II.8, are to be connected to the original system S, which we described as at rest, by motions with constant velocity q. Then since the system S' corresponds to a particular parameter p as well as to a definite velocity q, there must be a relation between p and q, which we write as

$$p = F(q) \qquad (68)$$

so that the parameter p appears as a function of the velocity q.

If we use the velocity q instead of the parameter p in equations (43) and (51) in virtue of (68), the group G_1 is not changed essentially when q is considered as the new group parameter.

To define the parameter q and also to determine the form of the function $F(q)$, we make the following requirement:

If the point M is moving with respect to the resting system S with velocity q, since the system S' also moves with velocity q with respect to S, then M has velocity $w' = 0$ in S'.

Thus we require that the pair

$$w = q \quad w' = 0 \qquad (69)$$

should obey equation (64), so that

$$p - p_0 = \frac{1}{\sqrt{\theta}} \log d(c_1, c_2, q, 0) \qquad (70)$$

This yields the explicit form of the function $F(q)$

$$p = F(q) = p_0 + \frac{1}{\sqrt{\theta}} \log \frac{c_2(c_1 - q)}{c_1(c_2 - q)} \qquad (71)$$

We use this expression in (67) to obtain the equation of transformation of the velocity w as

$$w' = \frac{c_1 c_2 (w - q)}{c_1 c_2 - (c_1 + c_2)q + qw} \qquad (72)$$

or, according to (58)

$$w' = \frac{-\alpha_{21}(w - q)}{-\alpha_{21} + (\alpha_{11} - \alpha_{22})q + \alpha_{12}qw} \qquad (73)$$

Furthermore, it follows from (71) that the parameter p_0 as the identity transformation corresponds to velocity $q = 0$. The system S at rest therefore singled out among the systems S', as the one which moves with velocity $q = 0$.

14. Let us from now on consider the velocity q as the group parameter, and set

$$\alpha_{ik}(F(q)) = b_{ik}(q) \quad (i, k = 1, 2) \qquad (74)$$

From (43) and (51) we obtain the equations

$$t' = b_{11}(q)t + b_{12}(q)x, \quad x' = b_{21}(q)t + b_{22}(q)x \qquad (43a)$$

and

$$w' = \frac{b_{21}(q) + b_{22}(q)w}{b_{11}(q) + b_{12}(q)w} \qquad (51a)$$

which now defines the group G_1. If we set $w = q$ in (51a) then necessarily $w' = 0$. Moreover this identity follows for every value of q:

$$b_{21}(q) + b_{22}(q)q = 0 \qquad (75)$$

If we now consider the new equations (43a) and (51a) as the basis for the group G_1, then the value $q = 0$ yields the identity and the value $q = \delta q$ the corresponding infinitesimal transformation. The equivalent result can be regarded as following from Eqn. (47) because the introduction of the new parameters q can alter the values of the coefficients α_{ik} but not their mutual relations -- and only these relations are relevant.

With our normalization of the group parameters, the coefficients α_{ik} acquire specific values, whereas previously only their relative behavior was fixed. In fact it follows from (45) and (46) that

$$b_{21}(\delta q) = \alpha_{21}\delta q, \quad b_{22}(\delta q) = 1 + \alpha_{22}\delta q$$

and therefore if we use the identity (75), with $q = \delta q$, and neglect terms of second order we obtain

$$(\alpha_{21} + 1)\delta q = 0$$

which means that

$$\alpha_{21} = -1 \qquad (76)$$

since $\delta q \neq 0$. Thereby, the coefficient α_{21}, which was previously constrained only by the inequality (55), is determined exactly.

According to (44) and (46) we obtain for the new coefficients b_{ik} in (43a) and (51a) the relations

$$\begin{aligned} b_{11}(0) &= 1, \quad b_{12}(0) = 0 \\ b_{21}(0) &= 1, \quad b_{22}(0) = 1 \end{aligned} \qquad (44a)$$

and

$$\begin{aligned} b'_{11}(0) &= \alpha_{11}, \quad b'_{12}(0) = \alpha_{12} \\ b'_{21}(0) &= -1, \quad b'_{22}(0) = \alpha_{22} \end{aligned} \qquad (46a)$$

