

# Oscillations in Lossy Transmission Lines Terminated by in Series Connected Nonlinear RCL-Loads

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**Abstract** We formulate conditions for an existence-uniqueness of oscillatory regimes in transmission lines terminated by in series connected nonlinear *RGLC*-loads. This is achieved by introducing suitable operator whose fixed point is a seeking solution. The method allows obtaining approximated solutions and an estimate of the rate of convergence.

**Keywords** Lossy Transmission Lines, *RCL*-Loads, Oscillatory Solution, Neutral Functional Differential Equation, Fixed Point Theorems

## 1. Introduction

In a recent paper[1] we have investigated a transmission line terminated by in series connected *RCL*-loads and we have reduced the mixed problem for the lossy transmission line equations to an initial value problem for a system of neutral equations on the boundary. Then we proved an existence-uniqueness result of periodic boundary value problem for the neutral system obtained but for one period. Global behaviour of the solution is declining rather than periodically. In contrast to parallel connected *RCL*-loads (cf.[2]) here the multiplier  $e^{-(R/L)t}$  could not be eliminated and this implies that the solution vanishes exponentially.

From the physical point of view this means that the signal vanishes in time. Moreover one can notice the voltage and current in the lossy transmission line are products of a periodic function and an exponential function, that is,

$$u(t) = V(t)e^{-(R/L)t}, \quad i(t) = I(t)e^{-(R/L)t}.$$

Therefore  $u(t)$ ,  $i(t)$  should be oscillating functions and vanish exponentially at infinity. This suggest us to state the problem for an existence-uniqueness of a global oscillatory solution on  $[T, \infty)$  vanishing exponentially at infinity. As in[1],[2] we assume the Heaviside condition is satisfied, that is,  $R/L = G/C$ . It implies that the transmission line is without distortion. Here we omit the reduction (cf.[1]) of the mixed problem for the hyperbolic system to a neutral system on the boundary and prove by the fixed point method (cf.[3]) an existence-uniqueness of the oscillatory solution for the neutral system. The main difficulty is generated by nonlinearities of *RLC*-loads (cf.[4]-[7]). We obtain

successive approximation of the seeking oscillatory solution in an explicit form and estimate the rate of convergence.

Especially we want to draw attention to a fundamental disadvantage of lot of papers – solving equations without guaranteed uniqueness of the solution. Whatever the method (numerical, approximated and so on) to apply without uniqueness is not known to which of the many solutions are coming.

We mention recent various approaches for an analysis of transmission lines[8]-[15]. We would like to point out that we obtain an approximated solution but in explicit form beginning with simple functions.

## 2. Mixed Problem for the Lossy Transmission Line System terminated by in Series Connected *RCL*-Loads

We proceed from the lossy transmission line equations system:

$$\begin{aligned} C \frac{\partial u(x, t)}{\partial t} + \frac{\partial i(x, t)}{\partial x} + Gu(x, t) &= 0 \\ L \frac{\partial i(x, t)}{\partial t} + \frac{\partial u(x, t)}{\partial x} + Ri(x, t) &= 0 \end{aligned} \quad (1)$$

$$(x, t) \in P = \{(x, t) \in R^2 : (x, t) \in [0, L] \times [0, \infty)\}$$

with initial conditions

$$u(x, 0) = u_0(x), \quad i(x, 0) = i_0(x), \quad x \in [0, L] \quad (2)$$

where  $L$ ,  $C$ ,  $R$  and  $G$  are prescribed specific parameters of the line and  $L > 0$  is its length and  $u_0(x)$ ,  $i_0(x)$  are prescribed functions. Here  $u(x, t)$ ,  $i(x, t)$  are unknown functions – voltage and current respectively. The boundary conditions are derived from the loads and sources at the ends of the line (cf.[1], Fig. 1) on the base of Kirchoff's laws.

For  $x = 0$  we have

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$$u_{Y_p} = \frac{d\tilde{L}_p(i)}{dt} \circ \frac{\partial dL_p(i)}{\partial i} + L_p(i) \frac{\partial di}{\partial t} \quad (p=0,1).$$

For the  $I$ - $V$  characteristics we assume that

$$R_p(i) = \sum_{n=1}^m r_n^{(p)} i^n, \quad (p=0,1).$$

The mixed problem (1)-(4) can be reduced to an initial value problem for a nonlinear neutral system. Recall some transformations from [1]:

$$u(x,t) = \frac{e^{-\frac{R}{L}t}}{2\sqrt{C}} W(x,t) - \frac{e^{-\frac{R}{L}t}}{2\sqrt{C}} J(x,t),$$

$$i(x,t) = \frac{e^{-\frac{R}{L}t}}{2\sqrt{L}} W(x,t) + \frac{e^{-\frac{R}{L}t}}{2\sqrt{L}} J(x,t)$$

which reduces (1) to the system

$$\frac{W}{t} + \frac{1}{\sqrt{LC}} \frac{W}{x} = 0, \quad \frac{J}{t} - \frac{1}{\sqrt{LC}} \frac{J}{x} = 0. \quad (7)$$

The initial conditions remain unchanged because

$$W(0,t) = V(0,t) = V_0(x), \quad J(0,t) = I(0,t) = I_0(x).$$

Assume that the unknown functions are (cf.[1])

$$W(L,t) = W(t), \quad J(0,t) = J(t).$$

and taking into account the relations obtained after integration along the characteristics

$$W(L,t+T) = W(0,t), \quad J(L,t) = J(0,t+T)$$

we obtain the system (cf.[1]):

$$\begin{aligned} & (e^{-\frac{R}{L}t} W(t) - e^{-\frac{R}{L}(t-T)} J(t-T)) / (2\sqrt{C}) \\ &= E_0(t-T) - R_0 \frac{\partial}{\partial t} (e^{-\frac{R}{L}t} W(t) + e^{-\frac{R}{L}(t-T)} J(t-T)) / (2\sqrt{L}) \\ & - \bar{C}_0^{-1} \frac{\partial}{\partial t} (e^{-\frac{R}{L}q} W(q) + e^{-\frac{R}{L}(q-T)} J(q-T)) / (2\sqrt{L}) \\ & - \frac{\partial}{\partial t} (e^{-\frac{R}{L}t} W(t) + e^{-\frac{R}{L}(t-T)} J(t-T)) / (2\sqrt{L}) \\ & + \frac{d}{dt} L_0 \frac{\partial}{\partial t} (e^{-\frac{R}{L}t} W(t) + e^{-\frac{R}{L}(t-T)} J(t-T)) / (2\sqrt{L}) \\ & + L_0 (e^{-\frac{R}{L}t} W(t) + e^{-\frac{R}{L}(t-T)} J(t-T)) / (2\sqrt{L}) \\ & + \frac{d}{dt} \frac{\partial}{\partial t} (e^{-\frac{R}{L}t} W(t) + e^{-\frac{R}{L}(t-T)} J(t-T)) / (2\sqrt{L}) \\ & (e^{-\frac{R}{L}(t-T)} W(t-T) - e^{-\frac{R}{L}t} J(t)) / (2\sqrt{C}) \\ &= E_1(t-T) + R_1 \frac{\partial}{\partial t} (e^{-\frac{R}{L}(t-T)} W(t-T) + e^{-\frac{R}{L}t} J(t)) / (2\sqrt{L}) \end{aligned}$$

$$\begin{aligned} & + \bar{C}_1^{-1} \frac{\partial}{\partial t} (e^{-\frac{R}{L}(q-T)} W(q-T) + e^{-\frac{R}{L}q} J(q)) / (2\sqrt{L}) \\ & + \frac{\partial}{\partial t} (e^{-\frac{R}{L}(t-T)} W(t-T) + e^{-\frac{R}{L}t} J(t)) / (2\sqrt{L}) \\ & + \frac{d}{dt} L_1 \frac{\partial}{\partial t} (e^{-\frac{R}{L}(t-T)} W(t-T) + e^{-\frac{R}{L}t} J(t)) / (2\sqrt{L}) \\ & + L_1 (e^{-\frac{R}{L}(t-T)} W(t-T) + e^{-\frac{R}{L}t} J(t)) / (2\sqrt{L}) \\ & + \frac{d}{dt} \frac{\partial}{\partial t} (e^{-\frac{R}{L}(t-T)} W(t-T) + e^{-\frac{R}{L}t} J(t)) / (2\sqrt{L}) \end{aligned}$$

**Principal Remark.** If  $W(t), J(t)$  are periodic functions

then the functions  $W(t) \circ e^{-\frac{R}{L}t} W(t), J(t) \circ e^{-\frac{R}{L}t} J(t)$  should be oscillatory ones. They should satisfy the following inequalities  $|W(t)| \leq W_0 e^{-\frac{R}{L}t}, |J(t)| \leq J_0 e^{-\frac{R}{L}t}$ . Further on we again denote them by  $W(t), J(t)$ , that is,

$$|W(t)| \leq W_0 e^{-\frac{R}{L}t}, |J(t)| \leq J_0 e^{-\frac{R}{L}t}.$$

So omitting some transformations given in details in [1] we reach the problem for existence-uniqueness of oscillatory solution  $(W(t), J(t))$  of the following system:

$$\begin{aligned} \frac{dW(t)}{dt} &= - \frac{dJ(t-T)}{dt} + \frac{2\sqrt{L}}{\tilde{L}_0(W,J)(t)} E_0(t-T) \\ & - \frac{2\sqrt{L}}{\tilde{L}_0(W,J)(t)} R_0 \frac{\partial}{\partial t} (W(t) + J(t-T)) / (2\sqrt{L}) \\ & - \frac{Z_0 W(t)}{\tilde{L}_0(W,J)(t)} + \frac{Z_0 J(t-T)}{\tilde{L}_0(W,J)(t)} \\ & - \frac{2\sqrt{L}}{\tilde{L}_0(W,J)(t)} \bar{C}_0^{-1} \frac{\partial}{\partial t} (W(q) + J(q-T)) / (2\sqrt{L}) U(W,J)(t); \\ \frac{dJ(t)}{dt} &= - \frac{dW(t-T)}{dt} - \frac{2\sqrt{L}}{\tilde{L}_1(W,J)(t)} E_1(t-T) \\ & - \frac{2\sqrt{L}}{\tilde{L}_1(W,J)(t)} R_1 \frac{\partial}{\partial t} (W(t-T) + J(t)) / (2\sqrt{L}) \\ & + \frac{Z_0 W(t-T)}{\tilde{L}_1(W,J)(t)} - \frac{Z_0 J(t)}{\tilde{L}_1(W,J)(t)} \\ & - \frac{2\sqrt{L}}{\tilde{L}_1(W,J)(t)} \bar{C}_1^{-1} \frac{\partial}{\partial t} (W(q-T) + J(q)) / (2\sqrt{L}) I(W,J)(t); \end{aligned}$$

$$W(t) = \tilde{W}_0(t), \quad \dot{W}(t) = \dot{\tilde{W}}_0(t), \quad J(t) = \tilde{J}_0(t), \quad \dot{J}(t) = \dot{\tilde{J}}_0(t) \quad (8)$$

