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Some Solutions to the Fractional and Relativistic Schrödinger Equations

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Abstract Laskin introduced the fractional quantum mechanics and several common problems were solved in a piecewise fashion. However, Jeng *et al* pointed out that it was meaningless to solve a nonlocal equation in a piecewise fashion and that all the solutions in publication were wrong except the solution for the delta function potential, which was obtained in the momentum space. Jeng's critique results in a crisis of fractional quantum mechanics, that is, in mathematics it is quite difficult to find solutions to the fractional Schrödinger equation and in physics there is no realization for the fractional quantum mechanics. In order to eliminate this crisis, this paper reports some analytic solutions to the fractional Schrödinger equation without using piecewise method, and introduces the relativistic Schrödinger equation as a realization of the fractional quantum mechanics. These two sister equations should be studied at the same time.

Keywords Fractional quantum mechanics, Fractional Schrödinger equation, Relativistic quantum mechanics, Relativistic Schrödinger equation

1. Introduction

In 2000, Laskin introduced the fractional quantum mechanics [1-3]. As the first example he solved the infinite square well problem in a piecewise fashion [3]. Since then, the piecewise method has been widely used in this field. In 2010, however, Jeng, *et al* [4] criticized that it was meaningless to solve a nonlocal equation in a piecewise fashion and they demonstrated thatit was impossible for the ground state function to satisfy the fractional Schrödinger equation near the boundary inside the well. In a series of papers [5-8], Bayin insisted that he explicitly completed the calculation in Jeng's paper and the wave functions did satisfy the fractional Schrödinger equation inside the well. Hawkins and Schwarz [9] claimed that Bayin's calculation contained serious mistakes. Luchko [10] provided some evidence that the solution did not satisfy the equation outside the well. On the other hand, Dong [11] re-obtained the Laskin's solution by solving the fractional Schrödinger equation with the path integral method. It is not easyfor readers to judge their mathematical argument [12, 13], but weagree with Jeng that the piecewise method to solve the equation is wrong, since recently we explicitly and inarguably showed that the Laskin's functions did not satisfy the fractional Schrödinger equation with $\alpha=1$ anywhere on the *x*-axis [14].

According to Jeng, *et al* [4], only the solution for the delta function potential [15-17] was acceptable and they themselves provided a solution for the one dimensional harmonic oscillator potential for the case $\alpha=1$ [4]. Readers have been looking forward to some other solutions to the fractional Schrödinger equation since the simple solutions for the infinite square well potential were disproved. Jeng *et al* [4] also showed their concern that there was no realization of the fractional quantum mechanics.

Jeng's critique resulted in a crisis within fractional quantum mechanics. In mathematics it is not easy to find a solution to the fractional Schrödinger equation, and in physics it is not easy to find realizations for the fractional quantum mechanics. In order to eliminate this crisis, this paper reports some solutions to the fractional Schrödinger equation without using the piecewise method, and introduces the relativistic Schrödinger equation [18-21], as a realization of the fractional quantum mechanics. Several solutions for the relativistic quantum mechanics are also presented.

2. The Relativistic Schrödinger Equation: A Realization of the Fractional Schrödinger Equation

In this section we will list the standard, fractional, and relativistic Schrödinger equations in one- and three-dimensional spaces, and explain why we claim the relativistic Schrödinger equation is an approximate realization for the fractional Schrödinger equation.

2.1. The Schrödinger Equation

In the standard quantum mechanics [22, 23], the time-independent Schrödinger equation is

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$$H\psi(\mathbf{r}) = E\psi(\mathbf{r}), \tag{1}$$

where $\psi(\mathbf{r})$ is a wave function defined in the 3 dimensional Euclideanspace R^3, E is an energy, and $\mathbf{r}$ is a vector in the 3 dimensional space.

The Hamiltonian operator

$$H = T + V(\mathbf{r}) \tag{2}$$

is the summation of the kinetic energy operator and the potential energy operator.

The standard kinetic energy operator is

$$T = \frac{\mathbf{p}^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2, \tag{3}$$

where $\mathbf{p} = -i\hbar\nabla$ is the momentum operator. As usual, m is the mass of a particle and $\hbar$ is the reduced Plank constant.

The one dimensional time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x), \tag{4}$$

where the wave function $\psi(x)$ and the potential $V(x)$ are defined on the x-axis.

2.2. The Fractional Schrödinger Equation and Its Scaling Property

In 2000 [1-3], Laskin generalized the classical kinetic energy and momentum relation (3) to

$$T_\alpha = D_\alpha |\mathbf{p}|^\alpha = \chi_\alpha mc^2 \left(\frac{|\mathbf{p}|}{mc}\right)^\alpha = D_\alpha \left(-\hbar^2\nabla^2\right)^{\alpha/2}, \tag{5}$$

where α is the fractional parameter, the coefficient $D_\alpha \equiv \chi_\alpha mc^2 / (mc)^\alpha$, χ_α is a positive number dependent on α, and c is the speed of the light. Originally Laskin [1-3] ever required the fractional parameter $1 < \alpha \le 2$, but in this paper we allow $0 < \alpha < \infty$, as in [4, 9], with an emphasis on the simplest nonlocal case $\alpha = 1$.

In the case $\alpha = 2$, taking $\chi_2 = 1/2$, the fractional kinetic energy is the same as the classical kinetic energy

$$T_2 = D_2 |\mathbf{p}|^2 = \frac{1}{2}mc^2\left(\frac{|\mathbf{p}|}{mc}\right)^2 = \frac{\mathbf{p}^2}{2m} = T \cdot \tag{6}$$

In the case $\alpha = 1$, taking $\chi_1 = 1$, the fractional kinetic energy is the approximate kinetic energy in the extremely relativistic case,

$$T_1 = D_1 |\mathbf{p}| = mc^2 \frac{|\mathbf{p}|}{mc} = c|\mathbf{p}| \cdot \tag{7}$$

The definition of the fractional kinetic energy operator is

$$T_\alpha \psi(\mathbf{r}) = \frac{1}{\sqrt{(2\pi\hbar)^3}} D_\alpha \int_{R^3} \varphi(\mathbf{p}) |\mathbf{p}|^\alpha \exp(i\frac{\mathbf{p}}{\hbar}\cdot\mathbf{r}) d^3\mathbf{p}, \tag{8}$$

where

$$\varphi(\mathbf{p}) = \frac{1}{\sqrt{(2\pi\hbar)^3}} \int_{R^3} \psi(\mathbf{r}) \exp(-i\frac{\mathbf{p}}{\hbar}\cdot\mathbf{r}) d^3\mathbf{r} \tag{9}$$

is called the wavefunction in the momentum space [22, 23].

The fractional Schrödinger equation is

$$H_{\alpha}\psi\left(\mathbf{r}\right)=E\psi\left(\mathbf{r}\right) \tag{10}$$

$$H_{\alpha}=T_{\alpha}+V(\mathbf{r}). \tag{11}$$

When $\alpha=2$, the fractional Schrödinger equation recovers the standard Schrödinger equation; when $\alpha=1$, the fractional Schrödinger equation is

$$H_{1}\psi\left(\mathbf{r}\right)=E\psi\left(\mathbf{r}\right). \tag{12}$$

For $\alpha=1$, the following scaling property is straightforward.

Scaling property. If a wave function $\psi\left(\mathbf{r}\right)$ and an energy E is a solution of the fractional Schrödinger equation for the potential $V\left(\mathbf{r}\right)$

$$D_{1}\,|\,\mathbf{p}\,|\,\psi(\mathbf{r})\;+\;V(\mathbf{r})\psi(\mathbf{r})\;=\;E\psi(\mathbf{r}), \tag{13}$$

then $\psi\left(\lambda\mathbf{r}\right)$ and λE is a solution of the fractional Schrödinger equation for the potential $\lambda V\left(\lambda\mathbf{r}\right)$, i.e.

$$D_{1}\,|\,\mathbf{p}\,|\,\psi(\lambda\mathbf{r})\;+\;\lambda V(\lambda\mathbf{r})\;\psi(\lambda\mathbf{r})\;=\;\lambda E\;\;\psi(\lambda\mathbf{r}). \tag{14}$$

In this paper λ is an arbitrary positive number.

The proof is trivial. From (13) we have

$$\frac{1}{\lambda}D_{1}\,|\,\mathbf{p}\,|\,\psi(\lambda\mathbf{r})\;+V(\lambda\mathbf{r})\psi(\lambda\mathbf{r})\;=\;E\psi(\lambda\mathbf{r}). \tag{15}$$

For a potential satisfying $\lambda V\left(\lambda\mathbf{r}\right)=V\left(\mathbf{r}\right)$, such as (1) the coulomb $V\left(\mathbf{r}\right)=-Ze^{2}/r$ with e the charge of an electron and Z the order number of an atom, or (2) the radial delta function potential $V\left(\mathbf{r}\right)=-V_{0}\delta(r)$ with the constant $V_{0}>0$, the scaling property can be described simply as follows.

Scaling property. For a potential $V\left(\mathbf{r}\right)$ with a property $V\left(\mathbf{r}\right)=\lambda V\left(\lambda\mathbf{r}\right)$, if a wave function $\psi\left(\mathbf{r}\right)$ and an energy E is a solution of the fractional Schrödinger equation $H_{1}\psi\left(\mathbf{r}\right)=E\psi\left(\mathbf{r}\right)$, then the wave function $\psi\left(\lambda\mathbf{r}\right)$ and the energy λE is also a solution.