Finally we insert the value (76) in equations (47) and (52) we obtain for the infinitesimal transformations of the group G

$$\delta t = (\alpha_{11}t + \alpha_{21}x)\,\delta q, \quad \delta x = (-t + \alpha_{22}x)\,\delta q \qquad (47a)$$

and for the infinitesimal transformation of velocity w

$$\delta w = -[1 + (\alpha_{11} - \alpha_{22})w + \alpha_{12}w^2]\delta q \qquad (52a)$$

or, from (59)

$$\delta w = -[1 - \frac{w}{c_1}][1 - \frac{w}{c_2}]\,\delta q \qquad (59a)$$

The finite equation (73) for the velocity becomes

$$w' = \frac{w - q}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}qw} \qquad (73a)$$

and from (57) directly

$$\theta = (\alpha_{11} - \alpha_{22})^2 - 4\alpha_{12} \qquad (57a)$$

15. When we consider (73a) with the parameter q and a second transformation of the same sort involving q', we obtain

$$w'' = \frac{w' - q'}{1 + (\alpha_{11} - \alpha_{22})q' + \alpha_{12}q'w'} \qquad (77)$$

From the group property of our transformations, it follows that w'' must be generated from w with some value q'' of the group parameter; that is

$$w'' = \frac{w - q''}{1 + (\alpha_{11} - \alpha_{22})q'' + \alpha_{12}q''w} \qquad (78)$$

where the parameter q'' is a function of q and q'.

In order to determine this function, it is only necessary to write the composition of (73a) and (77) explicitly

$$w'' = \frac{\dfrac{w-q}{1+(\alpha_{11}-\alpha_{22})q+\alpha_{12}qw}-q'}{1+(\alpha_{11}-\alpha_{22})q'+\dfrac{\alpha_{12}q'(w-q)}{1+(\alpha_{11}-\alpha_{22})q+\alpha_{12}qw}}$$

or

$$w'' =$$

$$\frac{w-\dfrac{q+q'+(\alpha_{11}-\alpha_{22})qq'}{1-\alpha_{12}qq'}}{1+(\alpha_{11}-\alpha_{22})\dfrac{q+q'+(\alpha_{11}-\alpha_{22})qq'}{1-\alpha_{12}qq'}+\alpha_{12}\dfrac{q+q'+(\alpha_{11}-\alpha_{22})qq'}{1-\alpha_{12}qq'}w}$$

$$(79)$$

By comparison with (78) it follows then

$$q'' = \frac{q+q'+(\alpha_{11}-\alpha_{22})qq'}{1-\alpha_{12}qq'} \qquad (80)$$

This equation, through which the parameter group P of our groups G and G_1 is given, [12] expresses the addition theorem of group velocities for the system S'', which moves with velocity q' with respect to S', which itself moves with velocity q with respect to system S.

If equation (77) is taken to represent the inverse transformation, then

$$w'' = w$$

so that the resulting transformation (78) should represent the identity, and $q'' = 0$. It follows from (80) that if we denote the parameter value of the inverse transformation by $\bar{q}$, we obtain

$$q+\tilde{q}+(\alpha_{11}-\alpha_{22})q\tilde{q}=0 \qquad (81)$$

so that

$$\tilde{q} = \frac{-q}{1+(\alpha_1-\alpha_{22})q} \qquad (82)$$

If we put this value in place of q' in (72), the inverse equation for (77) becomes

$$w = \frac{q+w'+(\alpha_{11}-\alpha_{22})qw'}{1-\alpha_{12}qw'} \qquad (83)$$

as one can also verify directly by solving (78a) for ω.

The formula (83) shows that w obtained from q and w' in exactly the same way as is obtained for q'' from q and q' -- an analogy that clarifies the kinematic meaning of equations (80) and (83).

6. Chapter V

16. Before we consider the general form for the finite group G, we shall exhibit as examples of the preceding development the two special one parameter linear homogeneous groups (2) and (1) mentioned in the introduction: these are the Galilei and Lorentz transformation groups.