$t \in [0, T]$  where

$$\begin{aligned} \mathbb{L}_0(W, J)(t) &= \frac{W(t) + J(t - T)}{2\sqrt{L}} \frac{d}{dt} L_0 \frac{\mathfrak{A}W(t) + J(t - T)}{\mathfrak{C}} \frac{\ddot{\mathfrak{O}}}{2\sqrt{L}} \\ &+ L_0 \frac{\mathfrak{A}W(t) + J(t - T)}{\mathfrak{C}} \frac{\ddot{\mathfrak{O}}}{2\sqrt{L}} = \mathfrak{A}_{n=0}^m (n+1) l_n^{(0)} \frac{\mathfrak{A}W(t) + J(t - T)}{\mathfrak{C}} \frac{\ddot{\mathfrak{O}}}{2\sqrt{L}}; \\ \mathbb{L}_1(W, J)(t) &= \frac{W(t - T) + J(t)}{2\sqrt{L}} \frac{d}{dt} L_1 \frac{\mathfrak{A}W(t - T) + J(t)}{\mathfrak{C}} \frac{\ddot{\mathfrak{O}}}{2\sqrt{L}} \\ &+ L_1 \frac{\mathfrak{A}W(t - T) + J(t)}{\mathfrak{C}} \frac{\ddot{\mathfrak{O}}}{2\sqrt{L}} = \mathfrak{A}_{n=0}^m (n+1) l_n^{(1)} \frac{\mathfrak{A}W(t - T) + J(t)}{\mathfrak{C}} \frac{\ddot{\mathfrak{O}}}{2\sqrt{L}}. \end{aligned}$$

The initial functions are obtained as in [1].

First we formulate the conditions for the initial functions

$$(\text{IN}) \quad |\tilde{W}_0(t)| \mathfrak{E} e^{-\frac{R}{L}t} W_0 e^{m(t-t_k)}; \quad |\tilde{J}_0(t)| \mathfrak{E} e^{-\frac{R}{L}t} J_0 e^{m(t-t_k)}.$$

Assume that one can find an interval  $|i| \mathfrak{E} i_0$  such that the inequalities

$$\begin{aligned} & \left| \frac{W(t) + J(t - T)}{2\sqrt{L}} \right| \mathfrak{E} e^{m(t-t_k)} e^{-\frac{R}{L}t} W_0 + e^{-\frac{R}{L}(t-T)} J_0 e^{-mT} \\ & \mathfrak{E} e^{m(t-t_k)} e^{-\frac{R}{L}t} W_0 + J_0 e^{-\frac{R}{L}t} e^{-\frac{R}{L}T} W_0 + J_0 e^{-\frac{R}{L}t} e^{-\frac{R}{L}T} J_0 \mathfrak{E} i_0; \quad (9) \\ & \left| \frac{W(t - T) + J(t)}{2\sqrt{L}} \right| \mathfrak{E} e^{m(t-t_k)} e^{-\frac{R}{L}(t-T)} W_0 e^{-mT} + e^{-\frac{R}{L}t} J_0 \\ & \mathfrak{E} e^{m(t-t_k)} e^{-\frac{R}{L}t} W_0 e^{-\frac{R}{L}T} J_0 + J_0 \mathfrak{E} i_0 \end{aligned}$$

imply

$$\begin{aligned} \mathbb{L}_0(t - T) &= \mathfrak{A}_{n=0}^m (n+1) l_n^{(0)} \frac{\mathfrak{A}W(t) + J(t - T)}{\mathfrak{C}} \frac{\ddot{\mathfrak{O}}}{2\sqrt{L}} \mathfrak{A} \hat{L}_0 > 0 \\ \mathfrak{P} \quad & \frac{1}{\mathbb{L}_0(W, J)(t)} \mathfrak{E} \frac{1}{\hat{L}_0}; \\ \mathbb{L}_1(t - T) &= \mathfrak{A}_{n=0}^m (n+1) l_n^{(1)} \frac{\mathfrak{A}W(t - T) + J(t)}{\mathfrak{C}} \frac{\ddot{\mathfrak{O}}}{2\sqrt{L}} \mathfrak{A} \hat{L}_1 > 0 \\ \mathfrak{P} \quad & \frac{1}{\mathbb{L}_1(W, J)(t)} \mathfrak{E} \frac{1}{\hat{L}_1}. \end{aligned}$$

This can be done if the polynomial has suitable properties (cf. Numerical example).

### 3. An Existence-Uniqueness of Oscillatory Solution for the Neutral System

Here we introduce an operator representation of the oscillatory problem and by a fixed point theorem in uniform spaces [3] we establish an existence-uniqueness of global oscillatory solution.

Now we are able to formulate the main problem: to find a solution of (8) with advanced prescribed zeros on an interval

$[t_0, \mathfrak{Y})$ ,  $(T \circ t_0)$  where  $\tilde{W}_0(t)$ ,  $\tilde{J}_0(t)$ ,  $t \hat{\mathfrak{I}} [0, T]$  are prescribed initial oscillating functions on the interval  $[0, t_0]$ .

Let  $S_T = \{t_k\}_{k=0}^n \hat{\mathfrak{I}} N$  (such that  $t_0 = 0$ ,  $t_n = T \circ t_0$ ) be the set of zeros of the initial functions, that is,  $\tilde{W}_0(t_k) = 0$ ,  $\tilde{J}_0(t_k) = 0$ .

Let  $S = \{t_k\}_{k=0}^\mathfrak{Y}$  be a strictly increasing sequence of real numbers satisfying the following conditions (C):

$$(\text{C1}) \quad \lim_{k \rightarrow \mathfrak{Y}} t_k = \mathfrak{Y};$$

(C2) for every  $k$  there is  $s < k$  such that  $t_k - T = t_s$  ( $t_k - T \mathfrak{A} t_0$ ) where  $t_s \hat{\mathfrak{I}} S_T \hat{\mathfrak{E}} S$ .

$$(\text{E}) \quad E_p(t_s) = 0 \quad (p = 0, 1).$$

It follows  $0 \mathfrak{E} \inf\{t_{k+1} - t_k : k = 0, 1, 2, \dots\}$

$$\mathfrak{E} \sup\{t_{k+1} - t_k : k = 0, 1, 2, \dots\} = T_0 < \mathfrak{Y};$$

Introduce the sets  $C^1[t_0, \mathfrak{Y})$  consisting of all continuous and bounded functions differentiable with bounded derivatives on every interval  $[t_k, t_{k+1}]$ .

**Remark 3.1.** Let us note that the left and right derivative at  $t_k$  of any  $W(\cdot), J(\cdot) \hat{\mathfrak{I}} C^1[t_0, \mathfrak{Y})$  may not coincide. This requires introducing of uniform spaces [3] with a suitable topology for continuous functions with piece wise continuous derivatives.

Introduce the sets

$$M_W = \{W(\cdot) \hat{\mathfrak{I}} C^1[t_0, \mathfrak{Y}) : W(t) = \tilde{W}_0(t), t \hat{\mathfrak{I}} [0, T] \dot{W}(t_k) = 0$$

$$\dot{W}(t) \mathfrak{E} W_0 e^{-\frac{R}{L}t} e^{m(t-t_k)}, t \hat{\mathfrak{I}} [t_k, t_{k+1}] \dot{W} \mathfrak{E} \mathfrak{b}$$

$$M_J = \{J(\cdot) \hat{\mathfrak{I}} C^1[t_0, \mathfrak{Y}) : J(t) = \tilde{J}_0(t), t \hat{\mathfrak{I}} [0, T] \dot{J}(t_k) = 0$$

$$\dot{J}(t) \mathfrak{E} J_0 e^{-\frac{R}{L}t} e^{m(t-t_k)}, t \hat{\mathfrak{I}} [t_k, t_{k+1}] \dot{J} \mathfrak{E} \mathfrak{b}$$

**Remark 3.2.** Conditions (C2) and (E) imply that if  $W(t_k) = 0$ ,  $E_p(t_k) = 0$  then  $W(t_k - T) = 0$ ,  $E_p(t_k) = 0$ .

**Remark 3.3.** Let us comment the conformity condition (CC). It could be obtained replacing  $t = t_0$  in (8) and in view of  $W(0) = W(t_0) = J(0) = J(t_0) = E_p(0) = E_p(t_0) = 0$  and  $\bar{C}_p^{-1}(0) = 0$ :

$$\dot{W}(t_0) = -\mathfrak{J}(0), \quad \mathfrak{J}(t_0) = -\dot{W}(0),$$

where  $\dot{W}(t) = dW(t)/dt$ . We notice that (CC) becomes a

relation between the initial functions  $\dot{W}_0(t_0) = -\mathfrak{J}_0(0)$ ,  $\mathfrak{J}_0(t_0) = -\dot{W}_0(0)$ . If the last condition is not satisfied then the jump of the derivative at  $t = t_0$  propagates to the right and it falls at some zero point because of  $t_k - T = t_s$ . We do not go beyond our function space because the derivative of our functions might have jumps at  $t = t_k$ .

**Remark 3.4.** It follows that the functions from  $M_W$  and  $M_J$  satisfy the inequalities

$|W(t)| \leq W_0 e^{m(t-t_k)}, |J(t)| \leq J_0 e^{m(t-t_k)}, t \in [t_k, t_{k+1}]$  ( $k=0,1,2,\dots$ )  
 where  $W_0, J_0, m, m_0 = m_0$  are positive constants and  $W_0 e^{m_0} \leq f_0 < \infty$ . Besides we point out that

$|W(t)| \leq W_0 e^{-\frac{R}{L}t} e^{m(t-t_k)}$  implies the global estimate  
 $|W(t)| \leq W_0 e^{-\frac{R}{L}t} e^{m(t-t_k)} \leq W_0 e^{-\frac{R}{L}t} e^{m_0}$ .

