International Journal of Theoretical and Mathematical Physics 2014, 4(3): 110-113
DOI: 10.5923/j.ijtmp.20140403.06

Reciprocal Symmetry and Classical Discrete Oscillator Incorporating Hall-Integral Energy Levels

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Abstract Classical oscillator differential equation is replaced by the corresponding (finite time) difference equation. The equation is, then, symmetrized so that it remains invariant under the change d→-d, where d is the smallest span of time. This symmetric equation has solutions, which come in reciprocally related pairs. One member of a pair agrees with the classical solution and the other is an oscillating solution and does not converge to a limit as d→0. This solution contributes to oscillator energy a term which is a multiple of half-integers.

Keywords Reciprocal Symmetry, Finite Difference Equation, Quantum Statistics, Classical Oscillator, Half-Integral Energy Levels

1. Introduction

In 1900 Max Planck presented his quantum hypothesis [1], which sets a lower limit to energy transfer. In 1905 Albert Einstein presented his relativistic postulate [2], which sets an upper limit to velocity. Apart from the fact that, they talk of different quantities (energy and velocity), there is a reciprocal relation between the two. Every number has a unique reciprocal. [3], [4]. There will be no ambiguity if, instead of representing distance, time etc. by *x, t,* etc., we represent them by their reciprocals (*x → 1/x, t → 1/t* etc.*).* Here *w*e can recall the principle of objectivity that physics should be independent of the quantities we define. The study of motion in terms of slowness (reciprocal of velocity) is as valid as the study in terms of velocity.

Velocity is defined as the distance covered in unit time,

V = x/t.

Reciprocally, we may define slowness V' as the reciprocal of V i.e.

V' = t/x

Physics does not depend on the quantities we define (objectivity). Therefore, it should be possible to describe motion in terms of slowness just as it is possible in terms of velocities [5].

Let us consider the well known function exp(iwt) which describes an oscillating motion. Energy of the oscillator is proportional to the square of w. exp(iwt) is the solution of a differential equation. Classical oscillator differential equation can be replaced by the corresponding (finite time) difference equation. The equation, then, can be symmetrized so that it remains invariant under the change d→-d, where d is the smallest span of time. This symmetric equation has solutions, which come in reciprocally related pairs. The angular speed w is modified to w` or w``. w` contains a part with an integer. W`` contains a part with a half integer. This corresponds to quantum mechanical oscillator energy levels. *f*=a.exp(iwt) describes oscillation between -a and +a. if we make w=0, f describes free oscillation between –*a* and +*a*.

The difference equation

$$\frac{df}{dt} = iwt \tag{1.1}$$

has a unique solution $f = a.\exp(iwt)$, f describe the motion of a harmonic oscillator if

$$w = \sqrt{\frac{k}{m}}$$

Where k is a constant and m is the mass.

Classical energy of the oscillator $E_{classical}$ is proportional to w^2

$$E_{classical} = \frac{1}{2}m(aw)^2 \tag{1.2}$$

The corresponding finite difference equation has more solutions [6], which come in reciprocal pairs. When the function represents a harmonic oscillator, different solutions will contribute to oscillator energy in different ways. We intend to study these contributions and compare them to the corresponding quantum mechanical values.

Let the oscillator having amplitude '*a*' oscillate between –*a* and +*a*. We place two perfectly rigid reflecting walls at –*a*

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Published online at http://journal.sapub.org/ijtmp

and *+a.* the presence of the walls does not influence the oscillation in any way. Now we make k=0 so that *w=0.* This makes the oscillator a free particle. Therefore, in the case (*w=0*) the non-vanishing energy term should the energy levels of a free particle bouncing between reflecting walls.

2. Oscillator Finite Difference Equation

Classical simple harmonic oscillator function *f* (with angular speed w) satisfies differential equation (1.1).

To exploit its symmetry properties we replace the above differential equation by the corresponding symmetric finite difference equation [7]

$$\frac{Dg_\pm}{D(t,\delta)} = iW.g_\pm \qquad (2.1)$$

Where,

$$\frac{Dg_\pm(w,t)}{D(t,\delta)} = \frac{g_\pm(w,t+\delta)-g_\pm(w,t-\delta)}{2\delta} \qquad (2.2)$$

The above difference quotient has the following symmetry under the change δ→-δ

$$\frac{Dg_\pm}{D(t,-\delta)} = \frac{Dg_\pm}{D(t,\delta)} \qquad (2.3)$$

We require that at least one of the solutions, g_+, of (2.1) should go over to (1.1) in the limit δ→0

$$\frac{Dg_+}{D(t,\delta)} = iW(w)g_+ \xrightarrow[\delta\to 0]{} \frac{df}{dt} = iwf \qquad (2.4)$$

With

$$g_+ \xrightarrow[\delta\to 0]{} f \quad \text{and} \quad \mathrm{W} \xrightarrow[\delta\to 0]{} w \qquad (2.5)$$