The coefficients b_{ik} of group (2), the Galilei transformations, are

$$b_{11}(q)=1, \qquad b_{12}(q)=0$$
$$b_{21}(q)=-q, \qquad b_{22}(q)=1 \qquad (84)$$

from which it follows that at $q=0$ they reduce to

$$b_{11}(0)=1, \quad b_{12}(0)=0$$
$$b_{21}(0)=0, \quad b_{22}(0)=1 \qquad (44b)$$

in agreement with (44a). Furthermore, it follows from (84) that

$$b'_{11}(q)=0, \qquad b'_{12}(q)=0$$
$$b'_{21}(q)=-1, \qquad b'_{22}(q)=0 \qquad (85)$$

and therefore from (46a) that

$$\alpha_{11}=b'_{11}(0)=0, \qquad \alpha_{12}=b'_{12}(0)=0$$
$$\alpha_{21}=b'_{21}(0)=-1, \qquad \alpha_{22}=b'_{22}(0)=0 \qquad (46a)$$

so that equation (76) is satisfied. For the infinitesimal transformation (47a) and (52a) we obtain

$$\delta t = 0 \qquad \delta x = -t\delta q \qquad (47b)$$

and

$$\delta w = -\delta q \qquad (52b)$$

Finally from (57a) we obtain

$$\theta = 0 \qquad (57b)$$

so that the two exceptional velocities are equal to one another, and in particular

$$c_1 = c_2 = \infty \qquad (86)$$

while the finite transformation (73a) of velocities becomes

$$w' = w-q \qquad (78b)$$

For the group (1) of Lorentz transformations, the coefficients b_{ik} are given by

$$b_{11}(q)=\frac{1}{\sqrt{1-q^2/c^2}}, \qquad b_{12}(q)=\frac{-q/c^2}{\sqrt{1-q^2/c^2}}$$
$$b_{21}(q)=\frac{-q}{\sqrt{1-q^2/c^2}}, \qquad b_{22}(q)=\frac{1}{\sqrt{1-q^2/c^2}} \qquad (87)$$

where for $q=0$ they again lead to equations (44a). For the derivatives $b'_{ik}(q)$ we find

$$b'_{11}(q)=\frac{q/c^2}{(1-q^2/c^2)^{3/2}}, \qquad b'_{12}(q)=\frac{-1/c^2}{1-q^2/c^2)^{3/2}}$$
$$b'_{21}(q)=\frac{-1}{1-q^2/c^2)^{3/2}}, \qquad b'_{22}(q)=\frac{q/c^2}{1-q^2/c^2)^{3/2}} \qquad (88)$$

and from this it follows that (46a) implies that

$$\alpha_{11}=b'_{11}(0)=0, \qquad \alpha_{12}=b'_{12}(0)=-1/q^2$$
$$\alpha_{21}=b'_{21}(0)=-1, \qquad \alpha_{22}=b'_{22}(0)=0 \qquad (46c)$$

so that equation (76) is also satisfied. Equations (47a) and (52a) yield the infinitesimal transformations

$$\delta t = -\frac{x}{q^2}\delta q \; ; \delta x = -t\delta q \tag{47c}$$

and

$$\delta w = -[1 - \frac{w^2}{q^2}]\delta q \tag{52c}$$

so that for the exceptional velocities for c_1 and c_2 we obain the values

$$c_1 = -c, \quad c_2 = +c \tag{89}$$

Now from (57a)

$$\theta = \frac{4}{c^2} \tag{57c}$$

and therefore we finally obtain from (73a) the finite transformation of velocity w in the form

$$w' = \frac{w - q}{1 - qw/c^2} \tag{73a}$$

7. Chapter VI

17. Now let us proceed to set up the general equations for the one parameter linear homogeneous group G; that is, the determination of the coefficients $b_{ik}(q)$ in (43a).

From equivalence of the two equations (51a) and (73a), it follows from their consistency that the four coefficients

$$\begin{matrix} b_{11}(q), & b_{12}(q) \\ b_{21}(q), & b_{22}(q) \end{matrix} \tag{90}$$

and the four quantities

$$\begin{matrix} 1 + (\alpha_{11} - \alpha 22q, & \alpha_{12}q \\ -q, & 1 \end{matrix} \tag{91}$$

must be proportional. Moreover the proportionality factor can depend only upon q. We write it as $\Omega(q)$, [** In the succeeding we write $\Omega(q)$ in many appropriate places in the text; this replaces $\omega(q)$ which through oversight appears in many expressions **] so that

$$\begin{matrix} b_{11}(q) \equiv \Omega(q)[1 + (\alpha_{11} - \alpha_{22})q], & b_{12}(q) \equiv \Omega(q)\alpha_{12}q \\ b_{21}(q) \equiv \Omega(q)[-q], & b_{22}(q) \equiv \Omega(q) \end{matrix} \tag{92}$$

where in addition the identity (75) is satisfied. By inserting the values (92) into equation (43a), we obtain

$$t' = \Omega(q)[(1 + (\alpha_{11} - \alpha_{22}q)t + \alpha_{12}qx] \\ x' = \Omega(q)[-qt + x] \tag{93}$$

where the function $\Omega(q)$ is unknown.