The one dimensional fractional Schrödinger equation is

$$H_{\alpha}\psi\left(x\right)=E\psi\left(x\right) \tag{16}$$

$$H_{\alpha}=T_{\alpha}+V(x). \tag{17}$$

When $\alpha=1$, we have

$$H_{1}\psi\left(x\right)=E\psi\left(x\right) \tag{18}$$

$$H_{1}=T_{1}+V(x)=D_{1}\,|\,p\,|+V(x)=D_{1}\hbar\frac{d}{dx}\mathbf{H}+V(x). \tag{19}$$

In this paper, the bold **H** denotes the Hilbert transform [24] while a normal *H* denotes the Hamiltonian operator.
From the definition of the fractional kinetic energy operator, we have

$$
\begin{aligned}
T_1\psi(x) &= D_1|p|\psi(x) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} D_1|p|\int_{-\infty}^{\infty}\psi(x')\exp(-ipx'/\hbar)dx' \ \exp(ipx/\hbar)dp \\
&= \frac{D_1\hbar}{2\pi}\int_{-\infty}^{\infty}|k|\int_{-\infty}^{\infty}\psi(x')\exp(-ikx')dx' \ \exp(ikx)dk \\
&= \frac{D_1\hbar}{2\pi}\int_{-\infty}^{\infty}(-i\,\mathrm{sgn}(k))(ik)\int_{-\infty}^{\infty}\psi(x')\exp(-ikx')dx' \ \exp(ikx)dk \\
&= \frac{D_1\hbar}{2\pi}\frac{d}{dx}\int_{-\infty}^{\infty}(-i\,\mathrm{sgn}(k))\int_{-\infty}^{\infty}\psi(x')\exp(-ikx')dx' \ \exp(ikx)dk \\
&= D_1\hbar\frac{d}{dx}H\psi(x) = D_1\hbar H\frac{d}{dx}\psi(x) \\
&= D_1\hbar\frac{d}{dx}\left(\psi(x)*\frac{1}{\pi x}\right) = D_1\hbar\left(\frac{d}{dx}\psi(x)\right)*\frac{1}{\pi x} \\
&= D_1\hbar\psi(x)*\frac{-1}{\pi x^2}
\end{aligned}
\tag{20}
$$

where $*$ denotes the convolution, and $1/x$ and $(1/x)' = -1/x^2$ are generalized functions [25]. We point out that the generalized function $-1/(2\pi^2 x^2)$ is the well-known ideal ramp filter, which plays an important role in the theory and applications of Computed Tomography [26, 27]. We will discuss the relationship between fractional quantum mechanics and the computed tomography in another paper.

The Dirac delta potential $V(x) = -V_0\delta(x)$ with $V_0 > 0$ satisfies $V(x) = \lambda V(\lambda x)$. Based on the scaling property, if a wave function $\psi(x)$ and an energy E is a solution of the fractional Schrödinger equation $H_1\psi = E\psi$ with a delta potential well, then the wave function $\psi(\lambda x)$ and the energy λE is also a solution. See Problems 3, 7, & 9.

2.3. The Relativistic Schrödinger Equation

According to the special relativity, the revised kinetic energy is [18]

$$T_r = \sqrt{\mathbf{p}^2c^2 + m^2c^4}\ , \tag{21}$$

where the subscript r means special **r**elativity.

For the case of low speed, the relativistic kinetic energy is approximately the summation of the rest energy and the classical kinetic energy ($\alpha = 2$)

$$T_r \approx mc^2 + \frac{\mathbf{p}^2}{2m} = mc^2 + T_2, \tag{22}$$

and for the case of extremely high speed, where the rest energy can be neglected, the relativistic kinetic energy is the fractional kinetic energy with $\alpha = 1$

$$T_r \approx |\mathbf{p}|c = T_1. \tag{23}$$

Generally speaking, if the speed of a particle increases from low to high, the relativistic kinetic energy T_r will approximately correspond to a fractional kinetic energy T_α, whose parameter α changes from 2 to 1. Therefore the relativistic kinetic energy is an approximate realization of the fractional kinetic energy.

The definition of the relativistic kinetic energy operator is

$$
\begin{aligned}
T_r\psi(\mathbf{r}) &= \sqrt{-\hbar^2c^2\nabla^2 + m^2c^4}\,\psi(\mathbf{r}) \\
&= \frac{1}{\sqrt{(2\pi\hbar)^3}}\int_{R^3}\varphi(\mathbf{p})\sqrt{\mathbf{p}^2c^2 + m^2c^4}\exp(i\frac{\mathbf{p}}{\hbar}\cdot\mathbf{r})d^3\mathbf{p}
\end{aligned}
\quad ,(24)
$$

where $\varphi(\mathbf{p})$ is the wave function in the 3D momentum space.

The relativistic Schrödinger equation is

$$H_r\psi\left(\mathbf{r}\right)=E\psi\left(\mathbf{r}\right) \tag{25}$$

$$H_r=T_r+V\ . \tag{26}$$

Accordingly, the 1D relativistic Schrödingere quationis

$$H_r\psi\left(x\right)=E\psi\left(x\right) \tag{27}$$

$$H_r=T_r+V=\sqrt{-\hbar^2c^2\frac{\partial^2}{\partial x^2}+m^2c^4}+V(x) \quad .(28)$$

As two sisters, the relativistic and fractional Schrödinger equations should be studied at the same time.

3. Solutions to the Fractional Schrödinger Equation

We will study the one dimensional problems first and then the 3D problems.

3.1. One Dimensional Problems

Problem 1. (Thefree particle.)

For $V(x)=0$, the fractional Schrödinger equation $H_\alpha\psi=E\psi$ has the solutions

$$\psi\left(x\right)=\exp(ikx) \tag{29}$$

$$E=D_\alpha(\hbar\,|\,k\,|)^\alpha \tag{30}$$

with $-\infty<k<\infty$, or equivalently

$$\psi\left(x\right)=\sin(kx),\ \text{or}\ \cos(kx) \tag{31}$$

$$E=D_\alpha(\hbar k)^\alpha \tag{32}$$

with $k\geq 0$.

Proof. According to the definition of the fractional kinetic energyoperator [1-3], we obviously have

$$\begin{aligned}H_\alpha\psi\left(x\right)&=D_\alpha T_\alpha\psi\left(x\right)=D_\alpha(-\hbar^2\nabla^2)^{\alpha/2}\exp(ikx)\\&=D_\alpha(\hbar\,|\,k\,|)^\alpha\psi\left(x\right)=E\psi\left(x\right)\end{aligned}\ , \tag{33}$$

with $-\infty<k<\infty$. Further, for $k>0$ we have

$$\begin{aligned}H_\alpha\sin\left(kx\right)&=H_\alpha\frac{1}{2i}\left(\exp\left(ikx\right)-\exp\left(-ikx\right)\right)\\&=D_\alpha(\hbar\,|\,k\,|)^\alpha\frac{1}{2i}\left(\exp\left(ikx\right)-\exp\left(-ikx\right)\right)\\&=D_\alpha(\hbar\,|\,k\,|)^\alpha\sin\left(kx\right)=E\sin\left(kx\right),\end{aligned}\ , \tag{34}$$

and

$$\begin{aligned}&H_\alpha\cos\left(kx\right)=E_\alpha\sin\left(kx\right),\\&H_\alpha 1=0.\end{aligned} \tag{35}$$

Generally, if $V(x)=V_0$ with V_0 a constant, the eigen-functions do not change but the new eigen-energies become

$$E=D_\alpha(\hbar\,|\,k\,|)^\alpha+V_0\,. \tag{36}$$

Problem 2. (A periodic potential.)
For the potential

$$V\left(x\right)=D_{\alpha}\left(\hbar k_0\right)^{\alpha}\frac{b}{b+\cos(k_0 x)}, \tag{37}$$

where $k_0=\pi/a$, a is a positive real number (as throughout this paper), b is a real number, and $0<\alpha<\infty$, the fractional Schrödinger equation $H_{\alpha}\psi=E\psi$ has a solution

$$\psi\left(x\right)=b+\cos(k_0 x) \tag{38}$$

$$E=D_{\alpha}(\hbar k_0)^{\alpha}. \tag{39}$$

Proof.
Since

$$\begin{aligned} |p|^{\alpha}\cos(k_0 x) &= (\hbar k_0)^{\alpha}\cos(k_0 x) \\ |p|^{\alpha}1 &= 0 \end{aligned} \tag{40}$$

We have

$$\begin{aligned} &D_{\alpha}|p|^{\alpha}\psi \\ &=D_{\alpha}|p|^{\alpha}\left(b+\cos(k_0 x)\right) \\ &=0+D_{\alpha}(\hbar k_0)^{\alpha}\cos(k_0 x) \\ &=D_{\alpha}(\hbar k_0)^{\alpha}\left(b+\cos(k_0 x)\right)-D_{\alpha}(\hbar k_0)^{\alpha}b \\ &=D_{\alpha}(\hbar k_0)^{\alpha}\left(b+\cos(k_0 x)\right)-D_{\alpha}(\hbar k_0)^{\alpha}\frac{b}{b+\cos(k_0 x)}\left(b+\cos(k_0 x)\right) \\ &=E\psi-V\psi. \end{aligned} \tag{41}$$

This completes the proof.
Further, we can calculate the average of the kinetic and potential energies of the particle.
Since

$$\begin{aligned} \int_{-a}^{a}\psi^*\left(x\right)\psi\left(x\right)dx &= \int_{-a}^{a}\left(b+\cos(k_0 x)\right)^2 dx \\ &=2a\left(b^2+1/2\right)=a(2b^2+1) \end{aligned} \tag{42}$$

the normalized function is

$$\Psi\left(x\right)=\frac{1}{\sqrt{a(2b^2+1)}}\left(b+\cos(k_0 x)\right). \tag{43}$$

The averages of the kinetic and the potential energy are

$$<T_{\alpha}>=\int_{-a}^{a}\Psi^*(x)D|p|^{\alpha}\Psi(x)dx=D_{\alpha}(\hbar k_0)^{\alpha}\frac{1}{2b^2+1}, \tag{44}$$

$$<V>=\int_{-a}^{a}\Psi^*(x)V\Psi(x)dx=D_{\alpha}(\hbar k_0)^{\alpha}\frac{2b^2}{2b^2+1}. \tag{45}$$

The average of the total energy is

$$<H> \;=\; <V>+<T_{\alpha}> \;=\; D_{\alpha}(\hbar k_0)^{\alpha} \;=\; E. \tag{46}$$

Problem 3. (The Delta potential well.)

