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Periodic Solutions of the Dirac-Lorentz Equation

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Abstract In a previous paper we have derived a new form of the radiation term without changing the Dirac physical assumptions. We have showed also that the fourth Dirac equation is a consequence of the first three ones, that implies the Dirac system is not overdetermined – three equations for three unknown functions. Here we replace the Dirac local expansions with nonlocal formulation of the problem. So we have obtained a system of first order neutral differential equations with respect to the unknown velocities containing both retarded and advanced arguments. Since the accelerator theory relies on the Dirac-Lorentz equation the obtained periodic solutions can be applied directly to the study of betatron equation.

Keywords Dirac-Lorentz equation, Fixed point method, Periodic operator, Periodic solution

1. Introduction

In a previous paper [1] we have derived a general form of the Dirac radiation term [2], [3] based on his original physical assumptions (cf. [3]). In the relativistic case the usually accepted radiation term leads to the well-known Dirac (or Lorentz-Dirac) equations [3]

$$m\ddot{x}_r = \frac{e}{c}F_{rn}\dot{x}_n + \frac{2}{3}(\dddot{x}_r - \frac{1}{c^2}\ddot{x}_n\ddot{x}_n\dot{x}_r)\frac{e^2}{c^3} \quad (r=1, 2, 3, 4) \quad (1)$$

where $(x_1(t), x_2(t), x_3(t), x_4(t) = ict)$ are the coordinates of the electron, e is its charge, m – its rest mass, c – the speed of the light, the dot is a differentiation with respect to the arc length, i.e. $\dot{x}_n = \dfrac{dx_n}{ds} = \dfrac{c\,dx_n}{\sqrt{c^2 - u^2}\,dt}$, and the Einstein summation convention is valid. The second term in (1) is the Abraham four-vector of radiation reaction derived also by Dirac [3].

Here we replace the radiation term in (1) by the one obtained in [1]. We consider just first three equations because in [1] is proved that the fourth one is a consequence of the rest ones. We have applied a similar form of the radiation terms to two-body problem of classical electrodynamics [4]- [6]. Many results concerning radiation terms are contained in [7]-[32]. They are based on various methods. Here we use the fixed point method from [33].

The derivation of the new form of the radiation term is based on the relativistic form of the retarded and advanced Lienard-Wiechert potentials [8]-[10], [34], [35]. We stand on the theory of differential equations of neutral type with both retarded and advanced arguments caused by the finite propagation of the interaction – the basic assumption of the Einstein relativistic theory. So Dirac equations become second order neutral equations.

The main goal of the present paper is to prove an existence-uniqueness of a periodic solution for Dirac equations. We use an operator formulation of the periodic problem from [36]. In view of [37]-[39] we are able to apply the results obtained to betatron radiation.

The paper consists of six sections. In Section 2 we derive the Dirac equations using retarded and advanced potentials. In Section 3 we derive a new form of the radiation term. In Section 4 we formulate a periodic problem and give some preliminary assertions. In Section 5 we give an operator formulation of the periodic problem and by a suitable fixed point theorem prove an existence-uniqueness of periodic solution for Dirac equations. Section 6 contains a conclusion remark.

2. Derivation of Dirac Equations Using Retarded and Advanced Potentials

First we recall some basic notions and denotations following the Synge formalism [35] (cf. also [34]). Consider a charge ***e*** describing any curve L in the space-time. Let $A(x_1(t), x_2(t), x_3(t), ict)$ be any event. The unit tangent vector to L at A is

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\frac{u_1(t)}{\Delta}, \frac{u_2(t)}{\Delta}, \frac{u_3(t)}{\Delta}, \frac{ic}{\Delta}\right) = \left(\frac{\vec{u}(t)}{\Delta}, \frac{ic}{\Delta}\right),$$

where $\Delta = \sqrt{c^2 - \sum_{\gamma=1}^{3} u_\gamma^2(t)} \equiv \sqrt{c^2 - \langle \vec{u}(t), \vec{u}(t)\rangle}$ and $\langle .,. \rangle_4$ is the scalar product in 4-dimensional Minkowski space,

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while $\langle .,. \rangle$ is the scalar product in 3-dimensional Euclidian subspace.

Let $A^{ret}(x_1(\check{t}), x_2(\check{t}), x_3(\check{t}), ic\,\check{t})$ be the intersection of L with the null-cone drawn into the past from A, and let $A^{adv}(x_1(\hat{t}), x_2(\hat{t}), x_3(\hat{t}), ic\,\hat{t})$ be the intersection of L with the null-cone drawn into the future from A where $\check{t} < t$ and $t < \hat{t}$.

Let $\check{\lambda} = \left(\check{\lambda}_1, \check{\lambda}_2, \check{\lambda}_3, \check{\lambda}_4\right) = \left(\frac{u_1(\check{t})}{\check{\Delta}}, \frac{u_2(\check{t})}{\check{\Delta}}, \frac{u_3(\check{t})}{\check{\Delta}}, \frac{ic}{\check{\Delta}}\right)$ be the unit tangent vector to the world line L at A^{ret}, where $\check{\Delta} = \sqrt{c^2 - \sum_{\gamma=1}^{3}\left(u_\gamma(\check{t})\right)^2} = \sqrt{c^2 - \left\langle \vec{u}(\check{t}), \vec{u}(\check{t}) \right\rangle}$ and let $\hat{\lambda} = \left(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4\right) = \left(\frac{u_1(\hat{t})}{\hat{\Delta}}, \frac{u_2(\hat{t})}{\hat{\Delta}}, \frac{u_3(\hat{t})}{\hat{\Delta}}, \frac{ic}{\hat{\Delta}}\right)$ be the unit tangent vector to L at A^{adv}, where

$$\hat{\Delta} = \sqrt{c^2 - \sum_{\gamma=1}^{3}\left(u_\gamma(\hat{t})\right)^2} = \sqrt{c^2 - \left\langle \vec{u}(\hat{t}), \vec{u}(\hat{t}) \right\rangle}.$$

Let

$$\xi^{ret} = \left(\xi_1^{ret}, \xi_2^{ret}, \xi_3^{ret}, \xi_4^{ret}\right)$$
$$= \left(x_1(t) - x_1(\check{t}), x_2(t) - x_2(\check{t}), x_3(t) - x_3(\check{t}),\ ic(t - \check{t})\right), \check{t} < t$$

be the retarded isotropic vector $A^{ret}A$ and let

$$\xi^{adv} = \left(\xi_1^{adv}, \xi_2^{adv}, \xi_3^{adv}, \xi_4^{adv}\right)$$
$$= \left(x_1(\hat{t}) - x_1(t), x_2(\hat{t}) - x_2(t),\ x_3(\hat{t}) - x_3(t),\ ic(\hat{t} - t)\right), \hat{t} > t$$

be the advanced isotropic vector AA^{adv}.

In accordance with the Dirac assumptions [3] the radiation term is defined as a half of the difference between both retarded and advanced potentials, that is,

$$F_{kn}^{rad} = \frac{1}{2}\left[\left(\frac{\partial A_n^{ret}}{\partial \check{x}_k} - \frac{\partial A_k^{ret}}{\partial \check{x}_n}\right) - \left(\frac{\partial A_n^{adv}}{\partial \hat{x}_k} - \frac{\partial A_k^{adv}}{\partial \hat{x}_n}\right)\right],$$

where $A_n^{ret} = -\frac{e\check{\lambda}_n}{\langle \check{\lambda}, \xi^{ret} \rangle}, A_n^{adv} = -\frac{e\hat{\lambda}_n}{\langle \hat{\lambda}, \xi^{adv} \rangle}$ are the Lienard-Wiechert retarded and advanced potentials (cf. [8]-[10], [34], [35]).

So that the Dirac physical assumptions lead to the following form of the equations of motion:

$$m\frac{d\lambda_k}{ds} = \frac{e}{c^2}\left(F_{kn}\lambda_n + F_{kn}^{rad}\lambda_n\right) \Leftrightarrow m\frac{d\lambda_k}{ds} =$$
$$\frac{e}{c^2}\left[F_{kn}\lambda_n + \frac{1}{2}\left(\frac{\partial A_n^{ret}}{\partial \check{x}_k} - \frac{\partial A_k^{ret}}{\partial \check{x}_n}\right)\lambda_n - \frac{1}{2}\left(\frac{\partial A_n^{adv}}{\partial \hat{x}_k} - \frac{\partial A_k^{adv}}{\partial \hat{x}_n}\right)\lambda_n\right]$$

or

$$\frac{d\lambda_k}{ds} = \frac{e}{mc^2}F_{kn}\lambda_n$$
$$+ \frac{e}{2mc^2}\left[\left(\frac{\partial A_n^{ret}}{\partial \check{x}_k} - \frac{\partial A_k^{ret}}{\partial \check{x}_n}\right) - \left(\frac{\partial A_n^{adv}}{\partial \hat{x}_k} - \frac{\partial A_k^{adv}}{\partial \hat{x}_n}\right)\right]\lambda_n.$$
$$(k = 1, 2, 3, 4).$$

Further on we assume (cf. [1]) that

(AR): $t - \check{t} = \tau^{ret}(t)$, $\hat{t} - t = \tau^{adv}(t)$.

In fact, postulating **(AR)** we extend the relation between the relativistic and Newton absolute time.

Since A^{ret} and A^{adv} lie on the trajectory L we obtain

$$\check{x}(t) = \left(x_1(\check{t}), x_2(\check{t}), x_3(\check{t}), x_4(\check{t}) = ic\,\check{t}\right)$$
$$= \left(x_1(t - \tau^{ret}(t)), x_2(t - \tau^{ret}(t)), x_3(t - \tau^{ret}(t)), ic(t - \tau^{ret}(t))\right),$$
$$\hat{x}(t) = \left(x_1(\hat{t}), x_2(\hat{t}), x_3(\hat{t}), x_4(\hat{t}) = ic\,\hat{t}\right)$$
$$= \left(x_1(t + \tau^{adv}(t)), x_2(t + \tau^{adv}(t)), x_3(t + \tau^{adv}(t)), ic(t + \tau^{adv}(t))\right),$$

and

$$\check{\lambda} = \left(\frac{u_1(t - \tau^{ret})}{\check{\Delta}}, \frac{u_2(t - \tau^{ret})}{\check{\Delta}}, \frac{u_3(t - \tau^{ret})}{\check{\Delta}}, \frac{ic}{\check{\Delta}}\right), \text{ where}$$
$$\check{\Delta} = \sqrt{c^2 - \left\langle \vec{u}(t - \tau^{ret}), \vec{u}(t - \tau^{ret}) \right\rangle};$$
$$\hat{\lambda} = \left(\frac{u_1(t + \tau^{adv})}{\hat{\Delta}}, \frac{u_2(t + \tau^{adv})}{\hat{\Delta}}, \frac{u_3(t + \tau^{adv})}{\hat{\Delta}}, \frac{ic}{\hat{\Delta}}\right), \text{ where}$$
$$\hat{\Delta} = \sqrt{c^2 - \left\langle \vec{u}(t + \tau^{ret}), \vec{u}(t + \tau^{ret}) \right\rangle}.$$

Therefore the isotropic vectors become

$$\xi^{ret} =$$
$$\left(x_1(t) - x_1(t - \tau^{ret}), x_2(t) - x_2(t - \tau^{ret}), x_3(t) - x_3(t - \tau^{ret}),\ ic\tau^{ret}(t)\right)$$

and

$$\xi^{adv} =$$
$$\left(x_1(t + \tau^{adv}) - x_1(t), x_2(t + \tau^{adv}) - x_2(t), x_3(t + \tau^{adv}) - x_3(t),\ ic\tau^{adv}(t)\right).$$

In general case the functions $\tau^{ret}(t), \tau^{adv}(t)$ can be obtained as solutions of the functional equations

$$\left\langle \xi^{ret}, \xi^{ret} \right\rangle_4 = 0, \quad \left\langle \xi^{adv}, \xi^{adv} \right\rangle_4 = 0$$

or

$$\sqrt{\left\langle \vec{x}(t) - \vec{x}(t-\tau^{ret}), \vec{x}(t) - \vec{x}(t-\tau^{ret}) \right\rangle} = c^2 \tau^{ret}(t),$$

$$\sqrt{\left\langle \vec{x}(t+\tau^{adv}) - \vec{x}(t), \vec{x}(t+\tau^{adv}) - \vec{x}(t) \right\rangle} = c^2 \tau^{adv}(t).$$

In what follows we briefly repeat the calculations from [1]:

$$\begin{aligned}
\frac{d\lambda_k}{ds} &= \frac{e}{mc^2} F_{kn}\lambda_n \\
&+ \frac{e^2}{2mc^2}\left[\frac{\xi_k^{ret}\langle \lambda, \check{\lambda}\rangle_4 - \check{\lambda}_k \langle \xi^{ret}, \lambda\rangle_4}{\langle \check{\lambda}, \xi^{ret}\rangle_4^3} \left(1 + \left\langle \xi^{ret}, \frac{d\check{\lambda}}{ds_{ret}} \right\rangle_4 \right) \right. \\
&\left. + \frac{\langle \xi^{ret}, \lambda\rangle_4 \, d\check{\lambda}_k / ds_{ret} - \left\langle \lambda, d\check{\lambda}/ds_{ret} \right\rangle_4 \xi_k^{ret}}{\langle \check{\lambda}, \xi^{ret}\rangle_4^2} \right] \\
&- \frac{e^2}{2mc^2}\left[\frac{\xi_k^{adv}\langle \lambda, \hat{\lambda}\rangle_4 - \hat{\lambda}_k \langle \xi^{adv}, \lambda\rangle_4}{\langle \hat{\lambda}, \xi^{adv}\rangle_4^3} \left(1 + \left\langle \xi^{adv}, \frac{d\hat{\lambda}}{ds_{adv}} \right\rangle_4 \right) \right. \\
&\left. + \frac{\langle \xi^{adv}, \lambda\rangle_4 \, d\hat{\lambda}_k / ds_{adv} - \left\langle \lambda, d\hat{\lambda}/ds_{adv} \right\rangle_4 \xi_k^{adv}}{\langle \hat{\lambda}, \xi^{adv}\rangle_4^2} \right]
\end{aligned} \tag{2}$$

(k=1,2,3,4),

where the elements of the electromagnetic tensor are:

$$\begin{pmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{pmatrix} = \begin{pmatrix} 0 & H_3 & H_2 & iE_1 \\ -H_3 & 0 & H_1 & iE_2 \\ -H_2 & -H_1 & 0 & iE_3 \\ -iE_1 & -iE_2 & -iE_3 & 0 \end{pmatrix}.$$

Here $F_{kn} = -F_{nk}$ and $\vec{E}(E_1, E_2, E_3)$ is the electric field intensity vector, and $\vec{H}(H_1, H_2, H_3)$ − the magnetic field intensity vector.