18. The kinematic interpretation of the factor $\Omega(q)$ can be inferred by means of equation (93), but we still have to determine its form. If we consider a point M which moves along the x-axis with constant velocity w and is located at position $x = a$ at time $t = 0$, its motion is described by the equation

$$x = a + wt \tag{94}$$

In order to determine the equation of motion of M with respect to a system S' moving with velocity q with respect to system S, let us solve equation (93) for t and x, to obtain

$$t = \frac{t' - \alpha_{11}qx'}{\Omega(q)[1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2]} \\ x = \frac{qt' + [1 - \alpha_{11} - \alpha_{22})x']}{\Omega(q)[1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2]} \tag{95}$$

for the inverse transformation of (92), and insert the expressions (95) into (94). We thereby obtain

$$qt' + [1 + (\alpha_{11} - \alpha_{22})q]x' \\ = a[1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2]\Omega(q) + wt' - \alpha_{12}qwx'$$

Therefore, when we solve this equation for x'

$$x' = a\frac{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}qw}\Omega(q) \\ + \frac{w - q}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}qw}t' \tag{96}$$

or

$$x' = a' + w't' \tag{97}$$

where

$$a' = a\frac{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}qw}\Omega(q) \tag{98}$$

is the value of x' at $t = 0$ and w' is the velocity of M in the system S', as given by (73a).

Let us consider two points M_1 and M_2 which have space time coördinates t_1, x_1 and t_2, x_2 in the resting system S, and which move with the same constant velocity w. At time

$$t_1 = t_2 = 0$$

the locations of M_1 and M_2 are given by

$$x_1 = a_1, \quad x_2 = a_2$$

so that the equations of motion of these two points with respect to S are

$$x_1 = a_1 + wt_1, \quad x_2 = a_2 + wt_2 \tag{99}$$

In the system S' moving with velocity q with respect to S, the equations of motion are

$$x'_1 = a'_1 + w't'_1, \quad x'_2 = a'_2 + w't'_2 \tag{100}$$

where t'_1, x'_1 and t'_2, x'_2 are the space-time coordinates of M_1 and M_2 in the system S'. Furthermore, from (98),

$$a_1' = a_1 \frac{1+(\alpha_{11}-\alpha_{22})q+\alpha_{12}q^2}{1+(\alpha_{11}-\alpha_{22})q+\alpha_{12}qw}\Omega(q)$$

$$a_2' = a_2 \frac{1+(\alpha_{11}-\alpha_{22})q+\alpha_{12}q^2}{1+(\alpha_{11}-\alpha_{22})q+\alpha_{12}qw}\Omega(q)$$

(101)

where w' is the velocity of the points in the system S', as given by (73a).

The two points M_1 and M_2, which move with the same velocity w along the x-axis, can be thought of as end points of a rigid rod, with length ℓ in the S system. [** The measurement of lengths is entirely compatible with the principles of special relativity; however the idealization of the existence of perfectly rigid bodies leads to paradoxes, and therefore these models of physical objects must be excluded. **] Therefore at equal times

$$t_1 = t_2$$

subtracting these two equations in (99), we obtain

$$\ell = x_2 - x_1 = a_2 - a_1 \qquad (102)$$

We can determine the length of the rigid rod ℓ', as measured in the system S', when at the same times

$$t_1 = t_2$$

we again obtain by subtracting the corresponding equations

$$\ell' = x_2' - x_1' = a'2 - a_1' \qquad (103)$$

Then from (101) and (102)

$$\ell' = \frac{1+(\alpha_{11}-\alpha_{22})q+\alpha_{12}q^2}{1+(\alpha_{11}-\alpha_{22})q+\alpha_{12}qw}\Omega(q)\,\ell \qquad (104)$$

Finally if we also assume that the slab is at rest in the system S'', that $w'=0$; then in accordance with (69) it moves in the system S with velocity $w=q$. It follows from (104) that

$$\ell' = \Omega(q)\,\ell \qquad (105)$$

In other words the function   is the factor with which the length ℓ of a rod at rest in system S must be multiplied to yield its length ℓ' in the system S', with respect to which it is also at rest, when S' is moving with velocity $w=q$ relative to S. The factor $\Omega(q)$ is called the contraction factor.