Introduce the following family of pseudo-metrics

$$\begin{aligned} r^{(k)}(W, \bar{W}) &= \max \left\{ |W(t) - \bar{W}(t)| : t \in [t_k, t_{k+1}] \right\}, \\ r^{(k)}(J, \bar{J}) &= \max \left\{ |J(t) - \bar{J}(t)| : t \in [t_k, t_{k+1}] \right\}, \\ \hat{r}^{(k)}(W, \bar{W}) &= \max \left\{ |W(t) - \bar{W}(t)| : t \in [t_0, t_{k+1}] \right\}, \\ \hat{r}^{(k)}(J, \bar{J}) &= \max \left\{ |J(t) - \bar{J}(t)| : t \in [t_0, t_{k+1}] \right\}, \\ r_m^{(k)}(W, \bar{W}) &= \max \left\{ e^{-m(t-t_k)} |W(t) - \bar{W}(t)| : t \in [t_k, t_{k+1}] \right\}, \\ r_m^{(k)}(J, \bar{J}) &= \max \left\{ e^{-m(t-t_k)} |J(t) - \bar{J}(t)| : t \in [t_k, t_{k+1}] \right\}, \\ \hat{r}_m^{(k)}(W, \bar{W}) &= \max \left\{ r_m^{(0)}(W, \bar{W}), r_m^{(1)}(W, \bar{W}), \dots, r_m^{(k)}(W, \bar{W}) \right\}, \\ \hat{r}_m^{(k)}(J, \bar{J}) &= \max \left\{ r_m^{(0)}(J, \bar{J}), r_m^{(1)}(J, \bar{J}), \dots, r_m^{(k)}(J, \bar{J}) \right\}, \\ r_m^{(k)}(\bar{W}, \bar{W}) &= \max \left\{ e^{-m(t-t_k)} |\bar{W}(t) - \bar{W}(t)| : t \in [t_k, t_{k+1}] \right\}, \\ r_m^{(k)}(\bar{J}, \bar{J}) &= \max \left\{ e^{-m(t-t_k)} |\bar{J}(t) - \bar{J}(t)| : t \in [t_k, t_{k+1}] \right\}, \\ \hat{r}_m^{(k)}(\bar{W}, \bar{W}) &= \max \left\{ r_m^{(0)}(\bar{W}, \bar{W}), r_m^{(1)}(\bar{W}, \bar{W}), \dots, r_m^{(k)}(\bar{W}, \bar{W}) \right\}, \\ \hat{r}_m^{(k)}(\bar{J}, \bar{J}) &= \max \left\{ r_m^{(0)}(\bar{J}, \bar{J}), r_m^{(1)}(\bar{J}, \bar{J}), \dots, r_m^{(k)}(\bar{J}, \bar{J}) \right\}. \end{aligned}$$

The following inequalities imply the equivalence of the both families of pseudo-metrics

$$\begin{aligned} r_m^{(k)}(W, \bar{W}) &\leq r^{(k)}(W, \bar{W}) \leq e^{m_0} r_m^{(k)}(W, \bar{W}), \quad (k=0,1,2,\dots) \\ r_m^{(k)}(J, \bar{J}) &\leq r^{(k)}(J, \bar{J}) \leq e^{m_0} r_m^{(k)}(J, \bar{J}), \quad (k=0,1,2,\dots). \end{aligned}$$

It is easy to verify that

$$\begin{aligned} \hat{r}^{(k)}(W, \bar{W}) &= \max \left\{ r^{(0)}(W, \bar{W}), r^{(1)}(W, \bar{W}), \dots, r^{(k)}(W, \bar{W}) \right\} \\ &\leq e^{m_0} \max \left\{ r_m^{(0)}(W, \bar{W}), r_m^{(1)}(W, \bar{W}), \dots, r_m^{(k)}(W, \bar{W}) \right\} \\ &= e^{m_0} \hat{r}_m^{(k)}(W, \bar{W}); \quad (10) \\ \hat{r}^{(k)}(J, \bar{J}) &= \max \left\{ r^{(0)}(J, \bar{J}), r^{(1)}(J, \bar{J}), \dots, r^{(k)}(J, \bar{J}) \right\} \\ &\leq e^{m_0} \max \left\{ r_m^{(0)}(J, \bar{J}), r_m^{(1)}(J, \bar{J}), \dots, r_m^{(k)}(J, \bar{J}) \right\} \\ &= e^{m_0} \hat{r}_m^{(k)}(J, \bar{J}). \end{aligned}$$

The set  $M_W \times M_J$  turns out into a complete uniform space with respect to the family of pseudo-metrics

$$\begin{aligned} \hat{r}_m^{(k)}((W, \bar{W}), (J, \bar{J})) &= \max \left\{ \hat{r}_m^{(k)}(W, \bar{W}), \hat{r}_m^{(k)}(J, \bar{J}), \hat{r}_m^{(k)}(\bar{W}, \bar{W}), \hat{r}_m^{(k)}(\bar{J}, \bar{J}) \right\} \\ &= \max \left\{ \hat{r}_m^{(k)}(W, \bar{W}), \hat{r}_m^{(k)}(J, \bar{J}), \hat{r}_m^{(k)}(\bar{W}, \bar{W}), \hat{r}_m^{(k)}(\bar{J}, \bar{J}) \right\} \\ &\quad (k=0,1,2,\dots). \end{aligned}$$

Define the operator  $B = (B_W(W, J), B_J(W, J))$  by the formulas

$$B_W(W, J)(t) = \tilde{W}_0(t), t \in [0, t_0],$$

$$\begin{aligned} B_W(W, J)(t) &= B_W^{(k)}(W, J)(t) := \int_{t_k}^t \mathcal{U}(W, J)(s) ds \\ &\quad - \frac{t - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \mathcal{U}(W, J)(s) ds, t \in [t_k, t_{k+1}], (k=0,1,2,\dots) \\ B_J(W, J)(t) &= \tilde{J}_0(t), t \in [0, t_0], \\ B_J(W, J)(t) &= B_J^{(k)}(W, J)(t) := \int_{t_k}^t \mathcal{J}(W, J)(s) ds \\ &\quad - \frac{t - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \mathcal{J}(W, J)(s) ds, t \in [t_k, t_{k+1}], (k=0,1,2,\dots). \end{aligned}$$

The sources  $E_0(t), E_1(t) : [0, \infty) \rightarrow (-\infty, \infty)$  are continuously differentiable oscillatory functions.

Further on the following assumptions will be hold:

**Assumption (IN):**  $|\tilde{W}_0(t)| \leq W_0 e^{-\frac{R}{L}t} e^{m(t-t_k)};$

$$|\tilde{J}_0(t)| \leq J_0 e^{-\frac{R}{L}t} e^{m(t-t_k)}, t \in [t_k, t_{k+1}]$$

**Assumption (E):**  $|E_p(t)| \leq W_0 e^{-\frac{R}{L}t} e^{m(t-t_k)} (p=0,1);$

**Assumption (II):**  $e^{m_0} \frac{W_0 + J_0}{2\sqrt{L}} \leq i_0; W_0 e^{m_0} \leq f_0; m > \frac{R}{L}.$

**Lemma 3.1.** Problem (8) has a solution  $(W, J) \in M_W \times M_J$  iff the operator  $B$  has a fixed point in  $M_W \times M_J$ , that is,

$$(W, J) = (B_W(W, J), B_J(W, J)). \quad (11)$$

**Proof:** Let  $(W, J) \in M_W \times M_J$  be a solution of (8). Then integrating every equation of (8) on  $[t_k, t] \subset [t_k, t_{k+1}]$  ( $k=0,1,2,\dots$ ) we obtain

$$\begin{aligned} W(t) - W(t_k) &= \int_{t_k}^t \mathcal{U}(W, J)(s) ds \quad \hat{=} \quad W(t) = \int_{t_k}^t \mathcal{U}(W, J)(s) ds, \\ J(t) - J(t_k) &= \int_{t_k}^t \mathcal{J}(W, J)(s) ds \quad \hat{=} \quad J(t) = \int_{t_k}^t \mathcal{J}(W, J)(s) ds \end{aligned}$$

and then

$$\begin{aligned} W(t) &= \int_{t_k}^t \mathcal{U}(W, J)(s) ds \quad \hat{=} \quad W(t_{k+1}) = \int_{t_k}^{t_{k+1}} \mathcal{U}(W, J)(s) ds = 0 \\ J(t) &= \int_{t_k}^t \mathcal{J}(W, J)(s) ds \quad \hat{=} \quad J(t_{k+1}) = \int_{t_k}^{t_{k+1}} \mathcal{J}(W, J)(s) ds = 0. \end{aligned} \quad (12)$$

Therefore the pair  $(W, J)$  satisfies

$$\begin{aligned} W(t) &= \int_{t_k}^t \mathcal{U}(W, J)(s) ds - \frac{t - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \mathcal{U}(W, J)(s) ds, \\ J(t) &= \int_{t_k}^t \mathcal{J}(W, J)(s) ds - \frac{t - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \mathcal{J}(W, J)(s) ds \end{aligned}$$