For a Dirac delta function potential $V(x) = -V_0\delta(x)$ with $V_0 > 0$, the fractional Schrödinger equation $H_1\psi = E\psi$ has a solution

$$\psi(x) = \delta(x), \quad \text{E= -}\infty \tag{47}$$

in the sense of a certain limit.

Proof.
The factional Schrödinger equation

$$D_1 \,|\, p \,|\, \psi(x) \text{ -}V_0\delta(\mathrm{x})\psi(x) \; = \; E\psi(x) \tag{48}$$

can be rewritten in the momentum space as [22]

$$D_1 \,|\, p \,|\, \varphi(p) \text{ -}\frac{V_0}{2\pi\hbar}\int_{-\infty}^{\infty}\varphi(p)\mathrm{dp} \; = \; E\varphi(p) \tag{49}$$

$$D_1 \,|\, p \,|\, \varphi(p) + |E\,|\,\varphi(p) \; = \; \frac{V_0}{2\pi\hbar}\int_{-\infty}^{\infty}\varphi(p)\mathrm{dp}\,, \tag{50}$$

where $\varphi(p)$ is the wavefunction in the momentum space.

We first change the integral limit in the above equation to a finite positive number p_0, and then let $p_0 \to \infty$. Thus we have

$$D_1 \,|\, p \,|\, \varphi(p) + |E\,|\,\psi(p) \; = \; \frac{V_0}{2\pi\hbar}\int_{-p_0}^{+p_0}\varphi(p)\mathrm{dp} \tag{51}$$

$$\varphi(p) \; = \; \frac{V_0}{2\pi\hbar}\frac{1}{D_1 \,|\, p \,|+|E\,|}\int_{-p_0}^{+p_0}\varphi(p)\mathrm{dp} \tag{52}$$

Taking an integral of the two sides, we have

$$\int_{-p_0}^{+p_0}\varphi(p)\mathrm{dp} \; = \; \frac{V_0}{2\pi\hbar}\int_{-p_0}^{+p_0}\frac{1}{D_1 \,|\, p \,|+|E\,|}dp\int_{-p_0}^{+p_0}\varphi(p)\mathrm{dp} \tag{53}$$

If

$$\int_{-p_0}^{+p_0}\varphi(p)\mathrm{dp} \; \neq 0 \tag{54}$$

we have

$$1 \; = \; \frac{V_0}{2\pi\hbar}\int_{-p_0}^{+p_0}\frac{1}{D_1 \,|\, p \,|+|E\,|}dp \tag{55}$$

$$1 \; = \; \frac{V_0}{\pi\hbar D_1}\ln\left(1+ p_0 D_1 / \,|\, E \,|\right) \tag{56}$$

$$\mathrm{E} = -\frac{D_1 p_0}{\exp(\pi\hbar D_1 / V_0) - 1} \tag{57}$$

$$\varphi(p) \; \simeq \; \frac{1}{|\, D_1 p \,|+|E\,|} \simeq \frac{1}{|\, D_1 p \,/\, E \,|+1}\,. \tag{58}$$

Here $\simeq$ means that the wave functions at both sides are equal except for a constant coefficient. Obviously, a constant coefficient is not important for an eigen-wavefunction.

Therefore we have

$$\lim_{p_0 \to \infty} \mathrm{E} = -\infty , \tag{59}$$

$$\lim_{p_0 \to \infty} \varphi(p) = 1 . \tag{60}$$

Accordingly, in real space we have

$$\lim_{p_0 \to \infty} \psi(x) \to \delta(x) . \tag{61}$$

We can simply write the solution as

$$\psi(x) = \delta(x), \quad \mathrm{E} = -\infty , \tag{62}$$

which completes the proof.

Let us compare this solution with the solution in the standard quantum mechanics. This solution indicates that the particle falls inside the potential well completely, while the solution to the standard Schrödinger equation with a delta potential well indicates that the particle appears outside the well mainly. Therefore we see that Laskin's particles are easier to be trapped than Schrödinger's particles. The second difference is that the delta potential well problem has a unique bound state in the standard quantum mechanics but has more than one bound states in the fractional quantum mechanics ($\alpha = 1$), as we will see soon.

Problem 4. (The linear potential.)

For a linear potential $V = Fx$ with $F > 0$, the solutions to the fractional Schrödinger equation $H_1\psi = E\psi$ are

$$\psi_1(\mathrm{x}) = \left(\sqrt{\frac{\pi}{8}} - \mathrm{C}(\xi/2)\right)\cos\frac{\xi^2}{4} + \left(\sqrt{\frac{\pi}{8}} - \mathrm{S}(\xi/2)\right)\sin\frac{\xi^2}{4} \tag{63}$$

where the functions C() and S() are Fresnel integrals, and

$$\xi = \sqrt{\left(\frac{2F}{D_1\hbar}\right)}\left(x - \frac{E}{F}\right) \tag{64}$$

with $E \in R$.

Proof.

The fractional Hamilton is

$$\mathrm{H}_1 = \mathrm{D}_1 |p| + Fx \tag{65}$$

In the momentum representation [22]

$$\mathrm{H}_1 = \mathrm{D}_1 |p| + i\hbar F \frac{d}{dp} \tag{66}$$

The Fractional Schrödinger equation is

$$\mathrm{D}_1 |p| \varphi(p) + i\hbar F \frac{d}{dp}\varphi(p) = E\varphi(p) \tag{67}$$

$$\frac{1}{\varphi(p)}\frac{d}{dp}\varphi(p) = i\frac{\mathrm{D}_1}{\hbar F}|p| - i\frac{E}{\hbar F} . \tag{68}$$

Its solution is

$$\varphi(p) = A\exp\left(i\frac{D}{2\hbar F}p|p| - i\frac{E}{\hbar F}p\right) \tag{69}$$

with A an arbitrary real number.

In the real space, the wavefunction is

$$
\begin{aligned}
\psi(\mathrm{x}) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\varphi(p)\exp(ipx/\hbar)dp \\
&= \frac{A}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\exp(\frac{iD_1}{2\hbar F}p\,|\,p\,|-\frac{iE}{\hbar F}p+\frac{ix}{\hbar}p)dp \\
&= \frac{2A}{\sqrt{2\pi\hbar}}\int_{0}^{\infty}\cos(\frac{D_1}{2\hbar F}p^2-\frac{E}{\hbar F}p+\frac{x}{\hbar}p)dp \\
&= \frac{2A}{\sqrt{2\pi\hbar}}\int_{0}^{\infty}\cos(\frac{D_1}{2\hbar F}p^2+\frac{1}{\hbar}(x-\frac{E}{F})p)dp \\
&= \frac{2A}{\sqrt{2\pi\hbar}}\sqrt{\frac{2\hbar F}{D_1}}\int_{0}^{\infty}\cos(u^2+\frac{1}{\hbar}\sqrt{\left(\frac{2\hbar F}{D_1}\right)}(x-\frac{E}{F})u)du \\
&= 2A\sqrt{\frac{F}{D_1\pi}}\int_{0}^{\infty}\cos(u^2+\xi u)du \\
&= C\int_{0}^{\infty}\cos(u^2+\xi u+\xi^2/4-\xi^2/4)du \\
&= C\int_{0}^{\infty}\cos\left[(u+\xi/2)^2-\xi^2/4\right]du \\
&= C\int_{0}^{\infty}\cos(u+\xi/2)^2\cos\xi^2/4+\sin(u+\xi/2)^2\sin\xi^2/4\;du \\
&= C\cos\frac{\xi^2}{4}\int_{0}^{\infty}\cos(u+\xi/2)^2du \;+\; C\sin\frac{\xi^2}{4}\int_{0}^{\infty}\sin(u+\xi/2)^2du \\
&= C\cos\frac{\xi^2}{4}\int_{\xi/2}^{\infty}\cos u^2du \;+\; C\sin\frac{\xi^2}{4}\int_{\xi/2}^{\infty}\sin u^2du \\
&= C\left(\sqrt{\frac{\pi}{8}}-\mathrm{C}(\xi/2)\right)\cos\frac{\xi^2}{4}+C\left(\sqrt{\frac{\pi}{8}}-\mathrm{S}(\xi/2)\right)\sin\frac{\xi^2}{4}
\end{aligned}
\tag{70}
$$

where the coefficient

$$C=2A\sqrt{\frac{F}{D_1\pi}}\,. \tag{71}$$

This completes the proof.
Furthermore, since

$$\mathrm{C}(+\infty)=\mathrm{S}(+\infty)=\sqrt{\pi/8},\quad \mathrm{C}(-\infty)=\mathrm{S}(-\infty)=-\sqrt{\pi/8}\,, \tag{72}$$

the limit behavior of the wavefunction is

$$\psi(\mathrm{x}) =\begin{cases} 0 & x\to+\infty \\ C\sqrt{\pi}\sin(\dfrac{\xi^2}{4}+\dfrac{\pi}{4}) & x\to-\infty \end{cases} \tag{73}$$

Problem 5. (A periodic potential.)
The periodic function X(x) defined by

$$X(x)=-|\,x\,|+a/2 \quad x\in[-a,a] \tag{74}$$

$$X(x+2a)=X(x) \quad x\in(-\infty,\infty) \tag{75}$$

is called a triangular wave, where a>0 is a real number, whose properties have been studied carefully in electronics [28-30].