19. In order to determine the specific analytical form of $\Omega(q)$, we use the parameter q in ther transformation (93) which converts the pair t,x into the pair t',x', together with a second transformation from the group G:

$$t'' = \Omega(q')\{[(1+(\alpha_{11}-\alpha_{22})q']t' + \alpha_{12}q'x'\}$$
$$x'' = \Omega(q')\{-q't' + x'\}$$

(106)

This transform converts t',x' into t'',x'' using the parameter q'. From the group multiplication property of the transformations, it follows that the resulting transformation, which converts t,x into t'',x'' has the form

$$t'' = \Omega(q'')\{[(1+(\alpha_{11}-\alpha_{22})q'']t + \alpha_{12}q''x\}$$
$$x'' = \Omega(q'')\{-q''t + x\}$$

(107)

where the parameter q'' is is given by equation (80) as a function of q and q'.

If we actually join the two equations (93) and (106) together, and use equation (80), we obtain

$$t'' = (1-\alpha_{12}qq')\Omega(q)\Omega(q')\{[(1+(\alpha_{11}-\alpha_{22})q'']t + \alpha_{12}q''x\}$$
$$x'' = (1-\alpha_{12}qq')\Omega(q)\Omega(q')\{-q''t + x\}$$

(108)

by comparison with (107) this implies

$$\Omega(q'') = (1-\alpha_{12}qq')\,\Omega(q)\,\Omega(q') \qquad (109)$$

Therefore from (80):

$$\Omega\{\frac{q+q'+(\alpha_{11}-\alpha_{22})qq'}{1-\alpha_{12}qq'}\} = (1-\alpha_{12}qq')\Omega(q)\Omega(q') \quad (110)$$

This is a functional equation, through which it is possible to determine the structure of the function $\Omega(q)$. For this purpose we differentiate (110) with respect to q' and then set $q'=0$, to obtain

$$\Omega'(q)[1+(\alpha_{11}-\alpha_{22})q+\alpha_{12}q^2] = \Omega(q)[\Omega'(0)-\alpha_{12}\Omega(0)q] \quad (111)$$

But from the last equation in (92)

$$\Omega(q) = b_{22}(q)$$

so that the contraction factor $\Omega(q)$ according to (44a) and (46a) also satisfies the conditions

$$\Omega(0) = b_{22}(0) = 1 \qquad (112)$$

and

$$\Omega'(0) = b_{22}(0) = \alpha_{22} \qquad (113)$$

by means of which we obtain the differential equation

$$\Omega'(q)[1+(\alpha_{11}-\alpha_{22})q+\alpha_{21}q^2] = \Omega(q)\,[\alpha_{22}-\alpha_{12}q] \quad (114)$$

with the initial conditions (112). It follows from (114) that

$$\frac{\Omega'(q)}{\Omega(q)} = \frac{\alpha_{22}-\alpha_{12}q}{[1+(\alpha_{11}-\alpha_{22})q+\alpha_{12}q^2]} \qquad (115)$$

and therefore

$$\int_0^q \frac{\Omega'(q)}{\Omega(q)}dq = \int_0^q \frac{\alpha_{22}-\alpha_{12}q}{[1+(\alpha_{11}-\alpha_{22})q+\alpha_{12}q^2]}dq \quad (116)$$

If we calculate the integrals on both sides of (116) and solve the resulting expression for $\Omega(q)$, we find the following expression for the contraction:

$$\Omega(q)$$

$$= \frac{1}{\sqrt{1+(\alpha_{11}-\alpha_{22})q+\alpha_{12}q^2}}\left[\frac{1+\dfrac{1+\alpha_{11}-\alpha_{22}+\sqrt{\theta}}{2}q}{1+\dfrac{1+\alpha_{11}-\alpha_{22}-\sqrt{\theta}}{2}q}\right]^{\frac{\alpha_{11}+\alpha_{22}}{2\sqrt{\theta}}}$$

(117)

which in fact also satisfies condition (113).