$$\begin{aligned}
& + \frac{2\sqrt{L}H_1}{\hat{L}_1} \frac{W_0 e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} + J_0 e^{\pi_0} \frac{t}{\ddot{\theta}} e^{-\frac{R}{L} s} \frac{L}{R} (e^{\frac{R}{L} T_0} - 1) ds}{2\sqrt{L}} \\
& \mathcal{E} e^{-\frac{R}{L} t} e^{m(t-t_k)} \frac{\dot{e}}{\ddot{\theta}} W_0 e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} + \frac{2\sqrt{L} W_0}{\hat{L}_1} e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} e^{m_0} \frac{L}{R} (e^{\frac{R}{L} T_0} - 1) \\
& + \frac{2\sqrt{L}}{\hat{L}_1} \frac{m}{n=1} \left| r_n^{(1)} \right| (i_0)^n e^{n m_0} \frac{L}{R} (e^{\frac{R}{L} T_0} - 1) + \frac{Z_0 J_0}{\hat{L}_1} e^{m_0} \frac{L}{R} (e^{\frac{R}{L} T_0} - 1) \\
& + \frac{Z_0 W_0}{\hat{L}_1} e^{m_0} e^{-(m \frac{R}{L} T)} \frac{L}{R} (e^{\frac{R}{L} T_0} - 1) + \frac{2\sqrt{L} H_1 i_0}{\hat{L}_1} e^{m_0} \frac{\mathcal{A} \ddot{\theta}^2}{\epsilon R \ddot{\theta}} (e^{\frac{R}{L} T_0} - 1)^2 \dot{u} \\
& \mathcal{E} e^{-\frac{R}{L} t} e^{m(t-t_k)} \left\{ W_0 e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} + e^{m_0} \frac{2L\sqrt{L}}{\hat{L}_0 R} (e^{\frac{R}{L} T_0} - 1) \right. \\
& \left. \frac{\dot{e}}{\ddot{\theta}} W_0 e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} + \frac{m}{n=1} \left| r_n^{(1)} \right| (i_0)^n e^{(n-1)m_0} + Z_0 i_0 + \frac{H_1 i_0}{R} (e^{\frac{R}{L} T_0} - 1) \dot{u} \right\}; \\
& J_2 \mathcal{E} \left| \frac{\ddot{\theta}}{\ddot{\theta}} \frac{dW(s-T)}{ds} \right|_{t_k} + \frac{2\sqrt{L}}{\hat{L}_1} \left| \frac{\ddot{\theta}}{\ddot{\theta}} E_1(s-T) \right|_{t_k} ds \\
& + \frac{2\sqrt{L}}{\hat{L}_1} \left| \frac{\ddot{\theta}}{\ddot{\theta}} \frac{dW(s-T) + J(s)}{2\sqrt{L}} \right|_{t_k} ds + \frac{Z_0}{\hat{L}_1} \left| \frac{\ddot{\theta}}{\ddot{\theta}} W(s-T) \right|_{t_k} ds \\
& + \frac{Z_0}{\hat{L}_1} \left| \frac{\ddot{\theta}}{\ddot{\theta}} J(s) \right|_{t_k} ds + \frac{2\sqrt{L}}{\hat{L}_1} \left| \frac{\ddot{\theta}}{\ddot{\theta}} \bar{C}_1 \frac{\mathcal{A} \ddot{\theta} W(q-T) + J(q)}{\epsilon \ddot{\theta}} \right|_{t_k} \frac{d\ddot{\theta}}{dq} ds \\
& \mathcal{E} \frac{2\sqrt{L} W_0}{\hat{L}_1} e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} e^{m_0} \frac{t_{k+1}}{\ddot{\theta}} e^{-\frac{R}{L} s} ds \\
& + \frac{2\sqrt{L}}{\hat{L}_1} \frac{m}{n=1} \left| r_n^{(1)} \right| \frac{\mathcal{A} W_0 e^{-\frac{R}{L}(s-T)} e^{m(s-T-t_k)} + J_0 e^{-\frac{R}{L} s} e^{m(s-t_k)}}{2\sqrt{L}} \frac{\ddot{\theta}^n}{\ddot{\theta}} ds \\
& + \frac{Z_0 W_0}{\hat{L}_1} e^{m_0} e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} e^{t_{k+1}} e^{-\frac{R}{L} s} ds + \frac{Z_0 J_0}{\hat{L}_1} e^{m_0} \frac{t_{k+1}}{\ddot{\theta}} e^{-\frac{R}{L} s} ds \\
& + \frac{2\sqrt{L}}{\hat{L}_1} H_1 \frac{t_{k+1}}{\ddot{\theta}} \left| \frac{\mathcal{A} W_0 e^{-\frac{R}{L}(q-T)} e^{m(q-T-t_k)} + J_0 e^{-\frac{R}{L} q} e^{m(q-t_k)}}{2\sqrt{L}} \right|_{t_k} \frac{\ddot{\theta}}{\ddot{\theta}} dq ds \\
& \mathcal{E} \frac{2\sqrt{L} W_0}{\hat{L}_1} e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} e^{m_0} \frac{t_{k+1}}{\ddot{\theta}} e^{-\frac{R}{L} s} ds \\
& + \frac{2\sqrt{L}}{\hat{L}_1} \frac{m}{n=1} \left| r_n^{(1)} \right| \frac{\mathcal{A} W_0 e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} + J_0}{2\sqrt{L}} \frac{\ddot{\theta}^n}{\ddot{\theta}} e^{t_{k+1}} e^{-\frac{R}{L} s} e^{n m(s-t_k)} ds \\
& + \frac{Z_0 W_0}{\hat{L}_1} e^{m_0} e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} e^{t_{k+1}} e^{-\frac{R}{L} s} ds + \frac{Z_0 J_0}{\hat{L}_1} e^{m_0} \frac{t_{k+1}}{\ddot{\theta}} e^{-\frac{R}{L} s} ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{2\sqrt{L} W_0}{\hat{L}_0} e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} + J_0 H_1 \frac{t_{k+1}}{\ddot{\theta}} e^{-\frac{R}{L} q} e^{m(q-t_k)} dq ds \\
& \mathcal{E} \frac{2\sqrt{L} W_0}{\hat{L}_1} e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} e^{m_0} \frac{t_{k+1}}{\ddot{\theta}} e^{-\frac{R}{L} s} ds \\
& + \frac{2\sqrt{L}}{\hat{L}_1} \frac{m}{n=1} \left| r_n^{(1)} \right| \frac{\mathcal{A} W_0 e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} + J_0}{2\sqrt{L}} \frac{\ddot{\theta}^n}{\ddot{\theta}} e^{n m_0} \frac{t_{k+1}}{\ddot{\theta}} e^{-\frac{R}{L} s} ds \\
& + \frac{Z_0 W_0}{\hat{L}_1} e^{m_0} e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} e^{t_{k+1}} e^{-\frac{R}{L} s} ds + \frac{Z_0 J_0}{\hat{L}_1} e^{m_0} \frac{t_{k+1}}{\ddot{\theta}} e^{-\frac{R}{L} s} ds \\
& + \frac{2\sqrt{L} W_0}{\hat{L}_1} e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} + J_0 e^{m_0} H_1 \frac{t_{k+1}}{\ddot{\theta}} e^{-\frac{R}{L} q} \frac{\ddot{\theta}}{\ddot{\theta}} dq ds \\
& \mathcal{E} e^{-\frac{R}{L} t} \frac{2L}{R} \text{sh} \frac{\mathcal{A} R}{\epsilon L} T_0 \frac{\ddot{\theta} 2\sqrt{L} W_0}{\ddot{\theta} \hat{L}_1} e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} e^{m_0} \\
& + \frac{2\sqrt{L}}{\hat{L}_1} \frac{m}{n=1} \left| r_n^{(1)} \right| \frac{\mathcal{A} W_0 e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} + J_0}{2\sqrt{L}} \frac{\ddot{\theta}^n}{\ddot{\theta}} e^{n m_0} e^{-\frac{R}{L} t} \frac{2L}{R} \text{sh} \frac{\mathcal{A} R}{\epsilon L} T_0 \frac{\ddot{\theta}}{\ddot{\theta}} \\
& + e^{-\frac{R}{L} t} \frac{2L}{R} \text{sh} \frac{\mathcal{A} R}{\epsilon L} T_0 \frac{\ddot{\theta} Z_0 W_0}{\ddot{\theta} \hat{L}_1} e^{m_0} e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} \\
& + e^{-\frac{R}{L} t} \frac{2L}{R} \text{sh} \frac{\mathcal{A} R}{\epsilon L} T_0 \frac{\ddot{\theta} Z_0 J_0}{\ddot{\theta} \hat{L}_1} e^{m_0} \\
& + \frac{2\sqrt{L} W_0}{\hat{L}_1} e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} + J_0 e^{m_0} H_1 \frac{t_{k+1}}{\ddot{\theta}} e^{-\frac{R}{L} s} \frac{\mathcal{A} R}{\epsilon L} T_0 \frac{\ddot{\theta}}{\ddot{\theta}} \\
& + \frac{2\sqrt{L}}{\hat{L}_1} \frac{m}{n=1} \left| r_n^{(1)} \right| \frac{\mathcal{A} W_0 e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} + J_0}{2\sqrt{L}} \frac{\ddot{\theta}^n}{\ddot{\theta}} e^{(n-1)m_0} \frac{2L}{R} \text{sh} \frac{\mathcal{A} R}{\epsilon L} T_0 \frac{\ddot{\theta}}{\ddot{\theta}} \\
& + \frac{2L \cdot 2\sqrt{L}}{\hat{L}_1 R} \text{sh} \frac{\mathcal{A} R}{\epsilon L} T_0 \frac{\ddot{\theta}}{\ddot{\theta}} \frac{Z_0 \mathcal{A} W_0 e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} + J_0}{2\sqrt{L}} \frac{\ddot{\theta}}{\ddot{\theta}} \\
& \mathcal{E} e^{-\frac{R}{L} t} e^{m_0} \text{sh} \frac{\mathcal{A} R}{\epsilon L} T_0 \frac{\ddot{\theta} 4L\sqrt{L}}{\ddot{\theta} R \hat{L}_1} \frac{\dot{e}}{\ddot{\theta}} W_0 e^{-\frac{\alpha}{\epsilon} m \frac{R}{L} \frac{\ddot{J}}{\ddot{\theta}}} \\
& + \frac{m}{n=1} \left| r_n^{(1)} \right| (i_0)^n e^{(n-1)m_0} + Z_0 i_0 + i_0 H_1 \frac{L}{R} (e^{\frac{R}{L} T_0} - 1) \dot{u}.
\end{aligned}$$

Therefore

$$|B_j^{(k)}(W, J)(t)| \mathcal{E}$$

$$\begin{aligned} & \mathbb{E} e^{-\frac{R}{L}t} e^{m(t-t_k)} \left| W_0 e^{-\frac{m}{L}T} + e^{mT} \frac{4L\sqrt{L}}{\hat{L}_1 R} \right. \\ & \cdot \frac{e^{\frac{R}{L}T_0} - 1}{2} + \operatorname{sh} \frac{mR}{L} T_0 e^{-\frac{m}{L}T} W_0 e^{-\frac{m}{L}T} + \hat{a} \left| r_n^{(1)} \right| (i_0)^n e^{(n-1)m} \\ & + Z_0 i_0 + \frac{H_1 i_0 L}{R} e^{\frac{R}{L}T_0} - 1 \mathbb{E} J_0 e^{-\frac{R}{L}t} e^{m(t-t_k)}. \end{aligned}$$

So the operator  $B$  maps the set  $M_W \times M_J$  into itself.

In what follows we show that  $B$  is contractive operator.