(2.1) has solutions in pairs. One of the pairs is [8]

$$g_+ = \exp\left(\frac{2\pi}{d}s_+ + \frac{sin^{-1}(wd)}{d}\right)it = \exp(w_+ it) \qquad (2.6)$$

$$g_- = \exp\left(\frac{2\pi}{d}s_- - \frac{sin^{-1}(wd)}{d}\right)it = \exp(w_- it) \qquad (2.7)$$

Where

$$S_+ = \text{integer} \qquad (2.8)$$

$$S_- = \text{Half-integer}$$

With the correspondence relation (for $s_+ = 0$)

$$g_+ \xrightarrow[d\to 0]{} f \qquad (2.9)$$

g_+ and g_- are related through the reciprocity relation

$$g_+.g_- = (-1)^{t/d} \qquad (2.10)$$

2.1. Assumption of Classical Physics

We consider an oscillator oscillating along *x* line between *–a* and *+a*

$$x = a.\sin wt \qquad (2.1.1)$$

x=0 at *t=0.* we measure at intervals of *d.* we are not able to measure at any interval less than *d*. After time *d* the value of *x* is

$$x = a.\sin wd \qquad (2.1.2)$$

What is the value *x* after time *d',* where *d'<d?* The classical assumption is

$$x = a.\sin wd' \qquad (2.1.3)$$

It is an assumption because no observations have been made for any interval *d'<d.* in (2.9) and (2.10) of this paper we have replaced assumption (2.1.3) by the less stringent assumptions below and we write for the displacement *x*

$$x_+ = a.\sin(\tfrac{2\pi}{d}s_+ + w)it = a.\sin(w_+ it) \qquad (2.1.4)$$

$$x_- = a.\sin(\tfrac{2\pi}{d}s_- - w)it = a.\sin(w_- it) \qquad (2.1.5)$$

We require that x_+ and x_- agree with the observed value at t=d so that

$$x_+ = a.\sin(w_+ id) = x_- = a.\sin(w_- id)$$
$$= x = a.\sin(wid) \qquad (2.1.6)$$

(2.1.4) and (2.1.5), therefore, express our ignorance about the values of *x* for *t<d.*

2.2. Classical and Quantum Energy Levels

The energies of the reciprocal symmetric oscillator $\boldsymbol{E_\pm^{R.S}}$ are [9]

$$\boldsymbol{E_+^{R.S} = \frac{1}{2}m(a.w_+)^2 =}$$
$$\boldsymbol{\frac{1}{2}ma^2[\left\{\frac{2\pi+}{d}\right\}^2 + 2[\left\{\frac{2\pi+}{d}\right\}w + w^2]} \qquad (2.2.1)$$

$$\boldsymbol{E_-^{R.S} = \frac{1}{2}m(a.w_-)^2 = \frac{1}{2}ma^2[\left\{\frac{2\pi s_-}{d}\right\}^2 + 2[\left\{\frac{2\pi s_-}{d}\right\}w + w^2]} \qquad (2.2.2)$$

For s_+=0, (2.2.1) gives the classical value (1.3).

The middle term of (2.2.2) is

$$\boldsymbol{E^{R.S}_{-middle\ term} = -ma^2}\left(\frac{\pi}{d}\right)(2s_-)\mathrm{w} \qquad (2.2.3)$$

It corresponds to quantum mechanical value. [10]

$$E^{oscillator}_{quantel} = \eta(2s_-)w \qquad (2.2.4)$$

The important difference is that there is no Planck's constant in (2.2.3).

3. Reciprocal Symmetry

Let $g_\pm$ be of the form $g_+ = (\pm a)^{\pm t/\delta}$ so that

$$g_+(w,t) = (-1)^{t/\delta} g_-(w,t) \qquad (3.1)$$

Consider equation (2.1)

$$\frac{g_+(w,t+\delta)-g_+(w,t-\delta)}{2\delta} = iW.g_+(w,t) \qquad (3.2)$$

Using (3.1) we find that g_-also satisfies the equation. This establishes reciprocal symmetry of (2.1), that the equation remains invariant under transformation (3.1).