For the potential

$$V(x) = \frac{2}{\pi}\frac{D_1\hbar}{X(x)}\ln\left|\operatorname{tg}\left(\frac{\pi x}{2a}\right)\right| \tag{76}$$

the fractional Schrödinger equation $H_1\psi = E\psi$ has the solution

$$\psi(x) = X(x) \tag{77}$$

$$E = 0. \tag{78}$$

Proof.
We have

$$\begin{aligned} H_1\psi(x) &= D_1 \,|\, p \,|\, X(x) \\ &= D_1\hbar\mathbf{H}\frac{d}{dx}\big(X(x)\big) \\ &= D_1\hbar\mathbf{H}(-\operatorname{sgn}(\sin\frac{\pi}{a}x)) \\ &= -D_1\hbar\frac{2}{\pi}\ln\left|\operatorname{tg}\left(\frac{\pi x}{2a}\right)\right| \\ &= -V(x)X(x) \\ &= -V(x)\psi(x). \end{aligned} \tag{79}$$

Therefore we have

$$H_1\psi(x) + V(x)\psi(x) = 0, \tag{80}$$

which completes the proof.
In the above proof, we used a formula

$$\mathbf{H}(\operatorname{sgn}(\sin\frac{\pi}{a}x)) = \frac{2}{\pi}\ln\left|\operatorname{tg}\left(\frac{\pi x}{2a}\right)\right| \tag{81}$$

which can be seen in book [23] (Equation 6.14, page 292, vol. 1).
Problem 5’. (The Dirac comb.)
The Schrödinger equation $H\psi = E\psi$ with the Dirac comb potential

$$V(x) = -\frac{2\hbar^2}{ma}\sum_{n=-\infty}^{\infty}\delta(x-na) \tag{82}$$

has a solution

$$\psi(x) = X(x) \tag{83}$$

$$E = 0. \tag{84}$$

Proof.
Since

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}X(x) = \frac{\hbar^2}{2m}\frac{d}{dx}\operatorname{sgn}(\sin\frac{\pi}{a}x) = \frac{\hbar^2}{m}\sum_{n=-\infty}^{\infty}(-1)^n\delta(x-na), \tag{85}$$

and

$$V(x)X\left(x\right)=-\frac{2\hbar^2}{ma}X\left(x\right)\sum_{n=-\infty}^{\infty}\delta(x-na)$$
$$=-\frac{2\hbar^2}{ma}\sum_{n=-\infty}^{\infty}\delta(x-na)X\left(na\right)=-\frac{\hbar^2}{m}\sum_{n=-\infty}^{\infty}(-1)^n\delta(x-na), \tag{86}$$

we have

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}X\left(x\right)+VX\left(x\right)=0\,, \tag{87}$$

which completes the proof.

We include this well-known result in standard quantum mechanics here so that the reader can compare the standard and the fractional Schrödinger equations conveniently.

Problem 6. For the potential

$$V\left(x\right)=-D_1\hbar\frac{2a}{x^2+a^2}\,, \tag{88}$$

with a>0, the fractional Schrödinger equation $H_1\psi = E\psi$ has two and only two bound states

$$\psi_1\left(x\right)=\frac{a}{x^2+a^2},\quad E_1=-D_1\hbar\,/\,a\,, \tag{89}$$

$$\psi_2\left(x\right)=\frac{x}{x^2+a^2},\quad E_2=0\,. \tag{90}$$

Proof.

We need a well-known Hilbert transform pair [24]

$$\mathbf{H}\frac{a}{x^2+a^2}=\frac{x}{x^2+a^2} \tag{91}$$

$$\mathbf{H}\frac{x}{x^2+a^2}=\frac{-a}{x^2+a^2}\,. \tag{92}$$

Taking the derivative of the above two equations, we have

$$\frac{d}{dx}\mathbf{H}\frac{a}{x^2+a^2}=-\frac{1}{x^2+a^2}+\frac{2a^2}{\left(x^2+a^2\right)^2} \tag{93}$$

$$\frac{d}{dx}\mathbf{H}\frac{x}{x^2+a^2}=\frac{2ax}{(x^2+a^2)^2} \tag{94}$$

Multiplying by the constant $D_1\hbar$, we have

$$D_1\hbar\frac{d}{dx}\mathbf{H}\frac{a}{x^2+a^2}=-\frac{D_1\hbar}{a}\frac{a}{x^2+a^2}+D_1\hbar\frac{2a}{x^2+a^2}\frac{a}{x^2+a^2} \tag{95}$$

$$D_1\hbar\frac{d}{dx}\mathbf{H}\frac{x}{x^2+a^2}=D_1\hbar\frac{2a}{x^2+a^2}\frac{x}{x^2+a^2} \tag{96}$$

Therefore we have

$$D_1\,|\,p\,|\,\psi_1(x)=E_1\psi_1(x)-V\psi_1(x) \tag{97}$$

$$D_1\,|\,p\,|\,\psi_2(x)=E_2-V\psi_2(x) \tag{98}$$

where

$$E_1 = -D_1\hbar / a, \quad E_2 = 0. \tag{99}$$

From the shapes of the wave functions [24], we know that $\psi_1(x)$ is the ground state, and $\psi_2(x)$ is the first excited state. Since the excited energy $E_2 = 0$, we know that there are no more excited states. The particle has only two bound states, the ground state and the excited state. This completes the proof.

Let us further calculate the averages of the kinetic and potential energies in this elegant problem.

The normalized functions are

$$\Psi_1(x) = \sqrt{\frac{2a}{\pi}} \frac{a}{x^2 + a^2} \tag{100}$$

$$\Psi_2(x) = \sqrt{\frac{2a}{\pi}} \frac{x}{x^2 + a^2} \tag{101}$$

Since the wavefunction Ψ_1 is even and Ψ_2 is odd, the two states obviously are orthogonal, i.e.

$$\int_{-\infty}^{\infty} \Psi_1^*(x)\Psi_2(x)dx = \int_{-\infty}^{\infty} \Psi_2^*(x)\Psi_1(x)dx = 0. \tag{102}$$

In the ground state, the averages of the kinetic and potential energies are

$$< V >= -\frac{3}{2}\frac{D_1\hbar}{a}, \quad < D_1 \mid p \mid > = \frac{1}{2}\frac{D_1\hbar}{a}. \tag{103}$$

In the excited state, the averages of the kinetic and potential energies are

$$< V >= -\frac{1}{2}\frac{D_1\hbar}{a}, \quad < D_1 \mid p \mid > = \frac{1}{2}\frac{D_1\hbar}{a}. \tag{104}$$

Problem 7. For the delta function potential

$$V(x) = -2D_1\hbar\pi\delta(x) \tag{105}$$

the fractional Schrödinger equation

$$H_1\psi(x) = E\psi(x) \tag{106}$$

has two solutions

$$\psi_1(x) = \pi\delta(x), \quad \mathrm{E}_1 = -\infty \tag{107}$$

$$\psi_2(x) = \frac{1}{x}, \qquad \mathrm{E}_2 = 0 \tag{108}$$

Proof. In the above example, let $a \to 0$, and notice that

$$\lim_{a\to 0} \frac{a}{x^2 + a^2} = \pi\delta(x), \qquad \lim_{a\to 0} \frac{x}{x^2 + a^2} = \frac{1}{x}. \tag{109}$$

This completes the proof.

The two solutions can also be written as

$$\psi_1(x) = \delta(x), \quad \mathrm{E}_1 = -\infty \tag{110}$$

$$\psi_2(x) = \frac{1}{\pi x}, \qquad \mathrm{E}_2 = 0 \tag{111}$$

This result is consistent with the solution for Problem 3 and the scaling property discussed in Sec. II.