The maximum of

$$\{r_m^{(0)}(W, \bar{W}), r_m^{(1)}(W, \bar{W}), \dots, r_m^{(k_0)}(W, \bar{W})\}$$

attends for some  $k \in \{0, 1, 2, \dots, k_0\}$ . Then

$$\begin{aligned} & |B_W^{(k)}(W, J)(t) - B_W^{(k)}(\bar{W}, \bar{J})(t)| \\ & \mathbb{E} \int_{t_k}^t |U(W, J)(s) - U(\bar{W}, \bar{J})(s)| ds \\ & + \frac{t - t_k}{t_{k+1} - t_k} \left| \int_{t_k}^{t_{k+1}} |U(W, J)(s) - U(\bar{W}, \bar{J})(s)| ds \right| \leq U_1 + U_2. \end{aligned}$$

We have

$$\begin{aligned} & U_1 \mathbb{E} \left| \int_{t_k}^t \frac{dJ(s-T)}{ds} - \frac{d\bar{J}(s-T)}{ds} \frac{\ddot{\phi}}{\phi} ds \right| \\ & + 2\sqrt{L} \left| \int_{t_k}^t \frac{E_0(s-T)}{\tilde{L}_0(W, J)(s)} - \frac{E_0(s-T)}{\tilde{L}_0(\bar{W}, \bar{J})(s)} \frac{\ddot{\phi}}{\phi} ds \right| \\ & + \int_{t_k}^t \left| \frac{1}{\tilde{L}_0(W, J)(t)} R_0 \frac{\ddot{\phi} W(s) + J(s-T)}{2\sqrt{L}} \right. \\ & - \frac{1}{\tilde{L}_0(\bar{W}, \bar{J})(t)} R_0 \frac{\ddot{\phi} \bar{W}(s) + \bar{J}(s-T)}{2\sqrt{L}} \frac{\ddot{\phi}}{\phi} ds \\ & + Z_0 \int_{t_k}^t \left| \frac{W(s)}{\tilde{L}_0(W, J)(t)} - \frac{\bar{W}(s)}{\tilde{L}_0(\bar{W}, \bar{J})(t)} \right| ds \\ & + Z_0 \int_{t_k}^t \left| \frac{J(s-T)}{\tilde{L}_0(W, J)(t)} - \frac{\bar{J}(s-T)}{\tilde{L}_0(\bar{W}, \bar{J})(t)} \right| ds \\ & + 2\sqrt{L} \int_{t_k}^t \left| \frac{1}{\tilde{L}_0(W, J)(t)} \bar{C}_0^{-1} \frac{\ddot{\phi} W(q) + J(q-T)}{2\sqrt{L}} dq \right. \\ & - \frac{1}{\tilde{L}_0(\bar{W}, \bar{J})(t)} \bar{C}_0^{-1} \frac{\ddot{\phi} \bar{W}(q) + \bar{J}(q-T)}{2\sqrt{L}} dq \frac{\ddot{\phi}}{\phi} ds \\ & \mathbb{E} r_m^{(k)}(J, \bar{J}) e^{-mT} \int_{t_k}^t e^{m(s-t_k)} ds \\ & \cdot W_0 \left| \int_{t_k}^t \frac{e^{m(s-T-t_k)}}{\tilde{L}_0^2(W, J)} \frac{d\tilde{L}_0}{di} |W(s) + J(s-T) - \bar{W}(s) - \bar{J}(s-T)| ds \right| \end{aligned}$$

$$\begin{aligned} & + 2\sqrt{L} \int_{t_k}^t \left| \frac{1}{\tilde{L}_0(W, J)(s)} R_0 \frac{\ddot{\phi} W(s) + J(s-T)}{2\sqrt{L}} \right. \\ & - \frac{1}{\tilde{L}_0(W, J)(s)} R_0 \frac{\ddot{\phi} \bar{W}(s) + \bar{J}(s-T)}{2\sqrt{L}} \frac{\ddot{\phi}}{\phi} ds \\ & + 2\sqrt{L} \int_{t_k}^t \left| \frac{1}{\tilde{L}_0(W, J)(s)} R_0 \frac{\ddot{\phi} \bar{W}(s) + \bar{J}(s-T)}{2\sqrt{L}} \right. \\ & - \frac{1}{\tilde{L}_0(\bar{W}, \bar{J})(s)} R_0 \frac{\ddot{\phi} \bar{W}(s) + \bar{J}(s-T)}{2\sqrt{L}} \frac{\ddot{\phi}}{\phi} ds \\ & + Z_0 \int_{t_k}^t \left| \frac{W(s)}{\tilde{L}_0(W, J)(t)} - \frac{\bar{W}(s)}{\tilde{L}_0(\bar{W}, \bar{J})(t)} \right| ds \\ & + Z_0 \int_{t_k}^t \left| \frac{\bar{W}(s)}{\tilde{L}_0(W, J)(t)} - \frac{\bar{W}(s)}{\tilde{L}_0(\bar{W}, \bar{J})(t)} \right| ds \\ & + Z_0 \int_{t_k}^t \left| \frac{J(s-T)}{\tilde{L}_0(W, J)(s)} - \frac{\bar{J}(s-T)}{\tilde{L}_0(W, J)(ss)} \right| ds \\ & + Z_0 \int_{t_k}^t \left| \frac{\bar{J}(s-T)}{\tilde{L}_0(W, J)(s)} - \frac{\bar{J}(s-T)}{\tilde{L}_0(\bar{W}, \bar{J})(s)} \right| ds \\ & + 2\sqrt{L} \int_{t_k}^t \left| \frac{1}{\tilde{L}_0(W, J)(s)} \bar{C}_0^{-1} \frac{\ddot{\phi} W(q) + J(q-T)}{2\sqrt{L}} dq \right. \\ & - \frac{1}{\tilde{L}_0(\bar{W}, \bar{J})(s)} \bar{C}_0^{-1} \frac{\ddot{\phi} W(q) + J(q-T)}{2\sqrt{L}} dq \frac{\ddot{\phi}}{\phi} ds \\ & + 2\sqrt{L} \int_{t_k}^t \left| \frac{1}{\tilde{L}_0(\bar{W}, \bar{J})(s)} \bar{C}_0^{-1} \frac{\ddot{\phi} W(q) + J(q-T)}{2\sqrt{L}} dq \right. \\ & - \frac{1}{\tilde{L}_0(\bar{W}, \bar{J})(s)} \bar{C}_0^{-1} \frac{\ddot{\phi} \bar{W}(q) + \bar{J}(q-T)}{2\sqrt{L}} dq \frac{\ddot{\phi}}{\phi} ds \\ & U_2 \mathbb{E} W_0 \int_{t_k}^{t_{k+1}} \frac{e^{m(s-T-t_k)}}{\left( \tilde{L}_0(W, J) \right)^2} \left| \frac{d\tilde{L}_0}{di} |W(s) + J(s-T) - \bar{W}(s) - \bar{J}(s-T)| ds \right| \\ & \mathbb{E} r_m^{(k)}(J, \bar{J}) e^{-mT} \frac{e^{m(t-t_k)} - 1}{m} + \left( r_m^{(k)}(W, \bar{W}) + e^{-mT} r_m^{(k)}(J, \bar{J}) \right) \\ & \cdot \frac{W_0}{\tilde{L}_0^2} \sup \left| \frac{d\tilde{L}_0}{di} \int_{t_k}^t e^{m(s-t_k)} ds + \frac{2\sqrt{L}}{\hat{L}_0} \hat{a} \left| r_n^{(0)} \right| n \right. \\ & \cdot \left| \int_{t_k}^t \frac{W_0 e^{m(s-t_k)} + J_0 e^{m(s-T-t_k)}}{2\sqrt{L}} \right|^{n-1} \\ & \cdot \left| \frac{W(s) + J(s-T)}{2\sqrt{L}} - \frac{\bar{W}(s) + \bar{J}(s-T)}{2\sqrt{L}} \right| ds \\ & + \left( r_m^{(k)}(W, \bar{W}) + e^{-mT} r_m^{(k)}(J, \bar{J}) \right) \frac{e^{m(t-t_k)} - 1}{2m\sqrt{L}} \frac{Z_0 J_0}{\hat{L}_0^2} \sup \left| \frac{d\tilde{L}_0}{di} \right| \end{aligned}$$



$$+ \frac{m}{n=1} \left| r_n^{(0)} \right| n i_0^n + Z_0 + \frac{2\sqrt{F_0}}{m_0 \sqrt{F_0 - j_0}} \frac{\dot{u} \ddot{u}}{\dot{u} \ddot{u}}.$$

Therefore

$$\begin{aligned} & \left| B_W^{(k)}(W, J)(t) - B_W^{(k)}(\bar{W}, \bar{J})(t) \right| \\ & \leq e^{m(t-t_k)} \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \bar{\hat{W}}, \bar{J}, \bar{\hat{J}})) e^{m(t-t_k)} \left\{ \frac{e^{-m\tau}}{m} + \frac{1+e^{-m\tau}}{m^2 \hat{L}_0} \right. \\ & \quad \cdot \left. \frac{\dot{e}}{\dot{e}} \frac{1}{\hat{L}_0} W_0 + \frac{m}{n=1} \left| r_n^{(0)} \right| i_0^n + Z_0 i_0 + \frac{H_0}{m} i_0 + \frac{\ddot{a}}{\ddot{a}} (n+1) \left| l_n^{(0)} \right| i_0^n \right. \\ & \quad + \frac{m}{n=1} \left| r_n^{(0)} \right| n i_0^{n-1} + Z_0 + \frac{2\sqrt{F_0}}{m_0 \sqrt{F_0 - j_0}} \frac{\dot{u} \ddot{u}}{\dot{u} \ddot{u}} \\ & \quad \left. \left. \varepsilon e^{m(t-t_k)} \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \bar{\hat{W}}, \bar{J}, \bar{\hat{J}})) \frac{1}{m} \left\{ e^{-m\tau} + \right. \right. \right. \\ & \quad \left. \left. e^{m_0} \frac{1+e^{-m\tau}}{m \hat{L}_0} \cdot \frac{\dot{e}}{\dot{e}} \frac{1}{\hat{L}_0} W_0 + \frac{m}{n=1} \left| r_n^{(0)} \right| (i_0)^n + Z_0 i_0 \right. \right. \\ & \quad \left. \left. + Z_0 i_0 + \frac{H_0}{m} i_0 + \frac{\ddot{a}}{\ddot{a}} (n+1) \left| l_n^{(0)} \right| (i_0)^n \right. \right. \\ & \quad \left. \left. + \frac{m}{n=1} \left| r_n^{(0)} \right| n (i_0)^{n-1} + Z_0 + \frac{2\sqrt{F_0}}{m_0 \sqrt{F_0 - j_0}} \frac{\dot{u} \ddot{u}}{\dot{u} \ddot{u}} \right. \right. \\ & \quad \left. \left. \left. \varepsilon e^{m(t-t_k)} K_W \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \bar{\hat{W}}, \bar{J}, \bar{\hat{J}})). \right. \right. \end{aligned}$$

It follows

$$\begin{aligned} & \left| B_W^{(k)}(W, \hat{W}, J, \hat{J})(t) - B_W^{(k)}(\bar{W}, \bar{\hat{W}}, \bar{J}, \bar{\hat{J}})(t) \right| \\ & \leq e^{m_0} K_W \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \bar{\hat{W}}, \bar{J}, \bar{\hat{J}})), \\ & \quad t \in [t_k, t_{k+1}] \quad (k = 0, 1, 2, \dots) \end{aligned}$$

and consequently

$$\begin{aligned} & r^{(k)}(B_W^{(k)}(W, \hat{W}, J, \hat{J}), B_W^{(k)}(\bar{W}, \bar{\hat{W}}, \bar{J}, \bar{\hat{J}})) \\ & \leq e^{m_0} K_W \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \bar{\hat{W}}, \bar{J}, \bar{\hat{J}})); \\ & \quad \hat{r}^{(k)}(B_W(W, \hat{W}, J, \hat{J}), B_W(\bar{W}, \bar{\hat{W}}, \bar{J}, \bar{\hat{J}})) \\ & \leq e^{m_0} K_W \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \bar{\hat{W}}, \bar{J}, \bar{\hat{J}})). \end{aligned}$$