3. Derivation of the Radiation Term

Following [1] we have to find the relations between the derivatives at past, present and future instants. The above system (2) can be split into "space-like" and "time-like" parts:

$$\begin{aligned}
\frac{d\lambda_\alpha}{ds} &= \frac{e}{mc^2} \sum_{n=1}^{4} F_{\alpha n}\lambda_n \\
&+ \frac{e^2}{2mc^2}\left[\frac{\xi_\alpha^{ret}\langle \lambda, \check{\lambda}\rangle_4 - \check{\lambda}_\alpha \langle \xi^{ret}, \lambda\rangle_4}{\langle \check{\lambda}, \xi^{ret}\rangle_4^3} \left(1 + \left\langle \xi^{ret}, \frac{d\check{\lambda}}{ds_{ret}} \right\rangle_4 \right) \right. \\
&\left. + \frac{\langle \xi^{adv}, \lambda\rangle_4 \, d\hat{\lambda}_\alpha / ds_{adv} - \left\langle \lambda, d\hat{\lambda}/ds_{adv} \right\rangle_4 \xi_\alpha^{adv}}{\langle \hat{\lambda}, \xi^{adv}\rangle_4^2} \right]
\end{aligned}$$

$(\alpha = 1, 2, 3)$;

$$\begin{aligned}
\frac{d\lambda_4}{ds} &= \frac{e}{mc^2} \sum_{n=1}^{4} F_{4n}\lambda_n \\
&+ \frac{e^2}{2mc^2}\left[\frac{\xi_4^{ret}\langle \check{\lambda}, \lambda\rangle_4 - \check{\lambda}_4 \langle \xi^{ret}, \lambda\rangle_4}{\langle \check{\lambda}, \xi^{ret}\rangle_4^3} \left(1 + \left\langle \xi^{ret}, \frac{d\check{\lambda}}{ds_{ret}} \right\rangle_4 \right) \right. \\
&\left. + \frac{\langle \xi^{ret}, \lambda\rangle_4 \, d\check{\lambda}_4 / ds_{ret} - \left\langle \lambda, d\check{\lambda}/ds_{ret} \right\rangle_4 \xi_4^{ret}}{\langle \check{\lambda}, \xi^{ret}\rangle_4^2} \right] \\
&- \frac{e^2}{2mc^2}\left[\frac{\xi_4^{adv}\langle \hat{\lambda}, \lambda\rangle_4 - \hat{\lambda}_4 \langle \xi^{adv}, \lambda\rangle_4}{\langle \hat{\lambda}, \xi^{adv}\rangle_4^3} \left(1 + \left\langle \xi^{adv}, \frac{d\hat{\lambda}}{ds_{adv}} \right\rangle_4 \right) \right. \\
&\left. . \frac{\langle \xi^{adv}, \lambda\rangle_4 \, d\hat{\lambda}_4 / ds_{adv} - \left\langle \lambda, d\hat{\lambda}/ds_{adv} \right\rangle_4 \xi_4^{adv}}{\langle \hat{\lambda}, \xi^{adv}\rangle_4^2} \right].
\end{aligned}$$

Differentiating the relation $t - \check{t} = (1/c)\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma(t) - x_\gamma(\check{t})\right]^2}$ with respect to $\check{t}$, considering $t = t(\check{t})$ we obtain

$$\frac{dt}{d\check{t}} - 1 = \left(\sum_{\gamma=1}^{3}\left[x_\gamma(t) - x_\gamma(\check{t})\right]\left[u_\gamma(t)\frac{dt}{d\check{t}} - u_\gamma(\check{t})\right] \right) / c\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma(t) - x_\gamma(\check{t})\right]^2}.$$

In a similar way differentiating $\hat{t} - t = (1/c)\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma(\hat{t}) - x_\gamma(t)\right]^2}$ with respect to $\hat{t}$ (considering $t = t(\hat{t})$) we obtain

$$1-\frac{dt}{d\hat{t}}=\left(\sum_{\gamma=1}^{3}\left[x_\gamma(\hat{t})-x_\gamma(t)\right]\left[u_\gamma(\hat{t})-u_\gamma(t)\frac{dt}{d\hat{t}}\right]\right)$$
$$/c\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma(\hat{t})-x_\gamma(t)\right]^2}.$$

We derive the radiation term under the Dirac assumptions **(D)**: $\tau^{ret}(t)=\tau^{adv}(t)=\tau>0$

where τ is a small parameter. The Dirac assumption is justified by the fact that, for example, the electron radiation time is 10^{-24} sec.

Applying the Taylor theorem under assumption **(D)** we get

$$u(t-\tau^{ret})=u(t-\tau)=u(t)-\tau\dot{u}(t)+\frac{\tau^2}{2}\ddot{u}(t)-\frac{\tau^3}{3!}\dddot{u}(t)+...;$$
$$u(t+\tau)=u(t)+\tau\dot{u}(t)+\frac{\tau^2}{2}\ddot{u}(t)+\frac{\tau^3}{3!}\dddot{u}(t)+...;$$
$$\xi_\alpha{}^{ret}=x_\alpha(t-\tau)-x_\alpha(t)\approx-\tau u_\alpha(t)+\frac{\tau^2}{2!}\dot{u}_\alpha(t)-...$$

It follows $\Delta\approx\check{\Delta}\approx\hat{\Delta}$. Consequently

$$\check{D}\equiv\frac{dt}{d\check{t}}=\frac{c^2\tau^{ret}-\left\langle\vec{\xi}^{ret},\vec{u}(t-\tau^{ret})\right\rangle}{c^2\tau^{ret}-\left\langle\vec{\xi}^{ret},\vec{u}(t)\right\rangle}$$
$$=\frac{c^2\tau-\left\langle\vec{\xi}^{ret},\vec{u}(t-\tau^{ret})\right\rangle}{c^2\tau-\left\langle\vec{\xi}^{ret},\vec{u}(t)\right\rangle}\approx\frac{c^2\tau-\left\langle\vec{\xi}^{ret},\vec{u}(t)\right\rangle}{c^2\tau-\left\langle\vec{\xi}^{ret},\vec{u}(t)\right\rangle}=1,$$

$$\hat{D}\equiv\frac{dt}{d\hat{t}}=\frac{c^2\tau^{adv}-\left\langle\vec{\xi}^{adv},\vec{u}(t+\tau^{adv})\right\rangle}{c^2\tau^{adv}-\left\langle\vec{\xi}^{adv},\vec{u}(t)\right\rangle}$$
$$\approx\frac{c^2\tau-\left\langle\vec{\xi}^{adv},\vec{u}(t)\right\rangle}{c^2\tau-\left\langle\vec{\xi}^{adv},\vec{u}(t)\right\rangle}=1$$

and then

$$\left\langle\lambda,\check{\lambda}\right\rangle_4=\frac{\left\langle\vec{u}(t),\vec{u}(t-\tau^{ret})\right\rangle-c^2}{\Delta\check{\Delta}}\approx-\frac{c^2-\left\langle\vec{u}(t),\vec{u}(t)\right\rangle}{\Delta^2}=-1;;$$
$$\left\langle\lambda,\hat{\lambda}\right\rangle_4=\left(\left\langle u(t),u(t+\tau^{adv})\right\rangle-c^2\right)/(\Delta\hat{\Delta})\approx-1;$$
$$\left\langle\xi^{ret},\lambda\right\rangle_4=\frac{\left\langle\vec{\xi}^{ret},\vec{u}(t)\right\rangle-c^2\tau}{\Delta}\approx\frac{\tau\left\langle\vec{u}(t),\vec{u}(t)\right\rangle-c^2\tau}{\Delta}=-\tau\Delta;$$
$$\left\langle\xi^{adv},\lambda\right\rangle_4=\frac{\left\langle\vec{\xi}^{adv},\vec{u}(t)\right\rangle-c^2\tau}{\Delta}\approx\frac{\tau\left\langle\vec{u}(t),\vec{u}(t)\right\rangle-c^2\tau}{\Delta}=-\tau\Delta;$$

$$\left\langle\xi^{ret},\check{\lambda}\right\rangle_4=\frac{\left\langle\xi^{ret},u(t-\tau)\right\rangle-c^2\tau}{\Delta}\approx\frac{\tau\left\langle u(t),u(t-\tau)\right\rangle-c^2\tau}{\Delta}\approx-\tau\Delta;$$
$$\left\langle\xi^{adv},\hat{\lambda}\right\rangle_4=\frac{\left\langle\vec{\xi}^{adv},\vec{u}(t+\tau)\right\rangle-c^2\tau}{\Delta}\approx\frac{\tau\left\langle\vec{u}(t),\vec{u}(t+\tau)\right\rangle-c^2\tau}{\Delta}\approx-\tau\Delta;$$
$$\frac{d\lambda_\alpha}{ds}=\frac{1}{\Delta}\frac{d}{dt}\left(\frac{u_\alpha(t)}{\Delta}\right)=\frac{\dot{u}_\alpha(t)}{\Delta^2}+\frac{\left\langle\vec{u}(t),\dot{\vec{u}}(t)\right\rangle u_\alpha(t)}{\Delta^4}$$
$(\alpha=1,2,3)$
$$\frac{d\lambda_4}{ds}=\frac{ic}{\Delta}\frac{d}{dt}\left(\frac{1}{\Delta}\right)=\frac{ic\left\langle\vec{u}(t),\dot{\vec{u}}(t)\right\rangle}{\Delta^4};\ \left(\dot{u}_\alpha(t)\equiv\frac{du_\alpha(t)}{dt}\right);$$
$$\frac{d}{ds_{ret}}=\frac{1}{\check{\Delta}}\frac{d}{d\check{t}}=\frac{1}{\check{\Delta}}\frac{dt}{d\check{t}}\frac{d}{dt}=\frac{1}{\check{\Delta}}\check{D}\frac{d}{dt}=\frac{1}{\Delta}\frac{d}{dt};$$
$$\frac{d}{ds_{adv}}=\frac{1}{\hat{\Delta}}\frac{d}{d\hat{t}}=\frac{1}{\hat{\Delta}}\frac{dt}{d\hat{t}}\frac{d}{dt}=\frac{1}{\hat{\Delta}}\hat{D}\frac{d}{dt}=\frac{1}{\Delta}\frac{d}{dt};$$
$$\frac{d\check{\lambda}_\alpha}{ds_{ret}}=\check{D}\left[\frac{\dot{u}_\alpha(t-\tau^{ret})}{\check{\Delta}^2}+\frac{u_\alpha(t-\tau^{ret})\left\langle\vec{u}(t-\tau^{ret}),\dot{\vec{u}}(t-\tau^{ret})\right\rangle}{\check{\Delta}^4}\right]$$
$$\approx\left(\dot{u}_\alpha(t-\tau)/\Delta^2\right)+\left(\left\langle\vec{u}(t-\tau^{ret}),\dot{\vec{u}}(t-\tau^{ret})\right\rangle u_\alpha(t-\tau)/\Delta^4\right);$$
$$\frac{d\check{\lambda}_4}{ds_{ret}}=\frac{ic\check{D}\left\langle\vec{u}(t-\tau^{ret}),\dot{\vec{u}}(t-\tau^{ret})\right\rangle}{\check{\Delta}^4}\approx\frac{ic\left\langle\vec{u}(t-\tau),\dot{\vec{u}}(t-\tau)\right\rangle}{\Delta^4},$$
$$\frac{d\hat{\lambda}_\alpha}{ds_{adv}}=\hat{D}\left[\frac{\dot{u}_\alpha(t+\tau^{adv})}{\hat{\Delta}^2}+\frac{u_\alpha(t+\tau^{adv})\left\langle\vec{u}(t+\tau^{adv}),\dot{\vec{u}}(t+\tau^{adv})\right\rangle}{\hat{\Delta}^4}\right];$$
$$\approx\left(\dot{u}_\alpha(t+\tau)/\Delta^2\right)+\left(\left\langle\vec{u}(t+\tau),\dot{\vec{u}}(t+\tau)\right\rangle u_\alpha(t+\tau)/\Delta^4\right);$$
$$\frac{d\hat{\lambda}_4}{ds_{adv}}=\frac{ic\hat{D}\left\langle\vec{u}(t+\tau^{adv}),\dot{\vec{u}}(t+\tau^{adv})\right\rangle}{\hat{\Delta}^4}\approx\frac{ic\left\langle\vec{u}(t+\tau),\dot{\vec{u}}(t+\tau)\right\rangle}{\Delta^4};$$
$$\left\langle\xi^{ret},\frac{d\check{\lambda}}{ds_{ret}}\right\rangle_4\approx\frac{\left\langle\vec{\xi}^{ret},\dot{\vec{u}}(t-\tau)\right\rangle}{\Delta^2}$$
$$+\left(\left(\left\langle\vec{\xi}^{ret},\vec{u}(t-\tau)\right\rangle-c^2\tau\right)\left\langle\vec{u}(t-\tau),\dot{\vec{u}}(t-\tau)\right\rangle\right)/\Delta^4$$
$$\approx\frac{\tau\left\langle\vec{u}(t),\dot{\vec{u}}(t-\tau)\right\rangle}{\Delta^2}+\frac{\tau\left(\left\langle\vec{u}(t),\vec{u}(t-\tau)\right\rangle-c^2\right)\left\langle\vec{u}(t-\tau),\dot{\vec{u}}(t-\tau)\right\rangle}{\Delta^4};$$
$$\left\langle\lambda,\frac{d\check{\lambda}}{ds_{ret}}\right\rangle_4=\left\langle\vec{u}(t),\dot{\vec{u}}(t-\tau)\right\rangle/\Delta^3$$
$$+\left(\left(\left\langle\vec{u}(t),\vec{u}(t-\tau)\right\rangle-c^2\right)\left\langle\vec{u}(t-\tau),\dot{\vec{u}}(t-\tau)\right\rangle\right)/\Delta^5$$
$$\approx\left(\left\langle\vec{u}(t),\dot{\vec{u}}(t-\tau)\right\rangle/\Delta^3\right)-\left(\Delta^2\left\langle\vec{u}(t-\tau),\dot{\vec{u}}(t-\tau)\right\rangle/\Delta^5\right)\approx0;$$
$$\left\langle\xi^{adv},\frac{d\hat{\lambda}}{ds_{adv}}\right\rangle_4=\frac{\left\langle\vec{\xi}^{adv},\dot{\vec{u}}(t+\tau)\right\rangle}{\Delta^2}$$

$$+\left(\left\langle \vec{\xi}^{adv},\dot{\vec u}(t+\tau)\right\rangle - c^2\tau\right)\left\langle \vec u(t+\tau),\dot{\vec u}(t+\tau)\right\rangle/\Delta^4$$

$$\approx \frac{\tau\left\langle \vec u(t),\dot{\vec u}(t+\tau)\right\rangle}{\Delta^2}+\frac{\tau\left(\left\langle \vec u(t),\vec u(t+\tau)\right\rangle-c^2\right)\left\langle \vec u(t+\tau),\dot{\vec u}(t+\tau)\right\rangle}{\Delta^4}$$

$$\approx\left(\tau\left\langle \vec u(t),\dot{\vec u}(t+\tau)\right\rangle/\Delta^2\right)+\left(\tau\left\langle \vec u(t+\tau),\dot{\vec u}(t+\tau)\right\rangle/\Delta^2\right);$$

$$\left\langle \lambda,\frac{d\hat{\lambda}}{ds_{adv}}\right\rangle_4=\frac{\left\langle \vec u(t),\dot{\vec u}(t+\tau)\right\rangle}{\Delta^3}$$

$$+\left(\left(\left\langle \vec u(t),\vec u(t+\tau)\right\rangle-c^2\right)\left\langle \vec u(t+\tau),\dot{\vec u}(t+\tau)\right\rangle/\Delta^5\right)$$

$$\approx\left(\left\langle u(t),\dot{\vec u}(t+\tau)\right\rangle/\Delta^3\right)-\left(\left\langle \vec u(t+\tau),\dot{\vec u}(t+\tau)\right\rangle/\Delta^3\right)\approx 0\,.$$