Problem 8. For the potential

$$V\left(x\right)=-D_1\hbar\frac{4a}{x^2+a^2} \tag{112}$$

with a>0, the fractional Schrödinger equation $H_1\psi\left(x\right)=E\psi\left(x\right)$ has a bound state

$$\psi\left(x\right)=\frac{2a^2x}{(x^2+a^2)^2},\quad \mathrm{E}=-\frac{D_1\hbar}{a} \tag{113}$$

Proof.
Taking the derivative of the two sides of Equation (95), we have

$$D_1\hbar\frac{d}{dx}\mathbf{H}\left(\frac{a}{x^2+a^2}\right)'=-\frac{D_1\hbar}{a}\left(\frac{a}{x^2+a^2}\right)'+D_1\hbar\frac{4a}{\left(x^2+a^2\right)}\left(\frac{a}{x^2+a^2}\right)' \tag{114}$$

$$D_1\,|\,p\,|\left(\frac{a}{x^2+a^2}\right)' = E\left(\frac{a}{x^2+a^2}\right)' - V(x)\left(\frac{a}{x^2+a^2}\right)' \tag{115}$$

This completes the proof.
Notice that the potential in this problem is just 2 times the potential in Problem5, but their solutions are completely different.

Problem 9. For the potential

$$V\left(x\right)=-4D_1\hbar\pi\delta(x) \tag{116}$$

the fractional Schrödinger equation

$$H_1\psi\left(x\right)=E\psi\left(x\right) \tag{117}$$

has a bound state

$$\psi\left(x\right)=\pi\delta\,'(x),\quad \mathrm{E}=-\infty\,. \tag{118}$$

Proof. By letting $a\to 0$ in Problem8, this statement follows immediately.

3.2. Three Dimensional Problems

Problem 10. (The Free particle.)
For $V(\mathbf{r})=0$, the solutions for the fractional Schrödinger equation $H_\alpha\psi_\alpha=E\psi_\alpha$ are

$$\psi_\alpha\left(\mathbf{r}\right)=\exp(i\mathbf{k}\cdot\mathbf{r}) \tag{119}$$

$$E_\alpha=D_\alpha(\hbar k)^\alpha\,. \tag{120}$$

with $\mathbf{k}$ any three dimensional vector. An alternative form of the eigen-functions is

$$\psi_\alpha\left(\mathbf{r}\right)=j_l(kr)Y_l^m(\theta,\varphi) \tag{121}$$

where j_l is the spherical Bessel function of order l, $Y_l^m(\theta,\varphi)$ is the spherical harmonic function of degree l and order m [22], (r,θ,φ) is the spherical coordinate system, and $k\equiv|\mathbf{k}|$ is the length of the vector **k.**

Proof. For $V(\mathbf{r})=0$, the standard Schrödinger equation

$$H\psi(\mathbf{r}) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) = EV(\mathbf{r}) \tag{122}$$

has the solutions

$$\psi(\mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r}) \tag{123}$$

$$E = \frac{\hbar^2 k^2}{2m}. \tag{124}$$

From the relationship between the fractional Hamiltonian and the standard Hamiltonian

$$H_\alpha = T_\alpha = D_\alpha |\mathbf{p}|^\alpha = D_\alpha(\mathbf{p}^2)^{\alpha/2} = D_\alpha(2mT)^{\alpha/2} = D_\alpha(2mH)^{\alpha/2}, \tag{125}$$

we see that the fractional Schrödinger equation

$$H_\alpha\psi_\alpha(\mathbf{r}) = E_\alpha\psi_\alpha(\mathbf{r}) \tag{126}$$

has the solutions

$$\psi_\alpha(\mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r}) \tag{127}$$

$$E_\alpha = D_\alpha(\hbar k)^\alpha. \tag{128}$$

In the spherical coordinate system, the classical Schrödinger equation

$$H\psi(\mathbf{r}) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) = EV(\mathbf{r}) \tag{129}$$

has solutions

$$\psi(\mathbf{r}) = j_l(kr)Y_l^m(\theta,\varphi). \tag{130}$$

Since

$$\begin{aligned} H_\alpha j_l(kr)Y_l^m(\theta,\varphi) &= D_\alpha(2mH)^{\alpha/2} j_l(kr)Y_l^m(\theta,\varphi) \\ &= D_\alpha(\hbar^2k^2)^{\alpha/2} j_l(kr)Y_l^m(\theta,\varphi) = D_\alpha(\hbar k)^\alpha j_l(kr)Y_l^m(\theta,\varphi) \end{aligned} \tag{131}$$

the solutions to the fractional Schrödinger equation has an alternative form

$$\psi_\alpha(\mathbf{r}) = j_l(kr)Y_l^m(\theta,\varphi), \tag{132}$$

$$E_\alpha = D_\alpha(\hbar k)^\alpha. \tag{133}$$

Problem 11. The function

$$Y(r) = \frac{1}{(2\pi\hbar)^3}\int_{R^3}\frac{1}{D_\alpha|\mathbf{p}|^\alpha}\exp(i\frac{\mathbf{p}}{\hbar}\cdot\mathbf{r})d^3\mathbf{p} \tag{134}$$

and E=0 is a solution of the fractional Schrödinger equation $H_\alpha\psi = E\psi$ with the potential

$$V(r) = -\delta(\mathbf{r})/Y(r), \tag{135}$$

where $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z) = \delta(r)/(2\pi r^2)$ is the Dirac's delata function in the 3D space.

Proof. It is easy to verify that

$$D_\alpha|\mathbf{p}|^\alpha\, Y(r) + V(r)Y(r) = \delta(\mathbf{r}) - \delta(\mathbf{r}) = 0. \tag{136}$$

This completes the proof.

There are two special cases, where the wave functions and the potential energy can be given explicitly:

(1) When $\alpha = 2$ we have

$$Y(r) = \frac{1}{(2\pi\hbar)^3}\int_{R^3}\frac{2m}{\mathbf{p}^2}\exp(i\frac{\mathbf{p}}{\hbar}\cdot\mathbf{r})d^3\mathbf{p}$$
$$= \frac{1}{(2\pi)^3}\int_{R^3}\frac{2m}{\hbar^2\mathbf{k}^2}\exp(i\mathbf{k}\cdot\mathbf{r})d^3\mathbf{k} = \frac{2m}{\hbar^2}\frac{1}{4\pi r} = \frac{m}{\hbar^2}\frac{1}{2\pi r} \tag{137}$$

and

$$V(r) = -\delta(\mathbf{r})/Y(r) = -\frac{\hbar^2}{m^2}\frac{1}{r}\delta(r). \tag{138}$$

Therefore we say that for the central potential $V(r) = -\frac{\hbar^2}{m^2}\frac{1}{r}\delta(r)$, the Schrödinger equation $H\psi = E\psi$ has a solution $\psi(r) = \frac{m}{\hbar^2}\frac{1}{2\pi r}$ and E=0.

(2) When $\alpha = 1$, we have

$$Y(r) = \frac{1}{(2\pi\hbar)^3}\int_{R^3}\frac{1}{D_1|\mathbf{p}|}\exp(i\frac{\mathbf{p}}{\hbar}\cdot\mathbf{r})d^3\mathbf{p} = \frac{1}{(2\pi)^3}\int_{R^3}\frac{1}{D_1\hbar|\mathbf{k}|}\exp(i\mathbf{k}\cdot\mathbf{r})d^3\mathbf{k}$$
$$= \frac{4\pi}{D_1\hbar}\frac{1}{(2\pi)^3}\int_{R^3}\frac{1}{4\pi|\mathbf{k}|}\exp(i\mathbf{k}\cdot\mathbf{r})d^3\mathbf{k} = \frac{1}{D_1\hbar}\frac{1}{2\pi^2 r^2}, \tag{139}$$

and

$$V(r) = -\delta(\mathbf{r})/Y(r) = -D_1\hbar\delta(r). \tag{140}$$

Therefore, for the central potential $V(r) = -D_1\hbar\delta(r)$, the fractional Schrödinger equation $H_1\psi = E\psi$ has a solution $\psi(r) = \frac{1}{D_1\hbar}\frac{1}{2\pi^2 r^2}$ and E=0.

Problem 12. (The harmonic oscillator potential.)
For a harmonic oscillator potential

$$V(\mathbf{r}) = k\mathbf{r}^2, \tag{141}$$

the fractional Schrödinger equation $H_1\psi = E\psi$ has solutions(in the momentum representation)

$$\varphi(\mathbf{p}) = \varphi(p) = \frac{1}{p}\mathrm{Ai}(\kappa p - r_n) \tag{142}$$

$$E_n = (\tfrac{1}{2}k\hbar^2 D_1^2)^{1/3}|r_n|. \tag{143}$$

where Ai(x) is the Airy function, r_n is its n-th zero point, and $\kappa \equiv (2D_1/(k\hbar^2))^{1/3}$.