Further on we have

$$\begin{aligned} & \left| B_J^{(k)}(W, J)(t) - B_J^{(k)}(\bar{W}, \bar{J})(t) \right| \\ & \leq \int_{t_k}^t |I(W, J)(s) - I(\bar{W}, \bar{J})(s)| ds \\ & \quad + \left| \int_{t_k}^{t_{k+1}} I(W, J)(s) ds - \int_{t_k}^{t_{k+1}} I(\bar{W}, \bar{J})(s) ds \right| = I_1 + I_2, \end{aligned}$$

We get

$$I_1 \leq \int_{t_k}^t \left| \dot{W}(s-T) - \dot{\bar{W}}(s-T) \right| ds$$

$$\begin{aligned} & + 2\sqrt{L} \left| \int_{t_k}^t \frac{\dot{W}(s-T)}{\dot{L}_1(W, J)(s)} - \frac{\dot{\bar{W}}(s-T)}{\dot{L}_1(\bar{W}, \bar{J})(s)} ds \right| \\ & + 2\sqrt{L} \left| \int_{t_k}^t \frac{R_1 \frac{\dot{W}(s-T) + J(s)}{2\sqrt{L}}}{\dot{L}_1(W, J)(s)} - \frac{R_1 \frac{\dot{\bar{W}}(s-T) + \bar{J}(s)}{2\sqrt{L}}}{\dot{L}_1(\bar{W}, \bar{J})(s)} ds \right| \\ & + Z_0 \left| \int_{t_k}^t \frac{W(s-T)}{\dot{L}_1(W, J)(s)} - \frac{\bar{W}(s-T)}{\dot{L}_1(\bar{W}, \bar{J})(s)} ds \right| \\ & + Z_0 \left| \int_{t_k}^t \frac{J(s)}{\dot{L}_1(W, J)(s)} - \frac{\bar{J}(s)}{\dot{L}_1(\bar{W}, \bar{J})(s)} ds \right| \\ & + 2\sqrt{L} \left| \int_{t_k}^t \frac{\bar{C}_1^{-1} \frac{\dot{W}(q-T) + J(q)}{2\sqrt{L}}}{\dot{L}_1(W, J)(s)} - \frac{\bar{C}_1^{-1} \frac{\dot{\bar{W}}(q-T) + \bar{J}(q)}{2\sqrt{L}}}{\dot{L}_1(\bar{W}, \bar{J})(s)} dq \right| ds \\ & \leq r_m^{(k)}(\hat{W}, \bar{\hat{W}}) e^{-m\tau} \frac{e^{m(t-t_k)} - 1}{m} \\ & + W_0 \left| \int_{t_k}^t \frac{1}{(\dot{L}_1(W, J))^2} \frac{d\dot{L}_1}{di} |W(s-T) + J(s) - \bar{W}(s-T) - \bar{J}(s)| ds \right| \\ & + 2\sqrt{L} \left| \int_{t_k}^t \frac{R_1 \frac{\dot{W}(s-T) + J(s)}{2\sqrt{L}}}{\dot{L}_1(W, J)(s)} - \frac{R_1 \frac{\dot{\bar{W}}(s-T) + \bar{J}(s)}{2\sqrt{L}}}{\dot{L}_1(\bar{W}, \bar{J})(s)} ds \right| \\ & + 2\sqrt{L} \left| \int_{t_k}^t \frac{R_1 \frac{\dot{\bar{W}}(s-T) + \bar{J}(s)}{2\sqrt{L}}}{\dot{L}_1(W, J)(s)} - \frac{R_1 \frac{\dot{\bar{W}}(s-T) + \bar{J}(s)}{2\sqrt{L}}}{\dot{L}_1(\bar{W}, \bar{J})(s)} ds \right| \\ & + Z_0 \left| \int_{t_k}^t \frac{W(s-T)}{\dot{L}_1(W, J)(t)} - \frac{\bar{W}(s-T)}{\dot{L}_1(W, J)(t)} ds \right| \\ & + Z_0 \left| \int_{t_k}^t \frac{\bar{W}(s-T)}{\dot{L}_1(W, J)(t)} - \frac{\bar{W}(s-T)}{\dot{L}_1(\bar{W}, \bar{J})(t)} ds \right| \\ & + Z_0 \left| \int_{t_k}^t \frac{J(s)}{\dot{L}_1(W, J)(s)} - \frac{\bar{J}(s)}{\dot{L}_1(W, J)(s)} ds \right| \\ & + Z_0 \left| \int_{t_k}^t \frac{\bar{J}(s)}{\dot{L}_1(W, J)(s)} - \frac{\bar{J}(s)}{\dot{L}_1(\bar{W}, \bar{J})(s)} ds \right| \\ & + 2\sqrt{L} \left| \int_{t_k}^t \frac{1}{\dot{L}_1(\bar{W}, \bar{J})(s)} \bar{C}_1^{-1} \frac{\dot{W}(q-T) + J(q)}{2\sqrt{L}} dq \right| \\ & - \frac{1}{\dot{L}_1(\bar{W}, \bar{J})(s)} \bar{C}_1^{-1} \frac{\dot{\bar{W}}(q-T) + \bar{J}(q)}{2\sqrt{L}} dq \right| ds \end{aligned}$$



$$\begin{aligned}
& + Z_0 \left| \frac{\partial}{\partial t} \left[ \frac{\bar{W}(s-T)}{\bar{L}_1(W, J)(t)} - \frac{\bar{W}(s-T)}{\bar{L}_1(\bar{W}, \bar{J})(t)} \right] \right| ds \\
& + Z_0 \left| \frac{\partial}{\partial t} \left[ \frac{J(s)}{\bar{L}_1(W, J)(s)} - \frac{\bar{J}(s)}{\bar{L}_1(\bar{W}, \bar{J})(s)} \right] \right| ds \\
& + Z_0 \left| \frac{\partial}{\partial t} \left[ \frac{\bar{J}(s)}{\bar{L}_1(W, J)(s)} - \frac{\bar{J}(s)}{\bar{L}_1(\bar{W}, \bar{J})(s)} \right] \right| ds \\
& + 2\sqrt{L} \left| \frac{\partial}{\partial t} \left[ \frac{1}{\bar{L}_1(W, J)(s)} \bar{C}_1^{-1} \frac{\partial}{\partial t} \frac{W(q-T) + J(q)}{2\sqrt{L}} \right] \right| ds \\
& - \frac{1}{\bar{L}_1(\bar{W}, \bar{J})(s)} \bar{C}_1^{-1} \frac{\partial}{\partial t} \frac{W(q-T) + J(q)}{2\sqrt{L}} \left| \frac{\partial}{\partial t} \right| ds \\
& + 2\sqrt{L} \left| \frac{\partial}{\partial t} \left[ \frac{1}{\bar{L}_1(\bar{W}, \bar{J})(s)} \bar{C}_1^{-1} \frac{\partial}{\partial t} \frac{W(q-T) + J(q)}{2\sqrt{L}} \right] \right| ds \\
& - \frac{1}{\bar{L}_1(\bar{W}, \bar{J})(s)} \bar{C}_1^{-1} \frac{\partial}{\partial t} \frac{\bar{W}(q-T) + \bar{J}(q)}{2\sqrt{L}} \left| \frac{\partial}{\partial t} \right| ds \\
& \leq e^{m(t-t_k)} \hat{r}_m^{(k)}((W, \bar{W}, J, \bar{J}), (\bar{W}, \bar{W}, \bar{J}, \bar{J})) \frac{e^{m_0} - 1}{m} \frac{1}{\hat{m}_1} + e^{-m\tau} \\
& \cdot \left\{ \frac{1}{\hat{L}_1} \frac{\partial}{\partial t} \frac{\partial}{\partial t} W_0 + \frac{m}{n=1} \left| r_n^{(1)} \right| i_0^n + Z_0 i_0 + \frac{H_1}{m} i_0 \frac{\partial}{\partial t} \frac{\partial}{\partial t} (n+1) \left| l_n^{(1)} \right| i_0^n \right. \\
& \quad \left. + \frac{m}{n=1} \left| r_n^{(1)} \right| n i_0^{n-1} + Z_0 + \frac{2\sqrt{F_1}}{m_1 \sqrt{F_1 - j_0}} \frac{\partial}{\partial t} \frac{\partial}{\partial t} \right\} \\
& \quad I_2 \leq \left| \frac{\partial}{\partial t} (W, J)(s) ds - \frac{\partial}{\partial t} (\bar{W}, \bar{J})(s) ds \right| \leq \\
& \leq \hat{r}_m^{(k)}((W, \bar{W}, J, \bar{J}), (\bar{W}, \bar{W}, \bar{J}, \bar{J})) \frac{e^{m_0} - 1}{m} e^{-m\tau} + \frac{1 + e^{-m\tau}}{\hat{m}_1} \\
& \cdot \left\{ \frac{1}{\hat{L}_1} \frac{\partial}{\partial t} \frac{\partial}{\partial t} W_0 + \frac{m}{n=1} \left| r_n^{(1)} \right| i_0^n + Z_0 i_0 + \frac{H_1}{m} i_0 \frac{\partial}{\partial t} \frac{\partial}{\partial t} (n+1) \left| l_n^{(1)} \right| i_0^n \right. \\
& \quad \left. + \frac{m}{n=1} \left| r_n^{(1)} \right| n i_0^{n-1} + Z_0 + \frac{2\sqrt{F_1}}{m_1 \sqrt{F_1 - j_0}} \frac{\partial}{\partial t} \frac{\partial}{\partial t} \right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \left| B_J^{(k)}(W, J)(t) - B_J^{(k)}(\bar{W}, \bar{J})(t) \right| \leq \\
& \leq e^{m(t-t_k)} \hat{r}_m^{(k)}((W, \bar{W}, J, \bar{J}), (\bar{W}, \bar{W}, \bar{J}, \bar{J})) \frac{1}{m} \\
& \cdot \left\{ e^{-m\tau} + e^{m_0} \frac{1 + e^{-m\tau}}{\hat{m}_1} \frac{1}{\hat{L}_1} \frac{\partial}{\partial t} \frac{\partial}{\partial t} W_0 + \frac{m}{n=1} \left| r_n^{(1)} \right| i_0^n + (Z_0 + \frac{H_1}{m}) i_0 \frac{\partial}{\partial t} \frac{\partial}{\partial t} \right. \\
& \quad \left. + \frac{m}{n=1} \left| r_n^{(1)} \right| n i_0^{n-1} + \frac{2\sqrt{F_1}}{m_1 \sqrt{F_1 - j_0}} \frac{\partial}{\partial t} \frac{\partial}{\partial t} \right\}
\end{aligned}$$

$$\leq e^{m(t-t_k)} K_J \hat{r}_m^{(k)}((W, \bar{W}, J, \bar{J}), (\bar{W}, \bar{W}, \bar{J}, \bar{J})).$$