Therefore the above system becomes

$$\frac{\dot u_\alpha(t)}{\Delta^2}+\frac{\left\langle \vec u(t),\dot{\vec u}(t)\right\rangle}{\Delta^4}u_\alpha(t)=\frac{e^2}{mc^2}\frac{1}{\Delta}\left[\sum_{\beta=1}^{3}F_{\alpha\beta}u_\beta(t)-E_\alpha\right]$$

$$+\frac{e^2}{2mc^2}\left[\left(-\tau u_\alpha+\frac{u_\alpha}{\Delta}\tau\Delta\right)/\left(\left(\frac{\tau\left\langle \vec u(t),\vec u(t-\tau)\right\rangle-c^2\tau}{\Delta}\right)^3\right)\right.$$

$$.\left(1+\tau\left(\left\langle \vec u(t),\dot{\vec u}(t-\tau)\right\rangle+\left\langle \vec u(t-\tau),\dot{\vec u}(t-\tau)\right\rangle\right)/\Delta^2\right)$$

$$+\left[1/\left(\left(\tau\left\langle \vec u(t),\vec u(t-\tau)\right\rangle-c^2\tau\right)/\Delta\right)^2\right]$$

$$.\left(\dot u_\alpha(t-\tau)+-\left(\left\langle \vec u(t-\tau),\dot{\vec u}(t-\tau)\right\rangle u_\alpha(t-\tau)/\Delta^2\right)\right)$$

$$\left..\left(\left(\left(\tau\left\langle \vec u(t),\vec u(t)\right\rangle-c^2\tau\right)/\Delta^3\right)-\left\langle \lambda,d\check{\lambda}/ds_{ret}\right\rangle_4\tau u_\alpha(t)\right)\right]$$

$$-\frac{e^2}{2mc^2}\left[\left(-\tau u_\alpha+\frac{u_\alpha}{\Delta}\tau\Delta\right)/\left(\left(\tau\left\langle \vec u(t),\vec u(t+\tau)\right\rangle-c^2\tau\right)/\Delta\right)^3\right.$$

$$.\left(1+\tau\left(\left\langle \vec u(t),\dot{\vec u}(t+\tau)\right\rangle+\left\langle \vec u(t+\tau),\dot{\vec u}(t+\tau)\right\rangle\right)/\Delta^2\right)\qquad(3.\alpha)$$

$$+\left[1/\left(\left(\tau\left\langle \vec u(t),\vec u(t+\tau)\right\rangle-c^2\tau\right)/\Delta\right)^2\right]$$

$$.\left(\left(\dot u_\alpha(t+\tau)+-\left(\left(\left\langle \vec u(t+\tau),\dot{\vec u}(t+\tau)\right\rangle u_\alpha(t+\tau)\right)/\Delta^2\right)\right)\right.$$

$$\left..\left(\frac{\tau\left\langle \vec u(t),\vec u(t)\right\rangle-c^2\tau}{\Delta^3}-\left\langle \lambda,d\hat{\lambda}/ds_{adv}\right\rangle_4\tau u_\alpha(t)\right)\right]$$

$(\alpha=1,2,3)$;

$$ic\left\langle \vec u(t),\dot{\vec u}(t)\right\rangle/\Delta^4=\left(e^2/mc^2\right)\left(-ic\left\langle \vec E,\vec u(t)\right\rangle/\Delta\right)$$

$$+\frac{e^2}{2mc^2}\left[\left(\left(-ic\tau+\frac{ic}{\Delta}\tau\Delta\right)/\left(\frac{\tau\left\langle \vec u(t),\vec u(t-\tau)\right\rangle-c^2\tau}{\Delta}\right)^3\right)\right.\qquad(3)$$

$$.\left(1+\left\langle \xi^{ret},d\check{\lambda}/ds_{ret}\right\rangle_4\right)$$

$$+\left(ic\frac{\left\langle \vec u(t-\tau),\dot{\vec u}(t-\tau)\right\rangle}{\Delta^4}\langle\xi^{ret},\lambda\rangle_4-ic\tau\left\langle \lambda,d\check{\lambda}/ds_{ret}\right\rangle_4\right)$$

$$\left./\left(\left(\tau\left\langle \vec u(t),\vec u(t-\tau)\right\rangle-c^2\tau\right)/\Delta\right)^2\right]$$

$$-\frac{e^2}{2mc^2}\left[\left(\left(-ic\tau+\frac{ic}{\Delta}\tau\Delta\right)/\left(\frac{\tau\left\langle \vec u(t),\vec u(t+\tau)\right\rangle-c^2\tau}{\Delta}\right)^3\right)\right.$$

$$.\left(1+\left\langle \xi^{adv},d\hat{\lambda}/ds_{adv}\right\rangle_4\right)$$

$$\left.+\frac{ic\frac{\left\langle \vec u(t+\tau),\dot{\vec u}(t+\tau)\right\rangle}{\Delta^4}\langle\xi^{adv},\lambda\rangle_4-ic\tau\left\langle \lambda,d\hat{\lambda}/ds_{adv}\right\rangle_4}{\left(\left(\tau\left\langle \vec u(t),\vec u(t+\tau)\right\rangle-c^2\tau\right)/\Delta\right)^2}\right].$$

The last equation should be divided by ic.

In [1] is proved that the 4-th equation (3) is a consequence of the first three ones (3.α).

Further on in view of

$$-\left\langle \vec u(t),\dot{\vec u}(t-\tau)\right\rangle u_\alpha+\left\langle \vec u(t-\tau),\dot{\vec u}(t-\tau)\right\rangle u_\alpha$$

$$\approx-\left\langle \vec u(t),\dot{\vec u}(t-\tau)\right\rangle u_\alpha+\left\langle \vec u(t),\dot{\vec u}(t-\tau)\right\rangle u_\alpha\approx 0;$$

$$\left\langle \vec u(t),\dot{\vec u}(t+\tau)\right\rangle u_\alpha-\left\langle \vec u(t+\tau),\dot{\vec u}(t+\tau)\right\rangle u_\alpha$$

$$\approx\left\langle \vec u(t),\dot{\vec u}(t+\tau)\right\rangle u_\alpha-\left\langle \vec u(t),\dot{\vec u}(t+\tau)\right\rangle u_\alpha\approx 0;$$

$$c^2-\left\langle \vec u(t),\vec u(t-\tau)\right\rangle\approx c^2-\left\langle \vec u(t),\vec u(t+\tau)\right\rangle$$

$$\approx c^2-\left\langle \vec u(t),\vec u(t)\right\rangle=\Delta^2;$$

$$\frac{\left\langle \vec u(t),\dot{\vec u}(t-\tau)\right\rangle}{\Delta^2}+\frac{\left(\left\langle \vec u(t),\vec u(t-\tau)\right\rangle-c^2\right)\left\langle \vec u(t-\tau),\dot{\vec u}(t-\tau)\right\rangle}{\Delta^4}$$

$$\approx\frac{\left\langle \vec u(t),\dot{\vec u}(t-\tau)\right\rangle\Delta^2}{\Delta^4}+\frac{\left(\left\langle \vec u(t),\vec u(t)\right\rangle-c^2\right)\left\langle \vec u(t),\dot{\vec u}(t-\tau)\right\rangle}{\Delta^4}=0;$$

$$\frac{\left\langle \vec u(t),\dot{\vec u}(t+\tau)\right\rangle}{\Delta^3}+\frac{\left(\left\langle \vec u(t),\vec u(t+\tau)\right\rangle-c^2\right)\left\langle \vec u(t+\tau),\dot{\vec u}(t+\tau)\right\rangle}{\Delta^3}$$

$$\approx\frac{\left\langle \vec u(t),\dot{\vec u}(t+\tau)\right\rangle\Delta^2}{\Delta^4}+\frac{\left(\left\langle \vec u(t),\vec u(t)\right\rangle-c^2\right)\left\langle \vec u(t),\dot{\vec u}(t+\tau)\right\rangle}{\Delta^4}=0$$

we obtain

$$\dot u_\alpha+\left(\left\langle \vec u,\dot{\vec u}\right\rangle/\Delta^2\right)u_\alpha=\left(e^2/mc^2\right)\left[\Delta\left(\sum_{\beta=1}^{3}F_{\alpha\beta}u_\beta-E_\alpha\right)\right.$$

$$\left.+\frac{1}{\Delta}\left(\frac{\dot u_\alpha(t+\tau)-\dot u_\alpha(t-\tau)}{2\tau}+\frac{u_\alpha(t)}{\Delta^2}\left\langle \vec u(t),\frac{\dot{\vec u}(t+\tau)-\dot{\vec u}(t-\tau)}{2\tau}\right\rangle\right)\right]$$

$(\alpha=1,2,3)$;

$$\frac{\left\langle \vec u(t),\dot{\vec u}(t)\right\rangle}{\Delta^3}=\frac{e^2}{mc^2}\left(-\left\langle \vec E,\vec u(t)\right\rangle+\frac{1}{\Delta^4}\left\langle \vec u(t),\frac{\dot{\vec u}(t+\tau)-\dot{\vec u}(t-\tau)}{2\tau}\right\rangle\right).$$

Denoting by

$$G_\alpha^{rad} = (1/\Delta)\left((\dot u_\alpha(t+\tau) - \dot u_\alpha(t-\tau))/2\tau\right)$$
$$+\left(1/\Delta^3\right)\left\langle \vec u(t), \left(\dot{\vec u}(t+\tau) - \dot{\vec u}(t-\tau)\right)/2\tau \right\rangle u_\alpha(t);$$

$$G_\alpha = \left(e^2/mc^2\right)\left[\Delta\left(\sum_{\beta=1}^{3} F_{\alpha\beta} u_\beta - E_\alpha\right) + G_\alpha^{rad}\right]$$

we write the system (3.α) in the form

$$\begin{aligned}
&\left(1+\frac{u_1^2(t)}{\Delta^2}\right)\dot u_1(t) + \frac{u_1(t)u_2(t)}{\Delta^2}\dot u_2(t) + \frac{u_1(t)u_3(t)}{\Delta^2}\dot u_3(t) = G_1\\
&\frac{u_1(t)u_2(t)}{\Delta^2}\dot u_1(t) + \left(1+\frac{u_2^2(t)}{\Delta^2}\right)\dot u_2(t) + \frac{u_2(t)u_3(t)}{\Delta^2}\dot u_3(t) = G_2\\
&\frac{u_1(t)u_3(t)}{\Delta^2}\dot u_1(t) + \frac{u_2(t)u_3(t)}{\Delta^2}\dot u_2(t) + \left(1+\frac{u_3^2(t)}{\Delta^2}\right)\dot u_3(t) = G_3
\end{aligned} \tag{4}$$

and we have to solve the last system with respect to $\dot u_1(t), \dot u_2(t), \dot u_3(t)$.

Assumption **(C):** $|\vec u(t)| = \sqrt{u_1^2(t)+u_2^2(t)+u_3^2(t)} \le \bar c < c$ for some constant $\bar c$.

Therefore $c^2 - |u|^2 \ge c^2 - \bar c^2 > 0$ and the determinant of the above system is obviously different from zero $\delta = c^2/\Delta^2 \Rightarrow 1 \le \delta \le c^2/\left(c^2 - \bar c^2\right)$. Consequently, the unique solution of (4) is

$$\begin{aligned}
\dot u_1 &= \left((c^2 - u_1^2)G_1 - u_1u_2G_2 - u_1u_3G_3\right)/c^2\\
\dot u_2 &= \left(-u_1u_2G_1 + (c^2 - u_2^2)G_2 - u_2u_3G_3\right)/c^2\\
\dot u_3 &= \left(-u_1u_3G_1 - u_2u_3G_2 + (c^2 - u_3^2)G_3\right)/c^2
\end{aligned} \tag{5}$$

where in view of

$$G_1 = \left(e^2/mc^2\right)\left[\Delta\left(\sum_{\beta=1}^{3} F_{1\beta}u_\beta - E_1\right) + G_1^{rad}\right]$$
$$= \left(e^2/mc^2\right)\left[\Delta\left(H_3u_2 + H_2u_3 - E_1\right) + G_1^{rad}\right],$$

$$G_2 = \left(e^2/mc^2\right)\left[\Delta\left(\sum_{\beta=1}^{3} F_{2\beta}u_\beta - E_2\right) + G_2^{rad}\right]$$
$$= \left(e^2/mc^2\right)\left[\Delta\left(-H_3u_1 + H_1u_3 - E_2\right) + G_2^{rad}\right],$$

$$G_3 = \left(e^2/mc^2\right)\left[\Delta\left(\sum_{\beta=1}^{3} F_{3\beta}u_\beta - E_3\right) + G_3^{rad}\right]$$
$$= \left(e^2/mc^2\right)\left[\Delta\left(-H_2u_1 - H_1u_2 - E_3\right) + G_3^{rad}\right].$$

we get

$$\begin{aligned}
\dot u_1 &= \frac{e^2}{mc^2}\left[\Delta\left(H_3u_2 + H_2u_3 - E_1 + \frac{u_1E_1 + u_2E_2 + u_3E_3}{c^2}u_1\right)\right.\\
&\left.+\frac{\left(c^2 - u_1^2\right)G_1^{rad} - u_1u_2G_2^{rad} - u_1u_3G_3^{rad}}{c^2}\right] \equiv F_1,\\
\dot u_2 &= \frac{e^2}{mc^2}\left[\Delta\left(-H_3u_1 + H_1u_3 - E_2 + \frac{u_1E_1 + u_2E_2 + u_3E_3}{c^2}u_2\right)\right.\\
&\left.+\frac{-u_1u_2G_1^{rad} + \left(c^2 - u_2^2\right)G_2^{rad} - u_2u_3G_3^{rad}}{c^2}\right] \equiv F_2,\\
\dot u_3 &= \frac{e^2}{mc^2}\left[\Delta\left(-H_2u_1 - H_1u_2 - E_3 + \frac{u_1E_1 + u_2E_2 + u_3E_3}{c^2}u_3\right)\right.\\
&\left.+\frac{-u_1u_3G_1^{rad} - u_2u_3G_2^{rad} + \left(c^2 - u_3^2\right)G_3^{rad}}{c^2}\right] \equiv F_3.
\end{aligned} \tag{6}$$

4. An Operator Formulation of the Periodic Problem and Preliminary Assertions

We formulate the main periodic problem: to find a T_0-periodic solution $(u_1(t), u_2(t), u_3(t))$ of the system (6) on the interval $t \in [0,T]$ with initial conditions $u_\alpha(0) = 0$, $\dot u_\alpha(0) = 0$ and $u_\alpha(t) = u_{0\alpha}(t), t \in (-\infty; 0]$ $(\alpha = 1,2,3)$ where $u_{0\alpha}(t)$ are prescribed T_0-periodic initial functions.