Proof.
The fractional Hamiltonian is

$$H_1 = D_1|\mathbf{p}| + k\mathbf{r}^2. \tag{144}$$

In the momentum representation, the Hamiltonian operator and its eigen-equation are

$$H_1 = -k\hbar^2\nabla_{\mathbf{p}}^2 + D_1|\mathbf{p}| \tag{145}$$

$$-k\hbar^2\nabla_{\mathbf{p}}^2\varphi(\mathbf{p}) + D|\mathbf{p}|\varphi(\mathbf{p}) = E\varphi(\mathbf{p}) \tag{146}$$

In the spherical coordinate system, the Laplace operator $\nabla_{\mathbf{p}}^2$ is expressed as

$$\nabla_{\mathbf{p}}^2 = \frac{1}{p}\frac{\partial^2}{\partial p^2} p + \frac{1}{p^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{p^2 \sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{147}$$

For a s-state wave function, $\psi(\mathbf{p}) = \psi(p)$, the eigen-equation becomes

$$-k\hbar^2 \frac{1}{p}\frac{\partial^2}{\partial p^2} p\psi(p) + D_1 p\psi(p) = E\psi(p). \tag{148}$$

Let $u(p) = p\psi(p)$. Then we have

$$-k\hbar^2 \frac{d^2}{dp^2} u(p) + D_1 p u(p) = Eu(p). \tag{149}$$

The solution of this equation under the condition $u(0) = u(\infty) = 0$ is

$$u(p) = \mathrm{Ai}(\kappa p - r_n) \tag{150}$$

$$E_n = (\tfrac{1}{2} k\hbar^2 D_1^2)^{1/3} \,|\, r_n \,|, \tag{151}$$

where Ai is the Airy function, r_n is its n-th zero point, and $\kappa \equiv (2D_1 / (k\hbar^2))^{1/3}$ [4]. This completes the proof.

Problem 13. (The Coulomb potential.)

For the Coulomb potential $V(r) = -Ze^2 / r$ with Z>0, and e is the charge of the electron, the fractional Schrödinger equation has a solution $\psi(\mathbf{r}) = 1/r$ with E=0 when $Ze^2 = 2D_1\hbar / \pi$.

Proof.
In fact we have

$$\begin{aligned} D_1 |\mathbf{p}| \frac{1}{4\pi r} &= D_1 |\mathbf{p}| \frac{1}{(2\pi)^3}\int_{R^3} \frac{1}{k^2}\exp(i\mathbf{k}\cdot\mathbf{r}) d^3\mathbf{k} \\ &= \frac{D_1\hbar}{(2\pi)^3}\int_{R^3}\frac{1}{k^2}|\mathbf{k}|\exp(i\mathbf{k}\cdot\mathbf{r})d^3\mathbf{k} = \frac{D_1\hbar}{(2\pi)^3}\int_{R^3}\frac{1}{k}\exp(i\mathbf{k}\cdot\mathbf{r})d^3\mathbf{k} \\ &= \frac{D_1\hbar 4\pi}{(2\pi)^3}\int_{R^3}\frac{1}{4\pi k}\exp(i\mathbf{k}\cdot\mathbf{r})d^3\mathbf{k} = \frac{D_1\hbar 4\pi}{(2\pi)^3}\frac{1}{r^2} \\ &= \frac{D_1\hbar}{2\pi^2}\frac{1}{r^2} \end{aligned} \tag{152}$$

and

$$V(r)\frac{1}{4\pi r} = -\frac{Ze^2}{r\cdot}\frac{1}{4\pi r} = -\frac{D\hbar}{2\pi^2}\frac{1}{r^2}. \tag{153}$$

Therefore we have

$$\begin{aligned} D_1 |\mathbf{p}| \frac{1}{4\pi r} + V\frac{1}{4\pi r} &= 0 \\ D_1 |\mathbf{p}| \psi(\mathbf{r}) + V\psi(\mathbf{r}) &= 0. \end{aligned} \tag{154}$$

This completes the proof.

4. Solutions to the Relativistic Schrödinger Equation

Again, let us study the one dimensional problems first and then the 3D problems.

4.1. One Dimensional Problems

Problem i. (The free particle.)

For V(x)=0, the solution for the relativistic Schrödinger equation $H_r\psi = E\psi$ is

$$\psi(x) = \exp(ikx) \tag{155}$$

$$E = \sqrt{(\hbar kc)^2 + m^2c^4}\ . \tag{156}$$

with $-\infty < k < \infty$.

Proof. According to the definition of the square root operator [18-20]

$$\sqrt{p^2c^2 + m^2c^4}\exp(ikx) = \sqrt{\hbar^2k^2c^2 + m^2c^4}\exp(ikx) \quad k \in R\,, \tag{157}$$

the above statement is obvious.

The solution can also be written as

$$\psi(x) = \sin(kx) \text{ or } \cos(\text{kx}) \tag{158}$$

$$E = \sqrt{(\hbar kc)^2 + m^2c^4}\ . \tag{159}$$

with $k \geq 0$.

Further, when $V(x)=V_0$ with V_0 a constant, the wavefunctions do not change but the new energy levels become

$$E = \sqrt{(\hbar kc)^2 + m^2c^4} + V_0 \tag{160}$$

Problem ii. (A periodic potential.)

For the potential

$$V(x) = \left(\sqrt{(\hbar k_0 c)^2 + m^2c^4} - mc^2\right)\frac{b}{b + \cos(k_0 x)} \tag{161}$$

where $k_0 = \pi / a$, a is a length, and b is a real number, the relativistic Schrödinger equation $H_r\psi = E\psi$ has a solution

$$\psi(x) = b + \cos(k_0 x) \tag{162}$$

$$E = \sqrt{(\hbar k_0 c)^2 + m^2c^4}\ . \tag{163}$$

Proof.

From the definition of the square root operator [18-20],

$$\sqrt{p^2c^2 + m^2c^4}\exp(ikx) = \sqrt{\hbar^2k^2c^2 + m^2c^4}\exp(ikx) \quad k \in R\,, \tag{164}$$

We get that

$$\sqrt{p^2c^2 + m^2c^4}\cos k_0 x = \sqrt{\hbar^2k_0^2c^2 + m^2c^4}\cos(k_0 x) \tag{165}$$

$$\sqrt{p^2c^2 + m^2c^4}\ \ 1 \ = \ mc^2\ \ 1. \tag{166}$$

Therefore we have

$$
\begin{aligned}
&T_r\psi_r \\
&=\sqrt{p^2c^2+m^2c^4}\left(b+\cos(k_0x)\right) \\
&=\sqrt{(\hbar k_0c)^2+m^2c^4}\left(\cos(k_0x)\right)+mc^2b \\
&=\sqrt{(\hbar k_0c)^2+m^2c^4}\left(b+\cos(k_0x)\right)-\left(\sqrt{(\hbar k_0c)^2+m^2c^4}-mc^2\right)b \\
&=\sqrt{(\hbar k_0c)^2+m^2c^4}\left(b+\cos(k_0x)\right)-\left(\sqrt{(\hbar k_0c)^2+m^2c^4}-mc^2\right)\frac{b}{b+\cos(k_0x)}\left(b+\cos(k_0x)\right) \\
&=E\psi-V\psi.
\end{aligned}
\tag{167}
$$

This completes the proof.

Further, we can calculate the average of the kinetic and potential energies.

The normalized function is

$$\Psi(x)=\frac{1}{\sqrt{a(2b^2+1)}}\left(b+\cos(k_0x)\right). \tag{168}$$

The averages of the kinetic and the potential energies are

$$<T_r>=\int_{-a}^{a}\Psi^*(x)T_r\Psi(x)dx=\frac{2b^2}{2b^2+1}mc^2+\frac{1}{2b^2+1}\sqrt{(\hbar k_0c)^2+m^2c^4}\ . \tag{169}$$

$$<V>=\int_{-a}^{a}\Psi^*(x)V\Psi(x)dx=\frac{2b^2}{2b^2+1}\left(\sqrt{(\hbar k_0c)^2+m^2c^4}-mc^2\right). \tag{170}$$

The total energy is

$$<H>=<T_r>+<V>=\sqrt{(\hbar k_0c)^2+m^2c^4}=E\ . \tag{171}$$

Problem iii. (The delta potential well.)

For a Dirac delta function potential $V=-V_0\delta(x)$ with $V_0>0$, the relativistic Schrödinger equation $H_r\psi=E\psi$ has a solution

$$\psi(x)=\delta(x),\quad \text{E}=-\infty \tag{172}$$

in the sense of a certain limit.

Proof. The proof is similar to Problem 3.

The relativistic Schrödinger equation

$$\sqrt{p^2c^2+m^2c^4}\psi(x)\ -V_0\delta(\text{x})\psi(x)\ =\ E_r\psi(x) \tag{173}$$

can be rewritten in the momentum space as [22]

$$\sqrt{p^2c^2+m^2c^4}\varphi(p)\ -\frac{V_0}{2\pi\hbar}\int_{-\infty}^{\infty}\varphi(p)\text{dp}\ =\ E_r\varphi(p) \tag{174}$$

$$\sqrt{p^2c^2+m^2c^4}\varphi(p)\ +|E_r|\varphi(p)\ =\ \frac{V_0}{2\pi\hbar}\int_{-\infty}^{\infty}\varphi(p)\text{dp} \tag{175}$$

We first change the integral limit to a positive number $p_0>mc$, and then let $p_0\to+\infty$. Thus we have

$$\sqrt{p^2c^2+m^2c^4}\varphi(p)\ +|E_r\,|\psi(p)\ =\ \frac{V_0}{2\pi\hbar}\int_{-p_0}^{+p_0}\varphi(p)\text{dp} \tag{176}$$

$$\varphi(p) = \frac{V_0}{2\pi\hbar}\frac{1}{\sqrt{p^2c^2+m^2c^4}+|E_r|}\int_{-p_0}^{+p_0}\varphi(p)\mathrm{dp} \tag{177}$$

Taking integral of the two sides, we have

$$\int_{-p_0}^{+p_0}\varphi(p)\mathrm{dp} = \frac{V_0}{2\pi\hbar}\int_{-p_0}^{+p_0}\frac{1}{\sqrt{p^2c^2+m^2c^4}+|E_r|}dp\cdot\int_{-p_0}^{+p_0}\varphi(p)\mathrm{dp} \tag{178}$$