3.1. Reciprocal Symmetric Solutions

(2.1) has a pair of solutions

$$g_\pm\left(\frac{w\delta}{2}\right) = (\pm\frac{1\pm isin\left(\frac{w\delta}{2}\right)}{1\mp isin\left(\frac{w\delta}{2}\right)})^{\frac{t}{\delta}} = (\pm 1)^{\frac{t}{\delta}}\exp(\pm iwt) \qquad (3.1.1)$$

g_+ and g_- satisfy (2.1) with

$$W = \frac{\sin(w\delta)}{\delta} \quad (3.1.2)$$

We may write

$$g_+ = \exp\left(\frac{(2n)\pi t}{\delta} i\right)\exp(iwt) = \exp(iw_+ t) \quad (3.1.3)$$

$$g_- = \exp\left(\frac{(2n+1)\pi t}{\delta} i\right)\exp(-iwt) = \exp(iw_- t) \quad (3.1.4)$$

Where

$$w_+ = \frac{(2n)\pi}{\delta} + w = y_+ + w \quad (3.1.5)$$

$$w_- = \frac{(2n+1)\pi}{\delta} - w = y_- - w \quad (3.1.6)$$

4. Classical and Hall-Integral Energy Levels

Simple harmonic oscillator function *f* satisfies the differential equation

$$\frac{df}{dt} -= \pm iwf$$

The corresponding finite difference symmetric equation is

$$\frac{Dg}{D(t,\delta)} = \pm iwg$$

It has two solutions $g = g_1$ and $g = g_2$ [10]

$$g_1 = A\left(\frac{1 - i\dfrac{1-\sqrt{1+(iw\delta)^2}}{w\delta}}{1 + i\dfrac{1-\sqrt{1+(iw\delta)^2}}{w\delta}}\right)^{t/\delta}$$

And

$$g_2 = A\left(\frac{1 + i\dfrac{1-\sqrt{1+(iw\delta)^2}}{w\delta}}{1 - i\dfrac{1-\sqrt{1+(iw\delta)^2}}{w\delta}}\right)^{t/\delta}$$

In the limit as $\delta \to 0, g_1$ gives the classical oscillator function f

$$g_1 = A\left(e^{2n\pi i}\frac{1 \pm i\dfrac{1-\sqrt{1+(iw\delta)^2}}{w\delta}}{1 \mp i\dfrac{1-\sqrt{1+(iw\delta)^2}}{w\delta}}\right)^{\frac{t}{\delta}} \xrightarrow[\delta\to 0]{}$$

$$A\exp\left(\frac{2n\pi}{\delta} \pm w\right)it$$

Half integral energy levels

g_2 gives

$$g_2 = A\left(-\frac{1 + i\dfrac{1-\sqrt{1+(iw\delta)^2}}{w\delta}}{1 - i\dfrac{1-\sqrt{1+(iw\delta)^2}}{w\delta}}\right)^{\frac{t}{\delta}} \xrightarrow[\delta\to 0]{} A\exp iw't$$

Where

$$w' = \frac{(2n+1)\pi}{\delta} + w = y + w$$

The energy of the oscillator is proportional to

$$(w')^2 = y^2 + 2yw + w^2$$
$$= \{\frac{(2n+1)\pi}{\delta}\}^2 + 2\left\{\frac{2(n+1)\pi}{\delta}\right\}w + w^2$$

The middle terms contain half-integral multiples. To this extent it corresponds to quantum mechanical value.

The first term of this equation, $y^2 = \{\frac{(2n+1)\pi}{\delta}\}^2$, corresponds to the energy levels of a free particle oscillating between reflecting walls. A similar term with (2n+1) replaced by 2n comes from g_1 solution. Adding terms, we get the total energy, Ey, of a free particle oscillating between reflecting walls as

$$E_y = \frac{1}{2}m\{\frac{An\pi}{\delta}\}^2$$

Where m is the mass, 2A is the width of the well and n is an integer.

The energy of the oscillator is proportional to

$$(w_+)^2 = {y_+}^2 + 2y_+ w + w^2$$
$$= \{\frac{2n\pi}{\delta}\}^2 + 2\left\{\frac{2n\pi}{\delta}\right\}w + w^2 \quad (4.1)$$
$$(w_-)^2 = {y_+}^2 + 2y_- w + w^2$$
$$= \{\frac{(2n+1)\pi}{\delta}\}^2 + 4\left\{\frac{\frac{n+1}{2}\pi}{\delta}\right\}w + w^2 (4.2)$$

For n=0 (4.1) gives the classical value. The middle term of (4.2) is a product term of half-integers and w. to this extent it corresponds to quantum mechanical value.

5. Conclusions

The pair of reciprocal symmetric functions $g_\pm$ of (2.9) and (2.10) describes a classical oscillator, which has discrete energy levels (2.2.1) and (2.2.2). One of the terms (2.2.3) compares well with the corresponding quantum mechanical term (2.2.2).

We have replaced oscillator differential equation by the corresponding symmetric discrete equation (2.1). This has brought to surface important parts of oscillator function, which were lost in the conventional solution. These parts contain discrete-integral and half-integral energy levels.

ACKNOWLEDGEMENTS

We are deeply indebted to Prof. Mushfiq Ahmad for his unwavering patience and thoughtful guidance to this research. His strong spirit of solving and meeting the challenging problems throughout the study have played very important role behind these achievements. We are very pleased for his moral affectionate behavior to the students.

We wish to express our gratitude to the authority of Rajshahi University, Rajshahi, Bangladesh for helping us to this research.

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