Let $W_{T_0}^\infty[0,\infty)$ be the set of all T_0-periodic functions from $L_{T_0}^\infty[0,\infty)$ whose derivatives of arbitrary order belong to $L_{T_0}^\infty[0,\infty)$. The functions from $W_{T_0}^\infty[0,\infty)$ are considered as all infinite differentiable functions on $(0,\infty)$ having continuous extensions on $[0,\infty)$. Introduce the function sets:

$$M = \left\{u(.) \in L_{T_0}^\infty[0,\infty) : \left|u^{(n)}(t)\right| \le U_0\omega^n e^{\mu(t-kT_0)}\right.$$
$$\left.\text{for a.e. } t \in [kT_0, (k+1)T_0]\ (k = 0,1,2,...\ ;\ n = 0,1,2,...)\right\}$$

where $U_0e^{\mu T_0} \le \bar c < c$; $mT_0 = T$,

$$M_0 = \left\{u(.) \in M : \int_0^{T_0} u(t)dt = 0;\ (n,k = 0,1,2,...)\right\}.$$

Remark 1. It is easy to verify that substituting $s = t - kT_0$ we get

$$\int_{kT_0}^{(k+1)T_0} u(t)dt = \int_0^{T_0} u(s + kT_0)ds = \int_0^{T_0} u(s)ds = 0.$$

We define the following family of pseudo-metrics

$\rho_{(k,n)}(u_\alpha^{(n)},\bar{u}_\alpha^{(n)}) = \text{ess sup}\left\{ e^{-\mu(t-kT_0)}\omega^{-n}\left|u_\alpha^{(n)}(t)-\bar{u}_\alpha^{(n)}(t)\right| : t\in[kT_0,(k+1)T_0],\ n=0,1,2,...\right\}.$

Since for $t\in[kT_0,(k+1)T_0]$ we have

$$e^{-\mu(t-kT_0)}\omega^{-n}\left|u^{(n)}(t)-\bar{u}^{(n)}(t)\right| \le e^{-\mu(t-kT_0)}\omega^{-n}2\omega^n U_0 e^{\mu(t-kT_0)} = 2U_0 .$$

It follows

$$\sup\left\{\rho_{(k,n)}(u_\alpha^{(n)},\bar{u}_\alpha^{(n)}) : n=0,1,2,...;, k=01,2,...,m\right\}<\infty$$

and then we put

$$\rho_{(k,\infty)}(u_\alpha,\bar{u}_\alpha)=\sup\left\{\rho_{(k,n)}(u_\alpha^{(n)},\bar{u}_\alpha^{(n)}) : n=0,1,2,...\right\}.$$

Further on we set

$$\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3)) = \max\left\{\rho_{(k,\infty)}(u_1,\bar{u}_1),\rho_{(k,\infty)}(u_2,\bar{u}_2),\rho_{(k,\infty)}(u_3,\bar{u}_3)\right\}.$$

Lemma 1. If $u(.)\in M_0$ then

$x(t)=x_0+\int_0^t u(s)ds$ is T_0-periodic function.

Proof: Let us set $s=p+T_0$ and then we obtain

$\int_{T_0}^{t+T_0} u(s)ds=\int_0^t u(p+T_0)dp=\int_0^t u(p)dp$. Therefore

$$x(t+T_0)=x_0+\int_0^{t+T_0}u(s)ds = x_0+\int_0^t u(s)ds+\int_t^{t+T_0}u(s)ds$$
$$=x_0+\int_0^t u(s)ds+\int_t^0 u(s)ds+\int_0^{T_0}u(s)ds+\int_{T_0}^{t+T_0}u(s)ds$$
$$=x_0+\int_0^t u(s)ds+\int_t^0 u(s)ds+\int_0^{T_0}u(s)ds+\int_0^t u(s)ds$$
$$=x_0+\int_0^t u(s)ds+\int_0^{T_0}u(s)ds=x_0+\int_0^t u(s)ds=x(t).$$

Lemma 1 is thus proved.

Define the operator

$$(Bu)_\alpha^{(k)}(t)=\int_{kT_0}^t F_\alpha(u)(s)ds-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}F_\alpha(u)(s)ds$$
$$-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}F_\alpha(u)(p)dpds,\ t\in[kT_0,(k+1)T_0]$$

$(Bu)_\alpha^{(k)}(t)=0,\ t=0$

$(\alpha=1,2,3;\ k=0,1,2,...,m)$ where by assumption $T=mT_0$ and F_α are the right-hand sides of (6).

Lemma 2. ([36]) For every $u_\alpha(.)\in M_0$ it follows

$$\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}F_\alpha(u)(p)dpds=\int_{(k+1)T_0}^{(k+2)T_0}\int_{(k+1)T_0}^{s}F_\alpha(u)(p)dpds.$$

Assumptions **(H-E):** The functions $H_\alpha(t,x_1,x_2,x_3)$, $E_\alpha(t,x_1,x_2,x_3)\,(\alpha=1,2,3)$ are T_0-periodic and smooth in *t* and

$$\left|\frac{\partial^p H_\alpha(t,x_1,x_2,x_3)}{\partial x_\alpha{}^p}\right|\le H_0e^{\mu(t-kT_0)};$$
$$\left|\frac{\partial^p E_\alpha(t,x_1,x_2,x_3)}{\partial x_\alpha{}^p}\right|\le E_0e^{\mu(t-kT_0)};$$
$$\left|\frac{\partial^p H_\alpha(t,x_1,x_2,x_3)}{\partial t^p}\right|\le H_0e^{\mu(t-kT_0)};$$
$$\left|\frac{\partial^p E_\alpha(t,x_1,x_2,x_3)}{\partial t^p}\right|\le E_0e^{\mu(t-kT_0)}$$

$(\alpha=1,2,3)$, $p=0,1,2,...,$ $t\in[kT_0,(k+1)T_0]$.

Lemma 3. (Main Lemma) The periodic problem (6) has a solution $(u_1,u_2,u_3)\in M_0\times M_0\times M_0$ iff the operator *B* has a fixed point belonging to $M_0\times M_0\times M_0$, provided

$$\left|H_\alpha(t,x_1,x_2,x_3)\right|\le H_0e^{\mu(t-kT_0)},\ \left|E_\alpha(t,x_1,x_2,x_3)\right|\le E_0e^{\mu(t-kT_0)}$$
$$t\in[kT_0,(k+1)T_0];\ \alpha=1,2,3$$

where H_0,E_0,μ are positive constants.

Proof: Let $(u_1,u_2,u_3)\in M_0\times M_0\times M_0$ be a T_0-periodic solution of the system $\dot{u}_\alpha=F_\alpha(u)$. Then after integration in view of $u_\alpha\in M_0$

(that is $u_\alpha(kT_0)=0$ and $\int_{kT_0}^{(k+1)T_0}u_\alpha(s)ds=0$) we obtain

$$u_\alpha(t)=\int_{kT_0}^t F_\alpha(u)ds\Rightarrow$$
$$0=u_\alpha((k+1)T_0)=\int_{kT_0}^{(k+1)T_0}F_\alpha(u)ds\Rightarrow\int_{kT_0}^{(k+1)T_0}F_\alpha(u)ds=0. \tag{7}$$

Therefore

$$u_\alpha(t)=\int_{kT_0}^t F_\alpha(u)ds\Rightarrow$$
$$u_\alpha(t)=\int_{kT_0}^t F_\alpha(u)ds-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}F_\alpha(u)(s)ds.$$

Besides in view of (7) we have

$$\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{p} F_\alpha(u)(s)dsdp = \int_{kT_0}^{(k+1)T_0}\left[(k+1)T_0 - s\right]F_\alpha(u)(s)ds$$

$$= (k+1)\int_{kT_0}^{(k+1)T_0} F_\alpha(u)(s)ds - \int_{kT_0}^{(k+1)T_0} sF_\alpha(u)(s)ds = -\int_{kT_0}^{(k+1)T_0} sF_\alpha(u)(s)ds.$$

But $\int_{kT_0}^{(k+1)T_0} sF_\alpha(u)(s)ds = \int_{kT_0}^{(k+1)T_0} s\,\dot{u}(s)ds = \int_{kT_0}^{(k+1)T_0} s\,du(s)$

$$= (k+1)T_0 u((k+1)T_0) - kT_0 u(kT_0) - \int_{kT_0}^{(k+1)T_0} u(s)ds = 0$$

that implies $\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{p} F_\alpha(u)(s)dsdp = 0$.

Consequently,

$$u_\alpha(t) = \int_{kT_0}^{t} F_\alpha(u)ds - \left(\frac{t-kT_0}{T_0} - \frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0} F_\alpha(u)(s)ds$$

can be written in the form

$$u_\alpha(t) = \int_{kT_0}^{t} F_\alpha(u)ds - \left(\frac{t-kT_0}{T_0} - \frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0} F_\alpha(u)(s)ds$$
$$- \frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{p} F_\alpha(u)(s)dsdp = (Bu)_\alpha^{(k)}(t),\ (\alpha = 1,2,3).$$

The last equalities mean that B has a fixed point in $M_0 \times M_0 \times M_0$.

Conversely, let $(u_1, u_2, u_3) \in M_0 \times M_0 \times M_0$ be a fixed point of B. Then the last equalities are satisfied and substituting $t = kT_0$ we get

$$0 = u_\alpha(kT_0) = \int_{kT_0}^{kT_0} F_\alpha(u)ds - \left(\frac{kT_0 - kT_0}{T_0} - \frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0} F_\alpha(u)(s)ds$$
$$- \frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s} F_\alpha(u)(\theta)d\theta ds$$

that implies

$$\frac{1}{2}\int_{kT_0}^{(k+1)T_0} F_\alpha(u)(s)ds - \frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s} F_\alpha(u)(\theta)d\theta ds = 0 \Rightarrow$$
$$\frac{1}{2}\int_{kT_0}^{(k+1)T_0} F_\alpha(u)(s)ds = \frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s} F_\alpha(u)(\theta)d\theta ds\ .$$

We show that $\int_{kT_0}^{(k+1)T_0} F_\alpha(u)(s)ds = 0$. Indeed, Let us suppose that $\int_{kT_0}^{(k+1)T_0} F_\alpha(u)(s)ds = \delta \neq 0$. Then we obtain

$$|G_1| \le \frac{e^2}{mc}\left(|F_{11}||u_1(t)| + |F_{12}||u_2(t)| + |F_{13}||u_3(t)| + |E_1|\right)$$
$$\le \frac{e^2}{mc}\left(\bar{c}|H_3| + \bar{c}|H_2| + |E_1| + \left|G_1^{rad}\right|\right),$$

$$|G_2| \le \frac{e^2}{mc}\left(|F_{21}||u_1(t)| + |F_{22}||u_2(t)| + |F_{23}||u_3(t)| + |E_2|\right),$$
$$\le \frac{e^2}{mc}\left(\bar{c}|H_3| + \bar{c}|H_1| + |E_2| + \left|G_2^{rad}\right|\right),$$

$$|G_3| \le \frac{e^2}{mc}\left(|F_{31}||u_1(t)| + |F_{32}||u_2(t)| + |F_{33}||u_3(t)| + |E_3|\right)$$
$$\le \frac{e^2}{mc}\left(\bar{c}|H_2| + \bar{c}|H_1| + |E_3| + \left|G_3^{rad}\right|\right)$$

which implies

$$\int_{kT_0}^{(k+1)T_0} |G_1|dt \le (e^2/mc)\int_{kT_0}^{(k+1)T_0}\left|\Delta\left(H_3u_2 + H_2u_3 - E_1\right) + G_1^{rad}\right|dt$$
$$\le (e^2/mc)\left(\int_{kT_0}^{(k+1)T_0}\left(\bar{c}|H_3| + \bar{c}|H_2| + |E_1|\right)dt + \left|\int_{kT_0}^{(k+1)T_0} G_1^{rad}dt\right|\right) \le$$
$$\le (e^2/mc)\left[\left(\bar{c}|H_0| + \bar{c}|H_0| + |E_0|\right)\int_{kT_0}^{(k+1)T_0} e^{\mu(t-kT_0)}dt\right.$$
$$+ \left|\int_{kT_0}^{(k+1)T_0}\left(\left(\dot{u}_1(t+\tau) - \dot{u}_1(t-\tau)\right)/2\tau\Delta\right)dt\right.$$
$$\left.\left. + \left\langle\vec{u}(t), \left(\dot{\vec{u}}(t+\tau) - \dot{\vec{u}}(t-\tau)\right)/2\tau\right\rangle u_1(t)/\Delta^3\right)dt\right|\right]$$
$$\le (e^2/mc)\left[\left(2\bar{c}|H_0| + |E_0|\right)\left((e^{\mu T_0} - 1)/\mu\right)\right.$$
$$\left. + \left|\frac{1}{(c^2 - \bar{c}^2)^{1/2}}\int_{kT_0}^{(k+1)T_0}\ddot{u}_1(t)dt + \frac{\bar{c}^2}{c^2 - \bar{c}^2)^{3/2}}\int_{kT_0}^{(k+1)T_0}\sum_{\gamma=1}^{3}\ddot{u}_\gamma(t)dt\right|\right]$$
$$= (e^2/mc)\left(2\bar{c}|H_0| + |E_0|\right)\left((e^{\mu T_0} - 1)/\mu\right)$$
$$\le (e^2/mc)\left[\left(2\bar{c}|H_0| + |E_0|\right)\left((e^{\mu T_0} - 1)/\mu\right)\right.$$
$$\left. + \left|\frac{1}{\sqrt{c^2 - \bar{c}^2}}\int_{kT_0}^{(k+1)T_0}\ddot{u}_1(t)dt + \frac{\bar{c}^2}{\left(\sqrt{c^2 - \bar{c}^2}\right)^3}\int_{kT_0}^{(k+1)T_0}\sum_{\gamma=1}^{3}\ddot{u}_\gamma(t)dt\right|\right]$$
$$= (e^2/mc)\left(2\bar{c}|H_0| + |E_0|\right)\left((e^{\mu T_0} - 1)/\mu\right)$$

since $\int_{kT_0}^{(k+1)T_0}\ddot{u}_\alpha(t)dt = 0$.