If $\int_{-p_0}^{+p_0}\varphi(p)\mathrm{dp} \neq 0$, we have

$$1 = \frac{V_0}{2\pi\hbar}\int_{-p_0}^{p_0}\frac{1}{\sqrt{p^2c^2+m^2c^4}+|E_r|}dp = \frac{V_0}{\pi\hbar}\int_{0}^{p_0}\frac{1}{\sqrt{p^2c^2+m^2c^4}+|E_r|}dp\,. \tag{179}$$

If the above integration is calculated only on a subinterval $[mc, p_0]\subset[0, p_0]$, we have

$$\frac{V_0}{\pi\hbar}\int_{mc}^{p_0}\frac{1}{\sqrt{p^2c^2+m^2c^4}+|E_r|}dp<1\,. \tag{180}$$

Since

$$\begin{aligned} & p>mc\Rightarrow p^2>m^2c^2\Rightarrow p^2c^2>m^2c^4 \\ & \Rightarrow 2p^2c^2>p^2c^2+m^2c^4 \\ & \Rightarrow \sqrt{2}pc>\sqrt{p^2c^2+m^2c^4} \\ & \Rightarrow \sqrt{2}pc+|E_r|>\sqrt{p^2c^2+m^2c^4}+|E_r| \\ & \Rightarrow \frac{1}{\sqrt{2}pc+|E_r|}<\frac{1}{\sqrt{p^2c^2+m^2c^4}+|E_r|}, \end{aligned} \tag{181}$$

we have

$$\frac{V_0}{\pi\hbar}\int_{mc}^{p_0}\frac{1}{\sqrt{2}pc+|E_r|}dp<1 \tag{182}$$

$$\frac{V_0}{\pi\sqrt{2}\hbar c}\ln\frac{p_0+|E_r|/(\sqrt{2}c)}{mc+|E_r|/(\sqrt{2}c)}<1 \tag{183}$$

Further we have

$$|\mathrm{E}_r|>\sqrt{2}\,\frac{p_0c-mc^2\exp(\sqrt{2}\pi\hbar c/V_0))}{\exp(\sqrt{2}\pi\hbar c/V_0)-1}\,. \tag{184}$$

We see

$$|\mathrm{E}_r|\to\infty,\quad \text{as}\ \ p_0\to\infty\,, \tag{185}$$

and hence the bound state energy

$$\mathrm{E}_r=-\infty\,. \tag{186}$$

The wave function

$$\varphi(p) \simeq \frac{1}{\sqrt{p^2c^2+m^2c^4}+|E_r|}\simeq\frac{1}{\sqrt{p^2c^2+m^2c^4}/|E_r|+1} \tag{187}$$

From Equation (185), we have

$$\varphi(p) \to 1 \text{ as } p_0 \to \infty. \tag{188}$$

Accordingly, in the real space we have

$$\psi(x) \to \delta(x) \text{ as } p_0 \to \infty. \tag{189}$$

Again, a particle with a relativistic kinetic energy is easier to be trapped than a particle with a Newtonian kinetic energy.

Problem iv. (The linear potential.)

For a linear potential $V = Fx$ with $F > 0$, the solution to the relativistic Schrödinger equation $H_r\psi = E\psi$ is

$$\psi(\mathrm{x}) = \int_0^\infty \cos(bu\sqrt{u^2+1} + b\ln(u+\sqrt{u^2+1}) - u\xi)du \tag{190}$$

$$\xi = (x - \frac{E}{F})\frac{mc}{\hbar}, b = \frac{m^2c^3}{2\hbar F} \tag{191}$$

with $E \in R$.

Proof.

The relativistic Hamilton is

$$\mathrm{H}_r = \sqrt{p^2c^2 + m^2c^4} + Fx. \tag{192}$$

In the momentum representation,

$$\mathrm{H}_r = \sqrt{p^2c^2 + m^2c^4} + iF\hbar\frac{d}{dp}. \tag{193}$$

The relativistic Schrödinger equation is

$$\sqrt{p^2c^2 + m^2c^4}\varphi(p) + i\hbar F\frac{d}{dp}\varphi(p) = E\varphi(p) \tag{194}$$

This equation can be solved easily

$$\frac{1}{\varphi(p)}\frac{d}{dp}\varphi(p) = \frac{i}{\hbar F}\sqrt{p^2c^2 + m^2c^4} - \frac{iE}{\hbar F}. \tag{195}$$

$$\ln\varphi(p) = \int \frac{i}{\hbar F}\sqrt{p^2c^2 + m^2c^4} - \frac{i}{\hbar F}Edp. \tag{196}$$

$$\begin{aligned}\ln\varphi(p) &= \frac{ic}{\hbar F}\int \left(\sqrt{p^2 + m^2c^2} - E/c\right)dp \\ &= \frac{ic}{2\hbar F}\left(p\sqrt{p^2+m^2c^2} + m^2c^2\ln\frac{p+\sqrt{p^2+m^2c^2}}{mc}\right) - \frac{iE}{\hbar F}p\end{aligned} \tag{197}$$

$$\varphi(p) = A\exp\left(\frac{ic}{2\hbar F}p\sqrt{p^2+m^2c^2} + \frac{im^2c^3}{2\hbar F}\ln\frac{p+\sqrt{p^2+m^2c^2}}{mc} - \frac{iE}{\hbar F}p\right), \tag{198}$$

where A is an arbitrary constant.

Via Fourier transform, we can get the wavefunction in the real space

$$\begin{aligned}
\psi_1(\mathrm{x}) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\varphi(p)\exp(ipx/\hbar)dp \\
&= \frac{A}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\exp(\frac{ic}{2\hbar F}p\sqrt{p^2+m^2c^2}+\frac{im^2c^3}{2\hbar F}\ln\frac{p+\sqrt{p^2+m^2c^2}}{mc}-\frac{iE}{\hbar F}p+\frac{ipx}{\hbar})dp \\
&= \frac{2A}{\sqrt{2\pi\hbar}}\int_{0}^{\infty}\cos(\frac{c}{2\hbar F}p\sqrt{p^2+m^2c^2}+\frac{m^2c^3}{2\hbar F}\ln\frac{p+\sqrt{p^2+m^2c^2}}{mc}+(x-\frac{E}{F})\frac{p}{\hbar})dp \\
&= \frac{2Amc}{\sqrt{2\pi\hbar}}\int_{0}^{\infty}\cos(\frac{m^2c^3}{2\hbar F}u\sqrt{u^2+1}+\frac{m^2c^3}{2\hbar F}\ln(u+\sqrt{u^2+1})+(x-\frac{E}{F})\frac{umc}{\hbar})du \\
&= C\int_{0}^{\infty}\cos(bu\sqrt{u^2+1}+b\ln(u+\sqrt{u^2+1})+u\xi)du
\end{aligned} \tag{199}$$

$$C=\frac{2Amc}{\sqrt{2\pi\hbar}},\ p=umc,\ \xi=(x-\frac{E}{F})\frac{mc}{\hbar}, \tag{200}$$

Further, we point out that

$$\psi(\mathrm{x})\to 0 \text{ as } \mathrm{x}\to+\infty. \tag{201}$$

It is because the value of $\psi(\mathrm{x})$ is equal to the sum of an alternating series and the absolute values of the terms in the series become smaller when $\mathrm{x}\to+\infty$.

Specifically, let us consider the integral

$$\mathrm{I}(\xi)=\int_0^{\infty}\cos G(u,\xi)du \tag{202}$$

with

$$\mathrm{G(u,}\xi)=(bu\sqrt{u^2+1}+b\ln(u+\sqrt{u^2+1})+u\xi\ . \tag{203}$$

For a fixed $\xi>0$, suppose that u_n with n=0,1,2,3 satisfy that

$$G(u_n,\xi)=n\pi+\pi/2 \tag{204}$$

$$\cos G(u_n,\xi)=0\,. \tag{205}$$

We have

$$\mathrm{I}(\xi)=I_0+I_1+I_2+\cdots+I_n+\cdots \tag{206}$$

With

$$\begin{aligned}
&\mathrm{I}_0(\xi)=\int_0^{u_0}\cos G(u,\xi)du \\
&\mathrm{I}_1(\xi)=\int_{u_0}^{u_1}\cos G(u,\xi)du \\
&\cdots \\
&\mathrm{I}_n(\xi)=\int_{u_{n-1}}^{u_n}\cos G(u,\xi)du
\end{aligned} \tag{207}$$

Since the terms I_n alternately change their sign, and

$$|I_1|<|I_2|<\cdots<|I_n|<\cdots. \tag{208}$$

The series of $\mathrm{I}(\xi)$ converges for any given $\xi>0$.

As $\xi\to\infty$, the interval between any two adjacent points, $u_{n+1}-u_n$, becomes closer to each other, every term

$I_n(\xi) \to 0$, and hence their alternating summation $I(\xi) \to 0$.

In other words, for any fixed $\xi \to \infty$, we have

$$\psi(\mathrm{x}) \to 0 \text{ as } \mathrm{x} \to +\infty. \tag{209}$$

It is relatively complicated to discuss the limit behavior of the wavefunction as $\mathrm{x} \to -\infty$, so we omit it temporarily. Interested readers can observe its behavior intuitively on a graph.