Thereby the final equations (98) for the general one parameter linear group, which through the infinitesimal transformation (47) is generated under conditions (55), is completely determined.

20. For the Galilei group one finds from (46b)

$$\Omega(q) = 1 \tag{117a}$$

and for the Lorentz group from (46c)

$$\Omega(q) = \frac{1}{\sqrt{1 + \alpha_{11}q^2}} = \frac{1}{\sqrt{1 - q^2/c^2}} \tag{117b}$$

in agreement with equations (2) and (1).

8. Chapter VII

21. Let us now proceed to analyze and investigate the constraints imposed by condition B., where the three continuous parameter groups which, through equations (93) and (117) give the contraction $\Omega(q)$, result in an even function of the velocity q; that is, properties which are independent of the direction of the x-axis.

For this situation it is necessary and sufficient that the odd order derivatives of Ωq vanish at $q = 0$; in particular

$$\left(\frac{d\Omega}{dq}\right)_{q=0} = \Omega'(0) = 0, \quad \left(\frac{d^3\Omega}{dq^3}\right)_{q=0} = \Omega'''(0) = 0 \tag{118}$$

We shall calculate the quantities $\Omega(0), \Omega'(0), \Omega''(0), \Omega'''(0)$. The first two are given through equations (112) and (113); we obtain the rest most simply by repeated differentiation of equation (114). The results, when we set $q = 0$, are

$$\Omega''(0) + (\alpha_{11} - 2\alpha_{22})\,\Omega'(0) + \alpha_{12}\,\Omega(0) = 0 \tag{119}$$

$$\Omega'''(0) + (2\alpha_{11} - 3\alpha_{22})\,\Omega''(0) + 4\alpha_{12}\,\Omega'(0) = 0 \tag{120}$$

It follows that

$$\Omega''(0) = -\alpha_{12} - \alpha_{22}(\alpha_{11} - 2\alpha_{22}) \tag{121}$$

and

$$\Omega'''(0) = 2\alpha_{11}\alpha_{12} + \alpha_{22}(2\alpha_{11}^2 - 7\alpha_{12} - 7\alpha_{11}\alpha_{22} + 6\alpha_{22}^2) \tag{122}$$

It follows from the first equation (118), taking into account equation (113) that

$$\alpha_{22} = 0 \tag{123}$$

and then from (122) that

$$\alpha_{11}\alpha_{12} = 0 \tag{124}$$

Equations (123) and (124) must be necessarily satisfied if the transformation equations for the contraction should obey requirement B. We will soon see that the requirements (123) and (124) are also sufficient conditions

22. There are three sub-cases that satisfy equation (124):

$$1.\,\alpha_{11} = 0\,, \alpha_{12} = 0 \tag{125a}$$

$$2.\,\alpha_{11} = 0\,, \alpha_{12} \neq 0 \tag{125b}$$

$$3.\,\alpha_{11} \neq 0\,, \alpha_{12} = 0 \tag{125c}$$

Each of these sub-cases corresponds to a particular type of transformation equation which satisfies the basic conditions A. and B.

For the first alternative, using equation (98) and taking into account the connection together with (117) and (117A), gives the Galilei transformation. The second alternative, as seen from equation (93) with (117) and (117A), gives the group of Lorentz transformations.

23. The third alternative leads to a group which has not yet been considered. It follows from equation (117) together with (123) and (125c) that

$$\Omega(q) = 1 \tag{126}$$

and from equation (93) that

$$\begin{aligned} t' &= (1 + \alpha_{11}q)t \\ x' &= -qt + x \end{aligned} \tag{127}$$

The exceptional velocities have the values

$$c_1 = -\frac{1}{\alpha_{11}}; \quad c_2 = \infty \tag{128}$$

since these are the roots of the quadratic equation (54), where the coefficients obey the conditions (76), (123), and (125c). The transformation (127) then has the form

$$\begin{aligned} t' &= (1 - \frac{q}{c_1})t \\ x' &= -qt + x \end{aligned} \tag{129}$$

The transformation of time in these equations can be given a physical interpretation, just as Einstein has given for the Lorentz transformation.