In a similar way we get

$$\int_{kT_0}^{(k+1)T_0}|G_2|dt \le (e^2/mc)\left(2\bar{c}|H_0| + |E_0|\right)(e^{\mu T_0} - 1)/\mu;$$

$$\int_{kT_0}^{(k+1)T_0} |G_3| dt \le (e^2/mc)\left(2\bar{c}|H_0|+|E_0|\right)(e^{\mu T_0}-1)/\mu .$$

Consequently

$$\left|\int_{kT_0}^{(k+1)T_0} F_1 dt\right| \le \frac{1}{c^2}\left[c^2\left|\int_{kT_0}^{(k+1)T_0}\left(G_1+G_1^{rad}\right)dt\right|+\bar{c}^2\left|\int_{kT_0}^{(k+1)T_0}\left(G_2+G_2^{rad}\right)dt\right|\right.$$
$$\left.+\bar{c}^2\left|\int_{kT_0}^{(k+1)T_0}\left(G_3+G_3^{rad}\right)dt\right|\right]$$
$$=(1/c^2)\left(c^2\int_{kT_0}^{(k+1)T_0}|G_1|dt+\bar{c}^2\int_{kT_0}^{(k+1)T_0}|G_2|dt+\bar{c}^2\int_{kT_0}^{(k+1)T_0}|G_3|dt\right)$$
$$\le (e^2/mc)\left(1+2\beta^2\right)\left(2\bar{c}|H_0|+|E_0|\right)(e^{\mu T_0}-1)/\mu;$$

$$\left|\int_{kT_0}^{(k+1)T_0} F_2 dt\right| \le \frac{1}{c^2}\left[\bar{c}^2\left|\int_{kT_0}^{(k+1)T_0}\left(G_1+G_1^{rad}\right)dt\right|+c^2\left|\int_{kT_0}^{(k+1)T_0}\left(G_2+G_2^{rad}\right)dt\right|\right.$$
$$\left.+\bar{c}^2\left|\int_{kT_0}^{(k+1)T_0}\left(G_3+G_3^{rad}\right)dt\right|\right]$$
$$\le \frac{e^2}{mc}\left(1+2\beta^2\right)\left(2\bar{c}|H_0|+|E_0|\right)\frac{e^{\mu T_0}-1}{\mu};$$

$$\left|\int_{kT_0}^{(k+1)T_0} F_3 dt\right| \le \frac{1}{c^2}\left[\bar{c}^2\left|\int_{kT_0}^{(k+1)T_0}\left(G_1+G_1^{rad}\right)dt\right|+\bar{c}^2\left|\int_{kT_0}^{(k+1)T_0}\left(G_2+G_2^{rad}\right)dt\right|\right.$$
$$\left.+c^2\left|\int_{kT_0}^{(k+1)T_0}\left(G_3+G_3^{rad}\right)dt\right|\right]$$
$$\le (e^2/mc)\left(1+2\beta^2\right)\left(2\bar{c}|H_0|+|E_0|\right)(e^{\mu T_0}-1)/\mu .$$

Since μT_0 = const for sufficiently large $\mu>0$ and small T_0 we can obtain $\delta<\delta$. Therefore

$$u_\alpha(t)=\int_{kT_0}^{t}F_\alpha(u)ds-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}F_\alpha(u)(s)ds$$

becomes $u_\alpha(t)=\int_{kT_0}^{t}F_\alpha(u)ds$. Differentiating the last equality we obtain the required assertion.

Lemma 3 is thus proved.

5. Existence-Uniqueness of the Periodic Problem

Theorem 1 (Main result) Let the following conditions be fulfilled:

1. The initial functions $u_{0\alpha}(t)$ $(\alpha=1,2,3)$ are defined on $[-T_0,0]$ and are such that their translations to the right on $[0,T]$ coincide with some functions from M_0 and $u^{(n)}{}_{0\alpha}(0)=0$ $(n=0,1,2,...;\alpha=1,2,3)$, where $T=mT_0$ for some positive integer *m*.
2. The components of intensity electric and magnetic vectors satisfy the assumptions **(H-E);**
3. The following inequalities are satisfied:

$$\frac{e^2\left(1+2\beta^2\right)}{m}\frac{1}{\mu}\left[\left(\frac{e^{\mu T_0}+1}{2}+\frac{e^{\mu T_0}-1}{\mu T_0}\right)\frac{2\bar{c}H_0+E_0}{c}\right.$$
$$\left.+\left(2\left(4-\beta^2\right)\omega^2U_0/c^3\left(1-\beta^2\right)^{3/2}\right)\right]\le U_0;$$

$$\left(1+2\beta^2\right)\frac{e^2}{m}\frac{1}{\mu}\left[\left(6H_0+\frac{3}{c}E_0\right)\omega+\frac{12\omega^3}{c^3\left(1-\beta^2\right)^{3/2}}U_0\right.$$
$$\left.+\left((e^{\mu T_0}-1)/\mu T_0\right)\left((2cH_0+E_0)/c\right)\right]\le \omega U_0 .$$

Then there is a unique T_0 -periodic solution of (6) $\left(u_1,u_2,u_3\right)\in M_0\times M_0\times M_0$.

Proof: In view of the Main Lemma 3 we have to prove that the operator *B* possesses a unique fixed point. This fixed point is a T_0 -periodic solution of (6).

The set $M_0\times M_0\times M_0$ turns out into a uniform space with a saturated family of pseudo-metrics for $k=0,1,2,...,m$:

$$\rho_{(k,n)}(u_\alpha^{(n)},\bar{u}_\alpha^{(n)})=\operatorname{ess\,sup}\left\{\omega^{-n}\left|u_\alpha^{(n)}(t)-\bar{u}_\alpha^{(n)}(t)\right|e^{-\mu(t-kT_0)}:\right.$$
$$\left. t\in[kT_0,(k+1)T_0]\right\}\ (\alpha=1,2,3),\ (n=0,1,2,...);$$
$$\rho_{(k,n)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))$$
$$=\max\left\{\rho_{(k,n)}(u_1^{(n)},\bar{u}_1^{(n)}),\rho_{(k,n)}(u_2^{(n)},\bar{u}_2^{(n)}),\rho_{(k,n)}(u_3^{(n)},\bar{u}_3^{(n)})\right\};$$
$$\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))$$
$$=\sup\left\{\rho_{(k,n)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3)):n=0,1,2,...\right\}<\infty$$

since

$$\rho_{(k,\infty)}(u_\alpha,\bar{u}_\alpha)=\sup\left\{\rho_{(k,n)}(u_\alpha^{(n)},\bar{u}_\alpha^{(n)}):n=0,1,2,...\right\}\le 2U_0$$

where the index set is

$$A=\{(0,0),(0,1),...,(0,\infty)\}\cup\{(1,0),(1,1),...,(1,\infty)\}$$
$$\cup...\cup\{(m,0),(m,1),...,(m,\infty)\}.$$

Define the operator

$$B:M_0\times M_0\times M_0\to M_0\times M_0\times M_0$$

by the formulas

$$(Bu)_\alpha^{(k)}(t)=\int_{kT_0}^{t}F_\alpha(u)(s)ds-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}F_\alpha(u)(s)ds$$
$$-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}F_\alpha(u)(p)dpds,\ t\in[kT_0,(k+1)T_0]$$
$$(Bu)_\alpha^{(k)}(t)=u_{0\alpha}(t),\ t\le 0$$

$(\alpha=1,2,3;\ k=0,12,...)$, where F_α are the right-hand sides of (6).

We show that B maps $M_0 \times M_0 \times M_0$ into itself. Since

$$(Bu)_\alpha^{(0)}(0) = \begin{cases} \int_0^0 F_\alpha(u)(s)ds + \frac{1}{2}\int_0^{T_0} F_\alpha(u)(s)ds - \frac{1}{T_0}\int_0^{T_0}\int_0^s F_\alpha(u)(p)dpds \\ u_{0\alpha}(0) = 0, \end{cases}$$

we have shown in the Main Lemma 3 that

$$\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t} F_\alpha(u)(s)dsdt = 0 \Rightarrow \int_0^{T_0}\int_0^t F_\alpha(u)(s)dsdt = 0.$$

Therefore $(Bu)_\alpha^{(0)}(0) = 0$.

First we check the following equality

$$\int_{kT_0}^{(k+1)T_0} (Bu)_\alpha^{(k)}(t)dt = \int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t} F_\alpha(u)(s)dsdt$$
$$-\int_{kT_0}^{(k+1)T_0}\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)dt\int_{kT_0}^{(k+1)T_0} F_\alpha(u)(s)ds$$
$$-\int_{kT_0}^{(k+1)T_0}\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s} F_\alpha(u)(p)dpdsdt$$
$$= \int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t} F_\alpha(u)(s)dsdt$$
$$-T_0\frac{\left(\frac{(k+1)T_0-kT_0}{T_0}-\frac{1}{2}\right)^2-\left(\frac{kT_0-kT_0}{T_0}-\frac{1}{2}\right)^2}{2}$$
$$\int_{kT_0}^{(k+1)T_0} F_\alpha(u)(s)ds$$
$$-\frac{T_0}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s} F_\alpha(u)(p)dpds = 0.$$

Further on for $t \in [kT_0,(k+1)T_0]$ we obtain

$$B_1 = \left|\int_{kT_0}^{t} F_\alpha(u)(s)ds\right| \le \frac{1}{c^2}\left[c^2\left|\int_{kT_0}^{t}\left(G_1+G_1^{rad}\right)dt\right|\right.$$
$$\left.+\bar{c}^2\left|\int_{kT_0}^{t}\left(G_2+G_2^{rad}\right)dt\right|+\bar{c}^2\left|\int_{kT_0}^{t}\left(G_3+G_3^{rad}\right)dt\right|\right]$$
$$\le (1/c^2)\left[c^2\int_{kT_0}^{t}|G_1|ds + c^2\int_{kT_0}^{t}\left|G_1^{rad}\right|ds + \bar{c}^2\int_{kT_0}^{t}|G_2|ds\right.$$
$$\left.+\bar{c}^2\int_{kT_0}^{t}\left|G_2^{rad}\right|ds + \bar{c}^2\int_{kT_0}^{t}|G_3|ds + \bar{c}^2\int_{kT_0}^{t}\left|G_3^{rad}\right|ds\right].$$

We have

$$\int_{kT_0}^{t}|G_1|ds \le \frac{e^2c}{mc^2}\int_{kT_0}^{t}\left|\sum_{\beta=1}^{3}F_{1\beta}u_\beta(s)-E_1\right|ds$$
$$\le \frac{e^2}{mc}\int_{kT_0}^{t}\left(\bar{c}|H_2|+\bar{c}|H_3|+|E_1|\right)ds \le \frac{e^2}{m}\frac{2\bar{c}H_0+E_0}{c}\frac{e^{\mu(t-kT_0)}-1}{\mu};$$

$$\int_{kT_0}^{t}|G_2|ds \le \frac{e^2c}{mc^2}\int_{kT_0}^{t}\left|\sum_{\beta=1}^{3}F_{2\beta}u_\beta(s)-E_2\right|ds$$
$$\le \frac{e^2}{mc}\int_{kT_0}^{t}\frac{e^2}{mc}\left(\bar{c}|H_3|+\bar{c}|H_1|+|E_2|\right)ds \le \frac{e^2}{m}\frac{2\bar{c}|H_0|+|E_0|}{c}\frac{e^{\mu(t-kT_0)}-1}{\mu};$$

$$\int_{kT_0}^{t}|G_3|ds \le \frac{e^2c}{mc^2}\int_{kT_0}^{t}\left|\sum_{\beta=1}^{3}F_{3\beta}u_\beta(s)-E_3\right|ds$$
$$\le \frac{e^2}{mc}\int_{kT_0}^{t}\left(\bar{c}|H_3|+\bar{c}|H_1|+|E_3|\right)ds \le \frac{e^2}{m}\frac{2\bar{c}|H_0|+|E_0|}{c}\frac{e^{\mu(t-kT_0)}-1}{\mu};$$

$$\left|\int_{kT_0}^{t}G_\alpha^{rad}ds\right| \le \frac{e^2}{mc^2}\left|\int_{kT_0}^{t}\left[\frac{|u_\alpha(s)|}{\left(c^2-\langle\vec{u}(s),\vec{u}(s)\rangle\right)^{3/2}}\sum_{\gamma=1}^{3}u_\gamma(s)\frac{\dot{u}_\gamma(s+\tau)-\dot{u}_\gamma(s-\tau)}{2\tau}\right.\right.$$
$$\left.\left.+\frac{1}{\left(c^2-\langle\vec{u}(s),\vec{u}(s)\rangle\right)^{1/2}}\frac{\dot{u}_\alpha(s+\tau)-\dot{u}_\alpha(s-\tau)}{2\tau}\right]ds\right|$$
$$\le \frac{e^2}{mc^2}\left[\frac{c^2}{\left(c^2-\bar{c}^2\right)^{3/2}}\sum_{\gamma=1}^{3}\int_{kT_0}^{t}\frac{\dot{u}_\gamma(s+\tau)-\dot{u}_\gamma(s-\tau)}{2\tau}ds\right.$$
$$\left.+\frac{1}{\left(c^2-\bar{c}^2\right)^{1/2}}\int_{kT_0}^{t}\frac{\dot{u}_\alpha(s+\tau)-\dot{u}_\alpha(s-\tau)}{2\tau}ds\right]$$
$$\le \frac{e^2}{mc^2}\left(\frac{1}{c\left(1-\beta^2\right)^{3/2}}\sum_{\gamma=1}^{3}\left|\int_{kT_0}^{t}\ddot{u}_\gamma(s)ds\right|+\frac{1-\beta^2}{c\left(1-\beta^2\right)^{3/2}}\left|\int_{kT_0}^{t}\ddot{u}_\alpha(s)ds\right|\right)$$
$$\le \frac{e^2}{mc^2}\frac{4-\beta^2}{c\left(1-\beta^2\right)^{3/2}}\omega^2U_0\frac{e^{\mu(t-kT_0)}-1}{\mu}.$$