4.2. Three Dimensional Problems

Problem v. (The free particle.)

For $V(\mathbf{r}) = 0$, the solutions to the relativistic Schrödinger equation $H_r\psi_r(\mathbf{r}) = E_r\psi_r(\mathbf{r})$ are

$$\psi_r(\mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r}) \tag{210}$$

$$E_r = \sqrt{\hbar^2 c^2 \mathbf{k}^2 + m^2 c^4} \tag{211}$$

with $\mathbf{k}$ a three dimensional vector. An alternative form of the eigen-functions is

$$\psi_r(\mathbf{r}) = j_l(kr)Y_l^m(\theta,\varphi). \tag{212}$$

Proof. For $V(\mathbf{r}) = 0$, the standard Schrödinger equation

$$H\psi(\mathbf{r}) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) = EV(\mathbf{r}) \tag{213}$$

has the solutions

$$\psi(\mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r}) \tag{214}$$

$$E = \frac{\hbar^2 k^2}{2m} \tag{215}$$

where $k = |\mathbf{k}|$ is the length of the vector **k.**

From the relationship between the relativistic Hamiltonian and the standard Hamiltonian

$$H_r = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} = \sqrt{2mc^2 T + m^2 c^4} = \sqrt{2mc^2 H + m^2 c^4} \tag{216}$$

we know that the relativistic Schrödinger equation

$$H_r\psi_r(\mathbf{r}) = E_r\psi_r(\mathbf{r}) \tag{217}$$

has the solutions

$$\psi_r(\mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r}) \tag{218}$$

$$E_r = \sqrt{\hbar^2 c^2 \mathbf{k}^2 + m^2 c^4}. \tag{219}$$

Obviously, the wavefunction can also be expressed in the spherical coordinate system.

Problem vi. The function

$$Y(r) = \frac{1}{(2\pi\hbar)^3}\int_{R^3} \frac{1}{\sqrt{\mathbf{p}^2 c^2 + m^2 c^4}} \exp(i\frac{\mathbf{p}}{\hbar}\cdot\mathbf{r})d^3\mathbf{p} \tag{220}$$

and E=0 is a solution of the relativistic Schrödinger equation $H_r\psi(\mathbf{r}) = E\psi(\mathbf{r})$ with the potential

$$V(r) = -\delta(\mathbf{r})/Y(r). \tag{221}$$

Proof. It is easy to verify that

$$\sqrt{\mathbf{p}^2c^2+m^2c^4}Y(r)+V(\mathbf{r})Y(\mathbf{r})=\delta(\mathbf{r})-\delta(\mathbf{r})=0\,. \tag{222}$$

This completes the proof.

Problem vii. (The harmonic oscillator potential)

For a harmonic oscillator potential $V(\mathbf{r})=k\mathbf{r}^2$, the s-state energies for the relativistic Schrödinger equation, E_n, satisfy

$$(\tfrac{1}{2}k\hbar^2c^2)^{1/3}\,|\,r_n\,|<E_n<(2n-\tfrac{1}{2})\hbar\omega+mc^2 \tag{223}$$

where $n=1,2,3,\cdots$, $\omega\equiv\sqrt{2k/m}$, and r_n is the n-th zero point of the Airy function Ai(x).

Proof.

In the momentum space, we have

$$H_r=\sqrt{\mathbf{p}^2c^2+m^2c^4}+k\mathbf{r}^2=\sqrt{\mathbf{p}^2c^2+m^2c^4}-k\hbar^2\nabla_{\mathbf{p}}^2 \tag{224}$$

The Schrödinger equation is

$$-k\hbar^2\nabla_{\mathbf{p}}^2\varphi(\mathbf{p})+\sqrt{\mathbf{p}^2c^2+m^2c^4}\,\varphi(\mathbf{p})=E_r\varphi(\mathbf{p})\,. \tag{225}$$

Up to some constants, this equation is mathematically the same as the Schrödinger equation in the real space with a square root potential

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r})+k\sqrt{r^2+r_0^2}\,\psi(\mathbf{r})=E\psi(\mathbf{r})\,, \tag{226}$$

with k and r_0 are positive numbers. We also know that in standard quantum mechanics the energy levels become higher if the potential becomes higher. Now let us return to the current problem.

Since

$$|\,\mathbf{p}\,|\,c<\sqrt{\mathbf{p}^2c^2+m^2c^4}<mc^2+\frac{\mathbf{p}^2}{2m}\,, \tag{227}$$

we see that the relativistic kinetic energy is smaller than the classical kinetic energy plus the rest energy mc^2, but greater than the energy levels of the fractional energy with $\alpha=1$ and $D_1=c$.

Specifically, for s-state, we have

$$(\tfrac{1}{2}k\hbar^2c^2)^{1/3}\,|\,r_n\,|<E_n<mc^2+(2n-\frac{1}{2})\hbar\omega \tag{228}$$

$n=1,2,3,\cdots$.

Here we used the result of Problem 12 and the energy formula for the classical harmonic oscillator [22].

Problem viii. (The Coulomb potential.)

For the Coulomb potential $V=-e^2/r$, wheree is the charge of the electron, the energy eigen value of the relativistic Schrödinger equation $H_r\psi(\mathbf{r})=E\psi(\mathbf{r})$ is

$$E_{nl}=mc^2\left[1-\frac{1}{2\mathrm{n}^2}\alpha^2-\frac{1}{2\mathrm{n}^4}\left(\frac{n}{l+1/2.}-\frac{3}{4}\right)\alpha^4+\frac{64}{15\pi}\frac{1}{n^3}\alpha^5\delta_{0l}+O\left(\alpha^6\right)\right] \tag{229}$$

where $\alpha=\mathrm{e}^2/\hbar c$ is the fine structure coefficient, n is the principle quantum number, *l* is the angular momentum quantum number. Only in this problem and its solution, α is not fractional parameter.

Proof.

The relativistic Hamiltonian is

$$H = \sqrt{\mathbf{p}^2c^2 + m^2c^4} - \frac{e^2}{r} \tag{230}$$

The Schrödinger equation

$$H\psi(\mathbf{r}) = \sqrt{\mathbf{p}^2c^2 + m^2c^4}\psi(\mathbf{r}) - \frac{e^2}{r}\psi(\mathbf{r}) = E\psi(\mathbf{r}) \tag{231}$$

has no analytic solutions by now. Since the relativistic effect is very small, we can use the perturbation method based on the classical Hamiltonian

$$H_0 = mc^2 + \frac{\mathbf{p}^2}{2m} - \frac{e^2}{r} \tag{232}$$

The classical Schrödinger equation

$$H_0\psi^0(\mathbf{r}) = mc^2\psi^0(\mathbf{r}) + \frac{\mathbf{p}^2}{2m}\psi^0(\mathbf{r}) - \frac{e^2}{r}\psi^0(\mathbf{r}) = E\psi^0(\mathbf{r}) \tag{233}$$

has the well-known wave function $\psi^0_{nlm}(\mathbf{r})$ and energy levels

$$\mathrm{E}^0_{\mathrm{n}} = mc^2\left(1 - \frac{1}{2\mathrm{n}^2}\alpha^2\right). \tag{234}$$

According to the perturbation theory [22], the first order approximation of the energy is

$$\begin{aligned}
\mathrm{E}_{\mathrm{nl}} &= \ \langle H \rangle \ = \ \langle\sqrt{\mathbf{p}^2c^2 + m^2c^4}\rangle \ - \langle\frac{e^2}{r}\rangle \\
&= \int_{R^3}\psi^{0*}_{nlm}(\mathbf{p})\sqrt{\mathbf{p}^2c^2 + m^2c^4}\psi^0_{nlm}(\mathbf{p})d^3\mathbf{p} \ - \ \frac{1}{\mathrm{n}^2}\alpha^2mc^2 \\
&= mc^2\left[1 - \frac{1}{2\mathrm{n}^2}\alpha^2 - \frac{1}{2\mathrm{n}^4}\left(\frac{n}{l+1/2} - \frac{3}{4}\right)\alpha^4 + \frac{64}{15\pi}\frac{1}{n^3}\alpha^5\delta_{0l} + O\left(\alpha^6\right)\right]
\end{aligned} \tag{235}$$

where

$$\phi^0_{nlm}(\mathbf{p}) = \frac{1}{\sqrt{(2\pi\hbar)^3}}\int_{R^3}\psi^0_{nlm}(\mathbf{r})\exp(-i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar})d^3\mathbf{r} \tag{236}$$

is the wavefunction in the momentum space[23]. The details of the calculation can be found in [19].

The new energy levels contain a valuable α^5 term, which is 41% of the observed Lamb shift [19]. We are trying to find the exact solutions for the relativistic Schrödinger equation with a Coulomb potential to see whether we can explain the Lamb shift better in the framework of quantum mechanics.

5. Conclusions

Jeng's critique resulted in a crisis of fractional quantum mechanics, that is, the fractional Schrödinger equation was difficult to solve in mathematics and had no realization in the real world. To eliminate this crisis, we present various solutions to the fractional Schrödinger equation, and introduce the relativistic Schrödinger equation as a realization of the fractional Schrödinger equation. Several solutions to the relativistic Schrödinger equation are also presented. The standard, fractional and relativistic Schrödinger equation should be studied together.

We wish that the winter of the fractional quantum mechanics could go away and its spring could come soon.

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