At time $t = 0$ a light ray originating at the origin and moving in the positive x-direction moves with velocity c_1. When a body moves with velocity q, then light moves with velocity $c_1 - q$ in the resting system. If we now wish the velocity of the light ray with respect to the time of the moving system is also c_1, we can accomplish this by changing the rate of the clock in a ratio proportional to the fraction $c_1 / (c_1 - q)$, so that for the moving body the time is t', given by

$$t' = \frac{c_1 - q}{c_1}t$$

as is given by the first equation (129). This time adjustment corresponds to the Doppler principle; and therefore we will call the transformation (129) the Doppler transformation.

The Doppler transformation is essentially different from the Lorentz transformation, in that for bodies moving with velocity q the same time prevails at all points; there is no space-time variation for them, and what is more important:

if we restrict the direction of production of a light ray to the x-direction, then all bodies moving in this direction possess the same light velocity c_1 (we take $c_1 > 0$); consequently light rays produced in the negative x-direction are not in alignment with all moving bodies.

When c_1 is an exceptional velocity, then c_2 is not one as well. According to equation (128) this would only be true for $c_1 = \infty$, or $\alpha_{11} = 0$, but then we would recover the Galilei transformation.

However, in the case of Lorentz transformation, because of equation (56) together with equations (76), (123), and (125b). we obtain

$$c_1 = -c_2 = c \ [see\ also\ equation\ (89)]$$

We can summarize the results of our investigation as follows:

Of all transformation equations that correspond to a one-parameter linear homogeneous group, there are three types for which the behavior of contraction does not depend upon the direction of motion in absolute space. Only one of these types yields a meaningful contraction of length; namely the Lorentz transformation [equation (1)]. The two other types, the Galilei transformation and the Doppler transformation [equation (2); especially (129)] leave the lengths unchanged. In the Lorentz transformation the velocity of light has the same value c for all systems moving in arbitrary directions. For the Doppler transformation this occurs only when the transformation lies in one direction; for the Galilei transformation only when the velocity of light is infinite.

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Translators Comments

The following article is translated from the original German text:

"Uber die Transformation der Raumzeitkoordinaten \\ von ruhenden auf bewegte Systeme" von Philipp Frank und Hermann Rothe. Ann. d. Phys. Vol 34 (series 4) 1911. pp 825-855.

This paper was of special importance in understanding the Lorentz transformation in the context of the special theory of relativity. Namely, only these two requirements were made:

(1) inertial invariance and
(2) that the product of two boosts in a given direction yielded a boost in the same direction.

It was shown that there are only two (consistent) possibilities: (a) a Galilean Transform or (b) a Lorentz Transform. It was not necessary to assume that the velocity of light is identical in all inertial frames. Note: deviations in translation are marked by [** **].

Thomas Erber and Porter Johnson*; Department of Physics; Illinois Institute of Technology; Chicago, Illinois, USA.

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[2]   Ph. Frank and H. Rothe, Sitzungberichte d. k. Acad. d. Wiss. in Wien, Kl. 119, Abt. IIsa, p. 615 ff, 1910.

[3]   Previous reference, equation (12) and the following unnumbered equation.

[4]   W. v. Ignatowsky, Berichte d. Deutsch. Phys. Ges. p. 788 ff and Arch. f. Math. u. Phys, 17 p. 1 ff 1910.

[5]   We have consistently employed the elementary theory of group representations as described by S. Lie and G. Scheffers, Lectures on Continuous Groups (1898) Leipzig.

[6]   The three variables t, x, p must necessarily be of limited ranges over the manifold being considered, which belongs with the coördinate system t, x, p.

[7]   The remainder of Chapter II is unnecessary for understanding this work, and serves only to make condition A. plausible.

[8]   S. Lie and G. Scheffers, loc cit p 32, Theorem 2.

[9]   Loc cit, p.57-58, Law 11.

[10]   S. Lie and G. Scheffers, loc cit p. 184.

[11]   Note that equations (64), (66), and (67) are independent of the $\pm$ sign in front of $\sqrt{\theta}$. Namely, if we replace $\sqrt{\theta}$ by $-\sqrt{\theta}$ in (56), the two invariant velocities c_1 and c_2 are exchanged, and the equations remain unchanged.

[1]   If one considers the original parameter p of the group, which for example appears in (67), then equation (13) gives the group parameter $p'' = p + p' + p_0$.