Therefore

$$B_1 = \left|\int_{kT_0}^{t}F_\alpha(u)(s)ds\right| \le \left(1+2\beta^2\right)\frac{e^2}{m}\frac{e^{\mu(t-kT_0)}-1}{\mu}$$
$$\cdot\left[\frac{2\bar{c}H_0+E_0}{c}+\frac{\left(4-\beta^2\right)\omega^2U_0}{c^3\left(1-\beta^2\right)^{3/2}}\right]$$
$$\le \frac{e^2\left(1+2\beta^2\right)}{m}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{\left(4-\beta^2\right)\omega^2U_0}{c^3\left(1-\beta^2\right)^{3/2}}\right)\frac{e^{\mu(t-kT_0)}}{\mu};$$

$$B_2 = \left|\frac{t-kT_0}{T_0}-\frac{1}{2}\right|\left|\int_{kT_0}^{(k+1)T_0}F_\alpha(u)(s)ds\right|,$$
$$\le \frac{1}{2}\frac{e^2\left(1+2\beta^2\right)}{m}\frac{2\bar{c}H_0+E_0}{c}\frac{e^{\mu T_0}-1}{\mu};$$

$$B_3=\frac{1}{T_0}\left|\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}F_\alpha(u)(p)dpds\right|$$

$$\le\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\left|\int_{kT_0}^{s}F_\alpha(u)(p)dp\right|ds\le$$

$$\le\frac{e^2\left(1+2\beta^2\right)}{T_0 m}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{\left(4-\beta^2\right)\omega^2U_0}{c^3\left(1-\beta^2\right)^{3/2}}\right)\frac{1}{\mu}\int_{kT_0}^{(k+1)T_0}e^{\mu(s-kT_0)}ds$$

$$=\frac{1}{T_0}\frac{e^2\left(1+2\beta^2\right)}{m}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{\left(4-\beta^2\right)\omega^2U_0}{c^3\left(1-\beta^2\right)^{3/2}}\right)\frac{1}{\mu}\frac{e^{\mu T_0}-1}{\mu}.$$

Finally we reach the estimate

$$\left|(Bu)_\alpha^{(k)}(t)\right|\le T_1+T_2+T_3$$

$$\le e^{\mu(t-kT_0)}\frac{e^2\left(1+2\beta^2\right)}{m}\frac{1}{\mu}\left[\left(\frac{e^{\mu T_0}+1}{2}+\frac{e^{\mu T_0}-1}{\mu T_0}\right)\frac{2\bar{c}H_0+E_0}{c}\cdot\right.$$

$$\left.+\left(2\left(4-\beta^2\right)\omega^2U_0\right)/\left(c^3\left(1-\beta^2\right)^{3/2}\right)\right]\le U_0e^{\mu(t-kT_0)}.$$

Let us estimate the derivatives

$$\frac{d(Bu)_\alpha^{(k)}(t)}{dt}=\begin{cases}F_\alpha(u)(t)-\dfrac{1}{T_0}\displaystyle\int_{kT_0}^{(k+1)T_0}F_\alpha(u)(s)ds,\ t\in[kT_0,(k+1)T_0]\\ \dot{u}_{0\alpha}(0)=0\end{cases}.$$

Then in view of

$$\left|\ddot{u}_\gamma(t)\right|=\left|\int_{kT_0}^{t}\int_{kT_0}^{s}\left(u_\gamma(t)\right)^{(4)}(p)dpds\right|\le\frac{\omega^4}{\mu^2}U_0e^{\mu(t-kT_0)}$$

we obtain

$$\left|G_\alpha^{rad}\right|\le\frac{e^2}{mc^2}\left[\frac{|u_\alpha(t)|}{\left(c^2-\langle\vec{u}(t),\vec{u}(t)\rangle\right)^{3/2}}\left|\sum_{\gamma=1}^{3}u_\gamma(t)\ddot{u}_\gamma(t)\right|\right.$$

$$\left.+\left|\ddot{u}_\alpha(t)\right|/\left(c^2-\langle\vec{u}(t),\vec{u}(t)\rangle\right)^{1/2}\right]$$

$$\le\frac{e^2}{mc^2}\left[\frac{c^2}{\left(c^2-\langle\vec{u}(t),\vec{u}(t)\rangle\right)^{3/2}}\left|\sum_{\gamma=1}^{3}\ddot{u}_\gamma(t)\right|\right.$$

$$\left.+\frac{c^2-\langle\vec{u}(t),\vec{u}(t)\rangle}{\left(c^2-\langle\vec{u}(t),\vec{u}(t)\rangle\right)^{3/2}}\left|\ddot{u}_\alpha(t)\right|\right]$$

$$\le\frac{e^2}{mc^2}\frac{c^2}{\left(c^2-\langle\vec{u}(t),\vec{u}(t)\rangle\right)^{3/2}}\left[\left|\sum_{\gamma=1}^{3}\ddot{u}_\gamma(t)\right|+\left|\ddot{u}_\alpha(t)\right|\right]$$

$$\le\frac{e^2}{m}\frac{1}{\left(c^2-\bar{c}^2\right)^{3/2}}4\frac{\omega^4}{\mu^2}U_0e^{\mu(t-kT_0)}$$

$$=\frac{4e^2}{m}\frac{1}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4U_0}{\mu^2}e^{\mu(t-kT_0)}$$

we obtain

$$\left|\frac{d(Bu)_\alpha^{(k)}(t)}{dt}\right|\le\left|F_\alpha(u)(t)\right|+\frac{1}{T_0}\left|\int_{kT_0}^{(k+1)T_0}F_\alpha(u)(s)ds\right|\le$$

$$\le\left(1+2\beta^2\right)\left(|G_1|+|G_2|+|G_3|+\left|G_1^{rad}\right|+\left|G_2^{rad}\right|+\left|G_3^{rad}\right|\right)$$

$$+\frac{1}{T_0}\frac{e^2\left(1+2\beta^2\right)}{m}\frac{2\bar{c}H_0+E_0}{c}\frac{e^{\mu T_0}-1}{\mu}$$

$$\le\left(1+2\beta^2\right)\left(\frac{e^2}{m}3\frac{2\bar{c}H_0+E_0}{c}+\left|G_1^{rad}\right|+\left|G_2^{rad}\right|+\left|G_3^{rad}\right|\right)$$

$$+\frac{e^2}{m}\frac{1+2\beta^2}{T_0}\frac{2\bar{c}H_0+E_0}{c}\frac{e^{\mu T_0}-1}{\mu}.$$

But

$$|G_1|+|G_2|+|G_3|\le\frac{e^2}{mc}\left|\sum_{\beta=1}^{3}F_{1\beta}u_\beta(t)-E_1\right|$$

$$+\frac{e^2}{mc}\left|\sum_{\beta=1}^{3}F_{2\beta}u_\beta(t)-E_2\right|+\frac{e^2}{mc}\left|\sum_{\beta=1}^{3}F_{3\beta}u_\beta(t)-E_3\right|$$

$$\le\frac{e^2}{mc}\left(c|F_{11}|+c|F_{12}|+c|F_{13}|+|E_1|+c|F_{21}|+c|F_{22}|\right.$$

$$\left.+c|F_{23}|+|E_2|+c|F_{31}|+c|F_{32}|+c|F_{33}|+|E_3|\right)$$

$$\le\frac{e^2}{mc}\left(2c|H_1|+2c|H_2|+2c|H_3|+|E_1|+|E_2|+|E_3|\right)$$

$$\le\frac{e^2}{mc}\left(2c\left|\int_{kT_0}^{t}\frac{\partial H_1}{\partial s}ds\right|+2c\left|\int_{kT_0}^{t}\frac{\partial H_2}{\partial s}ds\right|+2c\left|\int_{kT_0}^{t}\frac{\partial H_3}{\partial s}ds\right|\right.$$

$$\left.+\left|\int_{kT_0}^{t}\frac{\partial E_1}{\partial s}ds\right|+\left|\int_{kT_0}^{t}\frac{\partial E_2}{\partial s}ds\right|+\left|\int_{kT_0}^{t}\frac{\partial E_3}{\partial s}ds\right|\right)$$

$$\le\frac{e^2}{m}\left(6\frac{\omega H_0}{\mu}e^{\mu(t-kT_0)}+\frac{3}{c}\frac{\omega E_0}{\mu}e^{\mu(t-kT_0)}\right)$$

$$=3\left(e^2/m\right)\left(2H_0+(E_0/c)\right)(\omega/\mu)e^{\mu(t-kT_0)}$$

and consequently

$$\left|\frac{d(Bu)_\alpha^{(k)}(t)}{dt}\right|\le\left|F_\alpha(u)(t)\right|+\frac{1}{T_0}\left|\int_{kT_0}^{(k+1)T_0}F_\alpha(u)(s)ds\right|\le$$

$$\le\left(1+2\beta^2\right)\left(|G_1|+|G_2|+|G_3|+\left|G_1^{rad}\right|+\left|G_2^{rad}\right|+\left|G_3^{rad}\right|\right)$$

$$+\frac{e^2\left(1+2\beta^2\right)}{m}\frac{e^{\mu T_0}-1}{\mu T_0}\frac{2cH_0+E_0}{\mu c}e^{\mu(t-kT_0)}$$

$$\le e^{\mu(t-kT_0)}\left(1+2\beta^2\right)\frac{e^2}{m}\frac{1}{\mu}\left[\left(6H_0+\frac{3}{c}E_0\right)\omega+\frac{12\omega^3}{c^3\left(1-\beta^2\right)^{3/2}}U_0\right.$$

$$\left.+\left((2cH_0+E_0)(e^{\mu T_0}-1)/\mu T_0c\right)\right]\le\omega U_0e^{\mu(t-kT_0)}$$

and so on. Therefore $B\in M_0\times M_0\times M_0$.

Remark 2. In order to obtain suitable estimations for higher derivatives we use the chain of inequalities

$$\rho_{(k,0)}(u_\sigma,\bar{u}_\sigma)\le\frac{1}{\mu}\rho_{(k,1)}(\dot{u}_\sigma,\dot{\bar{u}}_\sigma)\le\frac{1}{\mu^2}\rho_{(k,2)}(\ddot{u}_\sigma,\ddot{\bar{u}}_\sigma)\le\ldots .$$

In this way we compensate the degree of ω in the nominator by the degree of μ in the denominator.

In what follows we show that B is contractive operator.

First we notice that the following Lipschitz estimate for the expression

$$G_\alpha(u)=\frac{e^2}{mc^2}\sqrt{c^2-\sum_{\gamma=1}^{3}u_\gamma{}^2(t)}\left(\sum_{\beta=1}^{3}F_{\alpha\beta}u_\beta(t)-E_\alpha\right)(\alpha=1,2,3)$$

is valid:

$$\left|G_\alpha(u_1,u_2,u_3)-G_\alpha(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\le\sum_{\beta=1}^{3}\left|\frac{\partial G_\alpha}{\partial u_\beta}\right|\left|u_\beta(t)-\bar{u}_\beta(t)\right|$$

$$\le\frac{e^2}{mc^2}\sum_{\sigma=1}^{3}\left(\frac{|-2u_\sigma|}{2\left(c^2-\langle\vec{u}(t),\vec{u}(t)\rangle\right)^{1/2}}\left|\sum_{\beta=1}^{3}F_{\alpha\sigma}u_\sigma(t)-E_\alpha\right|\right.$$

$$\left.+\left(c^2-\langle\vec{u}(t),\vec{u}(t)\rangle\right)^{1/2}\left|F_{\alpha\sigma}\right|\right)\left|u_\sigma(t)-\bar{u}_\sigma(t)\right|$$

$$\le e^{\mu(t-kT_0)}\frac{e^2}{mc^2}\left(\frac{2cH_0+E_0}{\sqrt{1-\beta^2}}+cH_0\right)\sum_{\sigma=1}^{3}\left|u_\sigma(t)-\bar{u}_\sigma(t)\right|;$$

and in a similar way from

$$G_\alpha^{rad}(u)=\frac{e^2}{mc^2}\left[\frac{u_\alpha(t)\sum_{\gamma=1}^{3}u_\gamma(t)\ddot{u}_\gamma(t)}{\left(c^2-\langle\vec{u}(t),\vec{u}(t)\rangle\right)^{3/2}}-\frac{\ddot{u}_\alpha(t)}{\left(c^2-\langle\vec{u}(t),\vec{u}(t)\rangle\right)^{1/2}}\right]$$

we obtain

$$\left|G_\alpha^{rad}(u_1,u_2,u_3)-G_\alpha^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|$$

$$\le\left|\frac{\partial G_\alpha^{rad}}{\partial u_\alpha}\right|\left|u_\alpha(t)-\bar{u}_\alpha(t)\right|+\left|\frac{\partial G_\alpha^{rad}}{\partial\ddot{u}_\alpha}\right|\left|\ddot{u}_\alpha(t)-\ddot{\bar{u}}_\alpha(t)\right|$$

$$\le\frac{e^2}{m}\left[e^{\mu(t-kT_0)}\frac{7\omega^2U_0}{c^4(1-\beta^2)^{5/2}}\left|u_\alpha(t)-\bar{u}_\alpha(t)\right|\right.$$

$$\left.+\left(2/c^3(1-\beta^2)^{3/2}\right)\left|\ddot{u}_\alpha(t)-\ddot{\bar{u}}_\alpha(t)\right|\right].$$