It follows

$$\begin{aligned}
& \left| B_J^{(k)}(W, \bar{W}, J, \bar{J})(t) - B_J^{(k)}(\bar{W}, \bar{W}, \bar{J}, \bar{J})(t) \right| \\
& \leq e^{m_0} K_J \hat{r}_m^{(k)}((W, \bar{W}, J, \bar{J}), (\bar{W}, \bar{W}, \bar{J}, \bar{J})), \\
& t \in [t_k, t_{k+1}] \quad (k = 0, 1, 2, \dots) \text{ and consequently} \\
& r^{(k)}(B_J^{(k)}(W, \bar{W}, J, \bar{J}), B_J^{(k)}(\bar{W}, \bar{W}, \bar{J}, \bar{J})) \\
& \leq e^{m_0} K_J \hat{r}_m^{(k)}((W, \bar{W}, J, \bar{J}), (\bar{W}, \bar{W}, \bar{J}, \bar{J})), \\
& \hat{r}^{(k)}(B_J(W, \bar{W}, J, \bar{J}), B_J(\bar{W}, \bar{W}, \bar{J}, \bar{J})) \\
& \leq e^{m_0} K_J \hat{r}_m^{(k)}((W, \bar{W}, J, \bar{J}), (\bar{W}, \bar{W}, \bar{J}, \bar{J})).
\end{aligned}$$

Finally we have to obtain an estimate of the derivatives for  $t \in [t_k, t_{k+1}]$ :

$$\begin{aligned}
& \left| \frac{\partial}{\partial t} (W, J)(t) - \frac{\partial}{\partial t} (\bar{W}, \bar{J})(t) \right| \leq \left| U(W, J)(t) - U(\bar{W}, \bar{J})(t) \right| \\
& + \frac{1}{t_{k+1} - t_k} \left| \frac{\partial}{\partial t} (W, J)(s) ds - \frac{\partial}{\partial t} (\bar{W}, \bar{J})(s) ds \right| \leq \vartheta_1 + \vartheta_2.
\end{aligned}$$

We get

$$\begin{aligned}
& \vartheta_1 \leq \left| \frac{dJ(t-T)}{dt} - \frac{d\bar{J}(t-T)}{dt} \right| \\
& + 2\sqrt{L} \left| \frac{R_0 \frac{\partial}{\partial t} \frac{\partial}{\partial t} (W(t) + J(t-T))}{\tilde{L}_0(W, J)(t)} - \frac{R_0 \frac{\partial}{\partial t} \frac{\partial}{\partial t} (\bar{W}(t) + \bar{J}(t-T))}{\tilde{L}_0(\bar{W}, \bar{J})(t)} \right| \\
& + Z_0 \left| \frac{W(t)}{\tilde{L}_0(W, J)(t)} - \frac{\bar{W}(t)}{\tilde{L}_0(\bar{W}, \bar{J})(t)} \right| \\
& + \frac{Z_0}{\tilde{L}_0} \left| \frac{J(t-T)}{\tilde{L}_0(W, J)(t)} - \frac{\bar{J}(t-T)}{\tilde{L}_0(\bar{W}, \bar{J})(t)} \right| \\
& + 2\sqrt{L} \left| \frac{1}{\tilde{L}_0(W, J)(t)} \bar{C}_0^{-1} \frac{\partial}{\partial t} \frac{\partial}{\partial t} \frac{W(q) + J(q-T)}{2\sqrt{L}} \right| ds \\
& - \frac{1}{\tilde{L}_0(\bar{W}, \bar{J})(t)} \bar{C}_0^{-1} \frac{\partial}{\partial t} \frac{\partial}{\partial t} \frac{\bar{W}(q) + \bar{J}(q-T)}{2\sqrt{L}} \left| \frac{\partial}{\partial t} \right| ds \\
& \leq r_m^{(k)}(\bar{J}, \bar{J}) e^{-m\tau} + \left( r_m^{(k)}(W, \bar{W}) + e^{-m\tau} r_m^{(k)}(J, \bar{J}) \right) \\
& \cdot \frac{W_0}{\hat{L}_0^2} \sup \left| \frac{d\bar{L}_0}{di} \right| + \frac{2\sqrt{L}}{\hat{L}_0} \frac{m}{n=1} \left| r_n^{(0)} \right| n \left| \frac{W_0 e^{m(t-t_k)} + J_0 e^{m(t-T-t_k)}}{2\sqrt{L}} \right|^{n-1} \\
& \cdot \left| \frac{W(t) + J(t-T)}{2\sqrt{L}} - \frac{\bar{W}(t) + \bar{J}(t-T)}{2\sqrt{L}} \right| \\
& + 2\sqrt{L} \frac{m}{n=1} \left| r_n^{(1)} \right| \frac{1}{\hat{L}_0^2} \left| \frac{d\bar{L}_0}{di} \right| \left| \frac{W_0 e^{m(t-t_k)} + J_0 e^{m(t-T-t_k)}}{2\sqrt{L}} \right|^{n-1} \\
& \cdot \left| \frac{W(t) + J(t-T)}{2\sqrt{L}} - \frac{\bar{W}(t) + \bar{J}(t-T)}{2\sqrt{L}} \right| + \frac{Z_0}{\hat{L}_0} |W(t) - \bar{W}(t)|
\end{aligned}$$





$$\begin{aligned}
& + \frac{(r_m^{(k)}(W, \bar{W})e^{-\mu t} + r_m^{(k)}(J, \bar{J}))}{2\sqrt{L}} \frac{e^{\mu(t-t_k)} - 1}{m} \frac{H_1}{\hat{L}_0^2} (W_0 + J_0) \sup \left| \frac{d\hat{L}_1}{dt} \right| \\
& + \frac{(r_m^{(k)}(W, \bar{W})e^{-\mu t} + r_m^{(k)}(J, \bar{J}))}{\hat{L}_1} \frac{e^{\mu(t-t_k)} - 1}{m} \frac{2\sqrt{F_1}}{c_1\sqrt{F_1 - j_0}} \\
& \mathbb{E} e^{\mu(t-t_k)} \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \hat{\bar{W}}, \bar{J}, \hat{\bar{J}})) \Big| e^{-\mu t} + \frac{1+e^{-\mu t}}{\hat{m}\hat{L}_1} \\
& \cdot \frac{\hat{e}}{\hat{e}} \frac{1}{\hat{L}_1} \frac{\hat{a}}{\hat{a}} W_0 + \hat{a} \Big| r_n^{(1)} \Big| \frac{\hat{a}}{\hat{e}} \frac{W_0 + J_0}{2\sqrt{L}} \frac{\hat{o}}{\hat{o}} + Z_0 \frac{W_0 + J_0}{2\sqrt{L}} + \frac{W_0 + J_0}{2\sqrt{L}} \frac{H_1}{m} \frac{\hat{o}}{\hat{o}} \\
& \cdot \hat{a} \Big| (n+1) \Big| l_n^{(1)} \Big| \frac{\hat{a}}{\hat{e}} \frac{W_0 + J_0}{2\sqrt{L}} \frac{\hat{o}}{\hat{o}} + \hat{a} \Big| r_n^{(1)} \Big| n \frac{\hat{a}}{\hat{e}} \frac{W_0 + J_0}{2\sqrt{L}} \frac{\hat{o}}{\hat{o}} \\
& + Z_0 + \frac{2\sqrt{F_1}}{m_1\sqrt{F_1 - j_0}} \frac{\hat{u}}{\hat{u}} \mathbb{E} e^{\mu(t-t_k)} \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \hat{\bar{W}}, \bar{J}, \hat{\bar{J}})) \\
& \cdot \Big| e^{-\mu t} + \frac{1+e^{-\mu t}}{\hat{m}\hat{L}_1} \frac{\hat{e}}{\hat{e}} \frac{1}{\hat{L}_1} \frac{\hat{a}}{\hat{a}} W_0 + \hat{a} \Big| r_n^{(1)} \Big| i_0^n + Z_0 i_0 + i_0 \frac{H_1}{m} \frac{\hat{o}}{\hat{o}} \\
& \cdot \hat{a} \Big| (n+1) \Big| l_n^{(1)} \Big| i_0^n + \hat{a} \Big| r_n^{(1)} \Big| n i_0^{n-1} + Z_0 + \frac{2\sqrt{F_1}}{m_1\sqrt{F_1 - j_0}} \frac{\hat{u}}{\hat{u}} \\
& \mathbb{E} e^{\mu(t-t_k)} \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \hat{\bar{W}}, \bar{J}, \hat{\bar{J}})) \frac{e^{\mu t} - 1}{m_0} \\
& \cdot \Big| \frac{1+e^{-\mu t}}{\hat{m}\hat{L}_1} \frac{\hat{e}}{\hat{e}} \frac{1}{\hat{L}_1} \frac{\hat{a}}{\hat{a}} W_0 + \hat{a} \Big| r_n^{(1)} \Big| i_0^n + Z_0 i_0 + i_0 \frac{H_1}{m} \frac{\hat{o}}{\hat{o}} \\
& \cdot \hat{a} \Big| (n+1) \Big| l_n^{(1)} \Big| i_0^n + \hat{a} \Big| r_n^{(1)} \Big| n i_0^{n-1} + Z_0 + \frac{2\sqrt{F_1}}{m_1\sqrt{F_1 - j_0}} \frac{\hat{u}}{\hat{u}}.
\end{aligned}$$

Then

$$\begin{aligned}
& \left| \hat{B}_J^{(k)}(W, J)(t) - \hat{B}_J^{(k)}(\bar{W}, \bar{J})(t) \right| \mathbb{E} \\
& \mathbb{E} e^{\mu(t-t_k)} r_m^{(k)}((W, J), (\bar{W}, \bar{J})) e^{\mu t} + \frac{e^{\mu t} - 1}{\hat{c}} \frac{e^{\mu t} - 1}{m_0} \frac{\hat{o}}{\hat{o}} \\
& \cdot \frac{1+e^{-\mu t}}{\hat{m}\hat{L}_1} \frac{\hat{e}}{\hat{e}} \frac{1}{\hat{L}_1} \frac{\hat{a}}{\hat{a}} W_0 + \hat{a} \Big| r_n^{(1)} \Big| i_0^n + Z_0 i_0 + \frac{H_1}{m} i_0 \frac{\hat{o}}{\hat{o}} \\
& \cdot \hat{a} \Big| (n+1) \Big| l_n^{(1)} \Big| i_0^n + \hat{a} \Big| r_n^{(1)} \Big| n i_0^{n-1} + Z_0 + \frac{2\sqrt{F_1}}{m_1\sqrt{F_1 - j_0}} \frac{\hat{u}}{\hat{u}}
\end{aligned}$$

$$\circ e^{\mu(t-t_k)} \hat{K}_J \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \hat{\bar{W}}, \bar{J}, \hat{\bar{J}})).$$