Then

$$\left|(Bu)_\alpha^{(k)}(t)-(B\bar{u})_\alpha^{(k)}(t)\right|\le\left|\int_{kT_0}^{t}\left(F_\alpha(u)(s)-F_\alpha(\bar{u})\right)(s)ds\right|$$

$$+\frac{1}{2}\left|\int_{kT_0}^{(k+1)T_0}\left(F_\alpha(u)(s)-F_\alpha(\bar{u})(s)\right)ds\right|$$

$$+\frac{1}{T_0}\left|\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}\left(F_\alpha(u)(p)-F_\alpha(\bar{u})(p)\right)dpds\right|;$$

$$\left|\int_{kT_0}^{t}\left(F_1(u_1,u_2,u_3)(s)-F_1(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right)(s)ds\right|$$

$$\le(1/c^2)\int_0^t\left[(c^2-u_1^2)\left(\left|G_1(u_1,u_2,u_3)-G_1(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)\right.$$

$$\left.+\left|G_1^{rad}(u_1,u_2,u_3)-G_1^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right]ds$$

$$+(1/c^2)\int_0^t\left[\left|u_1u_2\right|\left(\left|G_2(u_1,u_2,u_3)-G_2(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)\right.$$

$$\left.+\left|G_2^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)-G_2^{rad}(u_1,u_2,u_3)\right|\right]ds$$

$$+(1/c^2)\int_0^t\left|u_1u_3\right|\left[\left|G_3(u_1,u_2,u_3)-G_3(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right.$$

$$\left.+\left|G_3^{rad}(u_1,u_2,u_3)-G_3^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right]ds$$

$$+(1/c^2)\int_0^t\left|u_1^2-\bar{u}_1^2\right|\left(\left|G_1(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_1^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)ds$$

$$+(1/c^2)\int_0^t\left|u_1u_2-\bar{u}_1\bar{u}_2\right|\left(\left|G_2(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_2^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)ds$$

$$+(1/c^2)\int_0^t\left|u_1u_3-\bar{u}_1\bar{u}_3\right|\left(\left|G_3(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_3^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)ds$$

$$\le\sum_{\alpha=1}^{3}\int_0^t\left|G_\alpha(u_1,u_2,u_3)-G_\alpha(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|ds$$

$$+\sum_{\alpha=1}^{3}\int_0^t\left|G_\alpha^{rad}(u_1,u_2,u_3)-G_\alpha^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|ds$$

$$+(2/c)\int_0^t\left(\left|G_1(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_1^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)\left|u_1(s)-\bar{u}_1(s)\right|ds$$

$$+\left(1/c^2\right)\int_0^t\left[\left|G_2(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|\bar{u}_1\right|\left|u_2-\bar{u}_2\right|\right.$$

$$\left.+\left|G_2^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right]\left(\left|u_2\right|\left|u_1-\bar{u}_1\right|\right)ds$$

$$+\left(1/c^2\right)\int_0^t\left(\left|G_3(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_3^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)$$

$$\left(\left|u_1\right|\left|u_3-\bar{u}_3\right|+\left|\bar{u}_3\right|\left|u_1-\bar{u}_1\right|\right)ds$$

$$\le\frac{e^2}{mc^2}\left(\frac{2cH_0+E_0}{\sqrt{1-\beta^2}}+cH_0\right)\sum_{\alpha=1}^{3}\int_0^t e^{\mu(s-kT_0)}\sum_{\sigma=1}^{3}\left|u_\sigma(s)-\bar{u}_\sigma(s)\right|ds$$

$$+\frac{e^2}{m}\sum_{\alpha=1}^{3}\left[\frac{7\omega^2U_0}{c^4(1-\beta^2)^{5/2}}\int_0^t e^{\mu(s-kT_0)}\left|u_\alpha(s)-\bar{u}_\alpha(s)\right|\right.$$

$$\left.+\left(2/c^3(1-\beta^2)^{3/2}\right)\int_0^t\left|\ddot{u}_\alpha(s)-\ddot{\bar{u}}_\alpha(s)\right|ds\right]$$

$$+\frac{e^2}{m}\frac{2}{c}\left(\frac{2\overline{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$

$$\int_0^t e^{\mu(s-kT_0)}\left|u_1(s)-\overline{u}_1(s)\right|ds+$$

$$+\frac{e^2}{m}\frac{1}{c}\left(\frac{2\overline{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$

$$.\int_0^t e^{\mu(s-kT_0)}\left(\left|u_1(s)-\overline{u}_1(s)\right|+\left|u_2(s)-\overline{u}_2(s)\right|\right)ds$$

$$+\frac{e^2}{m}\frac{1}{c}\left(\frac{2\overline{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$

$$.\int_0^t e^{\mu(s-kT_0)}\left(\left|u_3(s)-\overline{u}_3(s)\right|+\left|u_1(s)-\overline{u}_1(s)\right|\right)ds$$

$$\le\frac{9e^2}{mc^2}\left(\frac{2cH_0+E_0}{\sqrt{1-\beta^2}}+cH_0\right)\frac{e^{2\mu(t-kT_0)}-1}{2\mu}$$

$$\rho_{(k,\infty)}((u_1,u_2,u_3),(\overline{u}_1,\overline{u}_2,\overline{u}_3))+$$

$$+\frac{3e^2}{m(1-\beta^2)^{5/2}}\left(\frac{7\omega^2U_0}{c^4}+\frac{2(1-\beta^2)}{c^3}\right)$$

$$.\left((e^{2\mu(t-kT_0)}-1)/2\mu\right)\rho_{(k,\infty)}((u_1,u_2,u_3),(\overline{u}_1,\overline{u}_2,\overline{u}_3))$$

$$+\frac{e^2}{m}\frac{2}{c}\left(\frac{2\overline{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$

$$.\left((e^{2\mu(t-kT_0)}-1)/2\mu\right)\rho_{(k,\infty)}((u_1,u_2,u_3),(\overline{u}_1,\overline{u}_2,\overline{u}_3))$$

$$+\frac{e^2}{m}\frac{2}{c}\left(\frac{2\overline{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$

$$.\,\rho_{(k,\infty)}((u_1,u_2,u_3),(\overline{u}_1,\overline{u}_2,\overline{u}_3))$$

$$+\frac{e^2}{m}\frac{2}{c}\left(\frac{2\overline{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$

$$.\left((e^{2\mu(t-kT_0)}-1)/2\mu\right)\rho_{(k,\infty)}((u_1,u_2,u_3),(\overline{u}_1,\overline{u}_2,\overline{u}_3))$$

$$\le e^{\mu(t-kT_0)}\frac{3e^2}{mc^2}\frac{e^{\mu T_0}}{2\mu}\left(\frac{13cH_0+5E_0}{\sqrt{1-\beta^2}}+\frac{7\omega^2U_0+2c(1-\beta^2)}{c^2(1-\beta^2)^{5/2}}\right.$$

$$\left.+\frac{8\omega^4U_0}{c^2\left(1-\beta^2\right)^{5/2}\mu^2}\right)\rho_{(k,\infty)}((u_1,u_2,u_3),(\overline{u}_1,\overline{u}_2,\overline{u}_3))\ ;$$

$$\left|\int_{kT_0}^t\left(F_2(u_1,u_2,u_3)(s)-F_2(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right)(s)ds\right|$$

$$\le(1/c^2)\int_0^t\left|u_1u_2\right|\left(\left|G_1(u_1,u_2,u_3)-G_1(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|\right.$$

$$\left.+\left|G_1^{rad}(u_1,u_2,u_3)-G_1^{rad}(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|\right)ds$$

$$+(1/c^2)\int_0^t\left|u_2u_3\right|\left[\left|G_3(u_1,u_2,u_3)-G_3(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|\right.$$

$$\left.+\left|G_3^{rad}(u_1,u_2,u_3)-G_3^{rad}(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|\right]ds$$

$$+(1/c^2)\int_0^t(c^2-u_2^2)\left[\left|G_2(u_1,u_2,u_3)-G_2(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|\right.$$

$$\left.+\left|G_2^{rad}(\overline{u}_1,\overline{u}_2,\overline{u}_3)-G_2^{rad}(u_1,u_2,u_3)\right|\right]ds$$

$$+(1/c^2)\int_0^t\left|u_1u_2-\overline{u}_1\overline{u}_2\right|\left(\left|G_1(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|+\left|G_1^{rad}(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|\right)ds$$

$$+(1/c^2)\int_0^t\left|u_2^2-\overline{u}_2^2\right|\left(\left|G_2(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|+\left|G_2^{rad}(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|\right)ds$$

$$+(1/c^2)\int_0^t\left|u_2u_3-\overline{u}_2\overline{u}_3\right|\left(\left|G_3(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|+\left|G_3^{rad}(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|\right)ds$$

$$\le\sum_{\alpha=1}^3\int_0^t\left|G_\alpha(u_1,u_2,u_3)-G_\alpha(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|ds$$

$$+\sum_{\alpha=1}^3\int_0^t\left|G_\alpha^{rad}(u_1,u_2,u_3)-G_\alpha^{rad}(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|ds$$

$$+(1/c^2)\int_0^t\left(\left|G_1(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|+\left|G_1^{rad}(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|\right)$$

$$.\left(\left|u_2\right|\left|u_1-\overline{u}_1\right|+\left|\overline{u}_1\right|\left|u_2-\overline{u}_2\right|\right)ds$$

$$+(2/c)\int_0^t\left(\left|G_2(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|+\left|G_2^{rad}(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|\right)\left|u_2(s)-\overline{u}_2(s)\right|ds$$

$$+(1/c^2)\int_0^t\left(\left|G_3(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|+\left|G_3^{rad}(\overline{u}_1,\overline{u}_2,\overline{u}_3)\right|\right)$$

$$.\left(\left|u_2\right|\left|u_3-\overline{u}_3\right|+\left|\overline{u}_3\right|\left|u_2-\overline{u}_2\right|\right)ds$$

$$\le\frac{e^2}{mc^2}\left(\frac{2cH_0+E_0}{\sqrt{1-\beta^2}}+cH_0\right)\sum_{\alpha=1}^3\int_0^t e^{\mu(s-kT_0)}\sum_{\sigma=1}^3\left|u_\sigma(s)-\overline{u}_\sigma(s)\right|ds$$

$$+\frac{e^2}{m}\sum_{\alpha=1}^3\frac{7\omega^2U_0}{c^4(1-\beta^2)^{5/2}}\left[\int_0^t e^{\mu(s-kT_0)}\left|u_\alpha(s)-\overline{u}_\alpha(s)\right|ds\right.$$

$$\left.+\left(2/c^3(1-\beta^2)^{3/2}\right)\int_0^t\left|\ddot{u}_\alpha(s)-\ddot{\overline{u}}_\alpha(s)\right|ds\right]$$

$$+\frac{e^2}{m}\frac{1}{c}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$

$$\cdot\int_0^t e^{\mu(s-kT_0)}\left(\left|u_1(s)-\bar{u}_1(s)\right|+\left|u_2(s)-\bar{u}_2(s)\right|\right)ds$$

$$+\frac{e^2}{m}\frac{2}{c}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$

$$\cdot\int_0^t e^{\mu(s-kT_0)}\left|u_2(s)-\bar{u}_2(s)\right|ds$$

$$+\frac{e^2}{m}\frac{1}{c}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$

$$\cdot\int_0^t e^{\mu(s-kT_0)}\left(\left|u_3(s)-\bar{u}_3(s)\right|+\left|u_2(s)-\bar{u}_2(s)\right|\right)ds$$

$$\leq e^{\mu(t-kT_0)}\frac{3e^2}{mc^2}\frac{e^{\mu T_0}}{2\mu}\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))$$

$$\cdot\left(\frac{13cH_0+5E_0}{\sqrt{1-\beta^2}}+\frac{7\omega^2U_0+2c(1-\beta^2)}{c^2(1-\beta^2)^{5/2}}+\frac{8\omega^4U_0}{c^2\left(1-\beta^2\right)^{5/2}\mu^2}\right);$$

$$\left|\int_{kT_0}^t\left(F_3(u_1,u_2,u_3)(s)-F_3(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right)(s)ds\right|$$

$$\leq\sum_{\alpha=1}^3\int_0^t\left|G_\alpha(u_1,u_2,u_3)-G_\alpha(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|ds$$

$$+\sum_{\alpha=1}^3\int_0^t\left|G_\alpha^{rad}(u_1,u_2,u_3)-G_\alpha^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|ds$$

$$+\frac{1}{c^2}\int_0^t\left(\left|G_1(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_1^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)\left(|u_3|\left|u_1-\bar{u}_1\right|+|\bar{u}_1|\left|u_3-\bar{u}_3\right|\right)ds$$

$$+\frac{2}{c}\int_0^t\left(\left|G_2(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_2^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)\left(|u_2|\left|u_3-\bar{u}_3\right|+|\bar{u}_3|\left|u_2-\bar{u}_2\right|\right)ds$$

$$+\frac{1}{c^2}\int_0^t\left(\left|G_3(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_3^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)\left|u_3(s)-\bar{u}_3(s)\right|ds\leq$$

$$\leq e^{\mu(t-kT_0)}\frac{3e^2}{mc^2}\frac{e^{\mu T_0}}{2\mu}\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))$$

$$\cdot\left(\frac{13cH_0+5E_0}{\sqrt{1-\beta^2}}+\frac{7\omega^2U_0+2c}{c^2(1-\beta^2)^{5/2}}+\frac{8\omega^4U_0}{c^2\left(1-\beta^2\right)^{5/2}\mu^2}\right).$$

For the second term we obtain

$$\frac{1}{2}\left|\int_{kT_0}^{(k+1)T_0}\left(F_\alpha(u)(s)-F_\alpha(\bar{u})(s)\right)ds\right|$$

$$\leq\frac{e^2}{m}\frac{e^{2\mu T_0}-1}{2\mu}\left[\frac{9}{2c^2}\left(\frac{2cH_0+E_0}{\sqrt{1-\beta^2}}+cH_0\right)\right.$$

$$+\left(3/2(1-\beta^2)^{5/2}\right)\left(\left(7\omega^2U_0/c^4\right)+\left(2(1-\beta^2)/c^3\right)\right)$$

$$\left.+\left((6\bar{c}H_0+3E_0)/c^2\right)+\left(12-3\beta^2\omega^4U_0/c^4\left(1-\beta^2\right)^{3/2}\mu^2\right)\right]$$

$$\cdot\,\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))$$

$$\leq\left(3e^2(e^{2\mu T_0}-1)/2\mu mc^2\right)\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3)\ .$$

$$\cdot\left(\frac{11cH_0+4E_0}{2\sqrt{1-\beta^2}}+\frac{7\omega^2U_0+2c}{2(1-\beta^2)^{5/2}c^2}+\frac{4}{c^2\left(1-\beta^2\right)^{5/2}}\frac{\omega^4U_0}{\mu^2}\right).$$

For the third term we get

$$\frac{1}{T_0}\left|\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}\left(F_\alpha(u)(p)-F_\alpha(\bar{u})(p)\right)dpds\right|\leq$$

$$\leq\frac{1}{T_0}\left|\int_{kT_0}^{(k+1)T_0}e^{\mu(t-kT_0)}dt\right|\frac{3e^2}{mc^2}\frac{e^{\mu T_0}}{2\mu}\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))$$

$$\cdot\left(\frac{13cH_0+5E_0}{\sqrt{1-\beta^2}}+\frac{7\omega^2U_0+2c}{c^2(1-\beta^2)^{5/2}}+\frac{8\omega^4U_0}{c^2\left(1-\beta^2\right)^{5/2}\mu^2}\right)$$

$$\leq\frac{e^{\mu T_0}-1}{\mu T_0}\frac{3e^2}{mc^2}\frac{e^{\mu T_0}}{2\mu}\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))$$

$$\cdot\left(\frac{13cH_0+5E_0}{\sqrt{1-\beta^2}}+\frac{7\omega^2U_0+2c}{c^2(1-\beta^2)^{5/2}}+\frac{8\omega^4U_0}{c^2\left(1-\beta^2\right)^{5/2}\mu^2}\right).$$

Therefore

$$\left|(Bu)_\alpha^{(k)}(t)-(B\bar{u})_\alpha^{(k)}(t)\right|\leq$$

$$\leq\left|\int_{kT_0}^t\left(F_\alpha(u)(s)-F_\alpha(\bar{u})\right)(s)ds\right|+\frac{1}{2}\left|\int_{kT_0}^{(k+1)T_0}\left(F_\alpha(u)(s)-F_\alpha(\bar{u})(s)\right)ds\right|$$

$$+\frac{1}{T_0}\left|\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}\left(F_\alpha(u)(p)-F_\alpha(\bar{u})(p)\right)dpds\right|$$

$$\leq e^{\mu(t-kT_0)}\frac{3e^2}{mc^2}\frac{e^{\mu T_0}}{2\mu}\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))$$

$$\cdot\left(\frac{13cH_0+5E_0}{\sqrt{1-\beta^2}}+\frac{7\omega^2U_0+2c}{c^2(1-\beta^2)^{5/2}}+\frac{8\omega^4U_0}{c^2\left(1-\beta^2\right)^{5/2}\mu^2}\right)$$

$$+\left(3e^2(e^{2\mu T_0}-1)/2\mu mc^2\right)\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3)$$

$$\cdot\left(\frac{11cH_0+4E_0}{2\sqrt{1-\beta^2}}+\frac{7\omega^2U_0+2c}{2(1-\beta^2)^{5/2}c^2}+\frac{4}{c^2\left(1-\beta^2\right)^{5/2}}\frac{\omega^4U_0}{\mu^2}\right)$$

$$+\frac{3e^2}{mc^2}\frac{e^{\mu T_0}-1}{\mu T_0}\frac{e^{\mu T_0}}{2\mu}\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))$$
$$.\left(\frac{13cH_0+5E_0}{\sqrt{1-\beta^2}}+\frac{7\omega^2U_0+2c}{c^2(1-\beta^2)^{5/2}}+\frac{8\omega^4U_0}{c^2\left(1-\beta^2\right)^{5/2}\mu^2}\right)$$
$$\le e^{\mu(t-kT_0)}\Big[\left(e^{\mu T_0}/2\mu\right)\Big[\left((13cH_0+5E_0)/(1-\beta^2)^{1/2}\right)$$
$$+\left(\left(7\omega^2U_0+2c\right)/c^2(1-\beta^2)^{5/2}\right)+\left(8\omega^4U_0/c^2(1-\beta^2)^{5/2}\mu^2\right)\Big]$$
$$+\left(\left(e^{2\mu T_0}-1\right)/2\mu\right)\Big[\left((11cH_0+4E_0)/2(1-\beta^2)^{1/2}\right)$$
$$+\left(\left(7\omega^2U_0+2c\right)/2(1-\beta^2)^{5/2}c^2\right)+\left(4\omega^4U_0/c^2(1-\beta^2)^{5/2}\mu^2\right)\Big]$$
$$+\left(\left(e^{2\mu T_0}-1\right)/2\mu\right)\left(\left((11cH_0+4E_0)/2(1-\beta^2)^{1/2}\right)\right.$$
$$\left.+\left(\left(7\omega^2U_0+2c\right)/2(1-\beta^2)^{5/2}c^2\right)+\left(4\omega^4U_0/\mu^2c^2(1-\beta^2)^{5/2}\right)\right)$$
$$+\frac{e^{\mu T_0}-1}{\mu T_0}\frac{e^{\mu T_0}}{2\mu}\left(\left((13cH_0+5E_0)/(1-\beta^2)^{1/2}\right)\right.$$
$$\left.+\left(\left(\mu^2\left(7\omega^2U_0+2c\right)+8\omega^4U_0\right)/c^2(1-\beta^2)^{5/2}\mu^2\right)\right)\Big]$$
$$.\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))$$
$$\le e^{\mu(t-kT_0)}\frac{3e^2}{mc^2}\frac{(\mu_0+1)e^{2\mu_0}+(\mu_0-1)e^{\mu_0}-\mu_0}{2\mu\mu_0}$$
$$.\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))$$
$$\left(\frac{13cH_0+5E_0}{(1-\beta^2)^{1/2}}+\frac{7\omega^2U_0+2c+\left(\left(8\omega^4U_0\right)/\mu^2\right)}{c^2(1-\beta^2)^{5/2}}\right).$$

Consequently

$$\rho_{(k,0)}(((Bu)_1^{(k)},(Bu)_2^{(k)},(Bu)_3^{(k)}),((B\bar{u})_1^{(k)},(B\bar{u})_2^{(k)},(B\bar{u})_3^{(k)}))$$
$$\le\frac{3e^2}{mc^2}\frac{(\mu_0+1)e^{2\mu_0}+(\mu_0-1)e^{\mu_0}-\mu_0}{2\mu\mu_0}\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))\,.$$
$$.\left(\frac{13cH_0+5E_0}{(1-\beta^2)^{1/2}}+\frac{7\omega^2U_0+2c+\left(8\omega^4U_0/\mu^2\right)}{c^2(1-\beta^2)^{5/2}}\right).$$

In what follows we make the same estimates for the derivatives of the operator functions B (for higher derivatives we recall Remark1). Indeed,

$$\left|\frac{d(Bu)_\alpha^{(k)}(t)}{dt}-\frac{d(B\bar{u})_\alpha^{(k)}(t)}{dt}\right|\le\left|F_\alpha(u)(t)-F_\alpha(\bar{u})(t)\right|$$
$$+\frac{1}{T_0}\left|\int_{kT_0}^{(k+1)T_0}\left(F_\alpha(u)(s)-F_\alpha(\bar{u})(s)\right)ds\right|$$

and

$$\left|F_1(u_1,u_2,u_3)(s)-F_1(\bar{u}_1,\bar{u}_2,\bar{u}_3)(s)\right|$$
$$\le((c^2-u_1^2)/c^2)\Big[\left|G_1(u_1,u_2,u_3)-G_1(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|$$
$$+\left|G_1^{rad}(u_1,u_2,u_3)-G_1^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\Big]$$
$$+(|u_1u_2|/c^2)\Big[\left|G_2(u_1,u_2,u_3)-G_2(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|$$
$$+\left|G_2^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)-G_2^{rad}(u_1,u_2,u_3)\right|\Big]$$
$$+(|u_1u_3|/c^2)\Big[\left|G_3(u_1,u_2,u_3)-G_3(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|$$
$$+\left|G_3^{rad}(u_1,u_2,u_3)-G_3^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\Big]$$
$$+(1/c^2)\Big[\left|u_1^2-\bar{u}_1^2\right|\left(\left|G_1(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_1^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)$$
$$+\left|u_1u_2-\bar{u}_1\bar{u}_2\right|\left(\left|G_2(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_2^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)$$
$$+\left|u_1u_3-\bar{u}_1\bar{u}_3\right|\left(\left|G_3(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_3^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)\Big]$$
$$\le\sum_{\alpha=1}^{3}\left|G_\alpha(u_1,u_2,u_3)-G_\alpha(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|$$
$$+\sum_{\alpha=1}^{3}\left|G_\alpha^{rad}(u_1,u_2,u_3)-G_\alpha^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|$$
$$+(2/c)\left(\left|G_1(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_1^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)\left|u_1(t)-\bar{u}_1(t)\right|$$
$$+(1/c^2)\left(\left|G_2(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_2^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)$$
$$.\left(|u_2(t)||u_1(t)-\bar{u}_1(t)|+|\bar{u}_1(t)||u_2(t)-\bar{u}_2(t)|\right)$$
$$+(1/c^2)\left(\left|G_3(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|+\left|G_3^{rad}(\bar{u}_1,\bar{u}_2,\bar{u}_3)\right|\right)$$
$$.\left(|u_1(t)||u_3(t)-\bar{u}_3(t)|+|\bar{u}_3(t)||u_1(t)-\bar{u}_1(t)|\right)$$
$$\le\frac{e^2}{mc^2}\left(\frac{2cH_0+E_0}{\sqrt{1-\beta^2}}+cH_0\right)\sum_{\alpha=1}^{3}e^{\mu(t-kT_0)}\sum_{\sigma=1}^{3}\left|u_\sigma(t)-\bar{u}_\sigma(t)\right|$$
$$+(e^2/m)\left[\frac{7\omega^2U_0}{c^4(1-\beta^2)^{5/2}}e^{\mu(t-kT_0)}\sum_{\alpha=1}^{3}\left|u_\alpha(t)-\bar{u}_\alpha(t)\right|\right.$$
$$\left.+\left(2/c^3(1-\beta^2)^{3/2}\right)\sum_{\alpha=1}^{3}\left|\ddot{u}_\alpha(t)-\ddot{\bar{u}}_\alpha(t)\right|\right]$$
$$+\frac{e^2}{m}\frac{2}{c}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$
$$.\,e^{\mu(t-kT_0)}\left|u_1(t)-\bar{u}_1(t)\right|$$
$$+\frac{e^2}{m}\frac{1}{c}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$
$$.\,e^{\mu(t-kT_0)}\left(|u_1(t)-\bar{u}_1(t)|+|u_2(t)-\bar{u}_2(t)|\right)$$
$$+\frac{e^2}{m}\frac{1}{c}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$
$$.\,e^{\mu(t-kT_0)}\left(|u_3(t)-\bar{u}_3(t)|+|u_1(t)-\bar{u}_1(t)|\right)$$
$$\le e^{2\mu(t-kT_0)}\frac{3e^2}{mc^2}\left(\frac{2cH_0+E_0}{(1-\beta^2)^{1/2}}+cH_0\right)\sum_{\sigma=1}^{3}\rho_{(k,0)}(u_\sigma,\bar{u}_\sigma)$$
$$+(e^2/m)\left[\left(7\omega^2U_0e^{2\mu(t-kT_0)}/c^4(1-\beta^2)^{5/2}\right)\sum_{\sigma=1}^{3}\rho_{(k,0)}(u_\sigma,\bar{u}_\sigma)\right.$$
$$\left..\left(2/c^3(1-\beta^2)^{3/2}\right)e^{\mu(t-kT_0)}\sum_{\alpha=1}^{3}\rho_{(k,0)}(\ddot{u}_\sigma,\ddot{\bar{u}}_\sigma)\right]$$

$$+(2e^2.mc)e^{2\mu(t-kT_0)}\rho_{(k,0)}(u_1,,\bar{u}_1)$$
$$.\left(\left((2\bar{c}H_0+E_0)/c\right)+\left((4-\beta^2)\,\omega^4/c^3\left(1-\beta^2\right)^{3/2}\mu^2\right)U_0\right)$$
$$+\frac{e^2}{m}\frac{1}{c}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$
$$.e^{2\mu(t-kT_0)}\left(\rho_{(k,0)}(u_1,,\bar{u}_1)+\rho_{(k,0)}(u_2,,\bar{u}_2)\right)$$
$$+\frac{e^2}{m}\frac{1}{c}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$
$$.e^{2\mu(t-kT_0)}\left(\rho_{(k,0)}(u_1,\bar{u}_1)+\rho_{(k,0)}(u_3,,\bar{u}_3)\right)$$
$$\le e^{2\mu(t-kT_0)}\frac{3e^2}{mc^2}\left(\frac{2cH_0+E_0}{\sqrt{1-\beta^2}}+cH_0\right)\frac{1}{\mu}\sum_{\sigma=1}^{3}\rho_{(k,1)}(\dot{u}_\sigma,\dot{\bar{u}}_\sigma)+$$
$$+\frac{e^2}{m}\left[\frac{7\omega^2U_0}{c^4(1-\beta^2)^{5/2}}e^{2\mu(t-kT_0)}\frac{1}{\mu}\sum_{\sigma=1}^{3}\rho_{(k,1)}(\dot{u}_\sigma,\dot{\bar{u}}_\sigma)\right.$$
$$\left.+\frac{2}{c^3(1-\beta^2)^{3/2}}e^{\mu(t-kT_0)}\frac{\omega^3}{\mu}\sum_{\sigma=1}^{3}\rho_{(k,3)}(\ddot{u}_\sigma,\dddot{\bar{u}}_\sigma)\right]$$
$$+\frac{e^2}{m}\frac{2}{c}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$
$$.\,e^{2\mu(t-kT_0)}\omega\,\rho_{(k,1)}(\dot{u}_1,\dot{\bar{u}}_1)/\mu$$
$$+\frac{e^2}{m}\frac{1}{c}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$
$$.e^{2\mu(t-kT_0)}\omega\left(\rho_{(k,1)}(\dot{u}_1,\dot{\bar{u}}_1)+\rho_{(k,1)}(\dot{u}_2,\dot{\bar{u}}_2)\right)/\mu$$
$$+\frac{e^2}{m}\frac{1}{c}\left(\frac{2\bar{c}H_0+E_0}{c}+\frac{4-\beta^2}{c^3\left(1-\beta^2\right)^{3/2}}\frac{\omega^4}{\mu^2}U_0\right)$$
$$.e^{2\mu(t-kT_0)}\omega\left(\rho_{(k,1)}(\dot{u}_1,\dot{\bar{u}}_1)+\rho_{(k,1)}(\dot{u}_3,\dot{\bar{u}}_3)\right)/\mu$$
$$\le e^{\mu(t-kT_0)}\frac{3e^2}{mc^2}\frac{e^{\mu T_0}}{2\mu}\rho_{(k,\infty)}((u_1,u_2,u_3),(\bar{u}_1,\bar{u}_2,\bar{u}_3))$$
$$.\left(\frac{13cH_0+5E_0}{\sqrt{1-\beta^2}}+\frac{7\omega^2U_0+2c(1-\beta^2)}{c^2(1-\beta^2)^{5/2}}+\frac{8\omega^4U_0}{c^2\left(1-\beta^2\right)^{5/2}\mu^2}\right).$$

Therefore the operator B is contractive in the sense of [35]. Its fixed point in view of the Main lemma is a T_0 – periodic solution of (6).

Theorem 1 is thus proved.

6. Conclusions

As an immediate consequence we obtain an existence-uniqueness of periodic solution for betatron equation (cf. [14], [27], [38[-[41]). Specific applications we will give in next papers.

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