Then

$$\begin{aligned}
& r_m^{(k)}(\hat{B}_J^{(k)}(W, J), \hat{B}_J^{(k)}(\bar{W}, \bar{J})) \\
& \mathbb{E} \hat{K}_J \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \hat{\bar{W}}, \bar{J}, \hat{\bar{J}})).
\end{aligned}$$

It follows

$$\begin{aligned}
& \hat{r}_m^{(k)}(\hat{B}_J^{(k)}(W, J), \hat{B}_J^{(k)}(\bar{W}, \bar{J})) \\
& \mathbb{E} \hat{K}_J \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \hat{\bar{W}}, \bar{J}, \hat{\bar{J}})).
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
& \hat{r}_m^{(k)}(B(W, J), \hat{B}(W, J), B(\bar{W}, \bar{J}), \hat{B}(\bar{W}, \bar{J})) \\
& \mathbb{E} K \hat{r}_m^{(k)}((W, \hat{W}, J, \hat{J}), (\bar{W}, \hat{\bar{W}}, \bar{J}, \hat{\bar{J}}))
\end{aligned}$$

where  $K = \max\{e^{\mu_0} K_W, e^{\mu_0} K_J, \hat{K}_W, \hat{K}_J\}$  might be chosen smaller than 1.

Therefore  $B$  is contractive operator and has a unique fixed point in  $M$  (cf.[3], Theorems 2.2.2 and 2.2.3). It is an oscillatory solution of (8).

Thus Theorem 3.1 is thus proved.

## 4. Numerical Example

Finally we demonstrate of how to apply the above theorem to engineering problems. We collect all inequalities implying an existence-uniqueness of oscillatory solution:

$$e^{\mu_0} \frac{W_0 + J_0}{2\sqrt{L}} \mathbb{E} i_0; \quad H_p = \frac{2F_p - f_0}{2c_p \sqrt{F_p} \sqrt{F_p - f_0}} \quad (p = 0, 1);$$

$$J_0 e^{-(\mu(R/L))T} + e^{\mu_0} \frac{4L\sqrt{L}}{\hat{L}_0 R} \frac{e^{(RT_0)/L} - 1}{2} + \text{sh}(RT_0/L) \frac{\hat{o}}{\hat{o}}$$

$$+ Z_0 i_0 + \frac{H_0 i_0 L}{R} (e^{(RT_0)/L} - 1) \frac{\hat{u}}{\hat{u}} \mathbb{E} W_0;$$

$$K_W =$$

$$\frac{e^{\mu_0}}{m} \Big| e^{-\mu t} + e^{\mu_0} \frac{1+e^{-\mu t}}{\hat{m}\hat{L}_0} \frac{\hat{e}}{\hat{e}} \frac{1}{\hat{L}_0} \frac{\hat{a}}{\hat{a}} W_0 + \hat{a} \Big| r_n^{(0)} \Big| i_0^n + Z_0 i_0 + \frac{H_0}{m} i_0$$

$$\cdot \hat{a} \Big| (n+1) \Big| l_n^{(0)} \Big| i_0^n + \hat{a} \Big| r_n^{(0)} \Big| n i_0^{n-1} + Z_0 + \frac{2\sqrt{F_0}}{m_0\sqrt{F_0 - j_0}} \frac{\hat{u}}{\hat{u}} < 1;$$

$$K_J =$$

$$\frac{e^{\mu_0}}{m} \Big| e^{-\mu t} + e^{\mu_0} \frac{1+e^{-\mu t}}{\hat{m}\hat{L}_1} \frac{\hat{e}}{\hat{e}} \frac{1}{\hat{L}_1} \frac{\hat{a}}{\hat{a}} W_0 + \hat{a} \Big| r_n^{(1)} \Big| i_0^n + Z_0 i_0 + \frac{H_1}{m} i_0 \frac{\hat{o}}{\hat{o}}$$

$$\cdot \hat{a} \Big| (n+1) \Big| l_n^{(1)} \Big| i_0^n + \hat{a} \Big| r_n^{(1)} \Big| n i_0^{n-1} + Z_0 + \frac{2\sqrt{F_1}}{m_1\sqrt{F_1 - j_0}} \frac{\hat{u}}{\hat{u}} < 1;$$

$$\hat{K}_W =$$

$$e^{-\mu t} + \frac{e^{\mu_0} - 1}{\hat{c}} \frac{1+e^{-\mu t}}{\hat{m}\hat{L}_0} \frac{\hat{e}}{\hat{e}} \frac{1}{\hat{L}_0} \frac{\hat{a}}{\hat{a}} W_0 + \hat{a} \Big| r_n^{(0)} \Big| i_0^n + Z_0 i_0 + \frac{H_0}{m} i_0 \frac{\hat{o}}{\hat{o}}$$

$$\cdot \hat{a} \Big| (n+1) \Big| l_n^{(0)} \Big| i_0^n + \hat{a} \Big| r_n^{(0)} \Big| n i_0^{n-1} + Z_0 + \frac{2\sqrt{F_0}}{m_0\sqrt{F_0 - j_0}} \frac{\hat{u}}{\hat{u}} < 1;$$

$$\hat{K}_J =$$

$$e^{-\mu t} + \frac{e^{\mu_0} - 1}{\hat{c}} \frac{1+e^{-\mu t}}{\hat{m}\hat{L}_1} \frac{\hat{e}}{\hat{e}} \frac{1}{\hat{L}_1} \frac{\hat{a}}{\hat{a}} W_0 + \hat{a} \Big| r_n^{(1)} \Big| i_0^n + Z_0 i_0 + \frac{H_1}{m} i_0 \frac{\hat{o}}{\hat{o}}$$

$$\cdot \hat{a} \Big| (n+1) \Big| l_n^{(1)} \Big| i_0^n + \hat{a} \Big| r_n^{(1)} \Big| n i_0^{n-1} + Z_0 + \frac{2\sqrt{F_1}}{m_1\sqrt{F_1 - j_0}} \frac{\hat{u}}{\hat{u}} < 1.$$

For a transmission line with length  $L = 1000 \text{ m}$ ;  
 $L = 0,45 \text{ nH/m}$ ;  $C = 80 \text{ pF/m}$ ;  $R = (r_c L) / S \gg 3 \text{ W}$ ;  
 $v = 1/\sqrt{LC} = 1,66 \cdot 10^8$ ;  $Z_0 = 75 \text{ W}$ ;  $T = L\sqrt{LC} = 6 \cdot 10^{-6}$ .

For waves with  $l_0 = (1/6)10^{-3} \text{ m}$  we have

$$f_0 = 1/(l_0\sqrt{LC}) = 10^{12} \text{ Hz}; T_0 = 1/f_0 = 10^{-12}.$$

Let us choose  $m = 10^{12}$ . Then  $mT_0 = m_0 = 1$ , and  
 $T = 6 \cdot 10^6 T_0$ . Consequently  $e^{-mT} = e^{-6 \cdot 10^6} \gg 0$ ;

$$\begin{aligned} R/L &\gg 6,6 \cdot 10^6; L/R = 1/(6,6 \cdot 10^6); RL/Z_0 = RT/L \gg 40; \\ (RT_0/L) &\gg 6,6 \cdot 10^{-6} \gg e^{(RT_0)/L} - 1 \gg (RT_0)/L \gg 6,6 \cdot 10^{-6}; \\ 2e^{(RT_0)/L} - 1 &\gg 13,2 \cdot 10^{-6}; L(e^{(RT_0)/L} - 1)/R = 10^{-12}; \\ L(2e^{(RT_0)/L} - 1)/R &= 2 \cdot 10^{-12}; \text{sh}((RT_0)/L) \gg 0; \\ e^{-(m(R/L)T)} &\gg e^{-6 \cdot 10^6}. \end{aligned}$$

We choose resistive elements with  $V$ - $I$  characteristics  
 $R_0(i) = R_1(i) = 0,028u - 0,125u^3$  i.e.  $r_1 = 0,028, r_2 = 0, r_3 = 0,125$   
and inductive element with  $L_0(i) = L_1(i) = 3i - (1/12)i^3$ .

Then  $\tilde{L}_0(i) = i(dL_0(i)/di) + L_0(i) = 6i - (1/3)i^3$  and

$$\tilde{a}_{n=1}^m(n+1) \left| l_n^{(0)} \right| = 2,3 + 4/12 = 19/3.$$

For  $i_0 = 1$  one obtains

$$\tilde{L}_0(i) = 6i - (1/3)i^3 > 6 - (1/3) = 17/3 = \hat{L} \text{ and consequently } 1/\hat{L} = 1/\tilde{L}_0 = 1/\hat{L}_1 = 3/17; \hat{m} = 5,66 \cdot 10^{12}; (2\sqrt{L})/\hat{L}_0 \gg 2,4 \cdot 10^{-4}.$$

Let us take

$$C_0(u) = C_1(u) = c/\sqrt{1 - (u/F_0)} = c\sqrt{F_0}/\sqrt{F_0 - u}, \text{ where}$$

$$h = 2; c = c_0 = c_1 = 50 \text{ pF} = 5 \cdot 10^{-11} \text{ F}; F = 0,9 \text{ V};$$

$$W_0 e^{-6 \cdot 10^6} + 6,5 \cdot 10^{-16} \frac{\partial}{\partial t} W_0 e^{-6 \cdot 10^6} + 76 \frac{\partial^2}{\partial t^2} W_0 \gg 0,037;$$

$$\begin{aligned} H_0 = H_1 &= (2F_0 - f_0)/(2c_0\sqrt{F_0}\sqrt{F_0 - f_0}) = 2 \cdot 10^{10}; \\ (2\sqrt{F_0})/(m_0\sqrt{F_0 - f_0}) &= 0,0424. \end{aligned}$$

We can take  $W_0 = J_0 = 0,5 \cdot 10^{-3}$ . Then the above inequalities become

$$e(W_0 + J_0)/(2\sqrt{L}) \ll 1; W_0 + J_0 \ll \frac{2}{e} \sqrt{0,45 \cdot 10^{-6}} = 0,493;$$

$$J_0 e^{-6 \cdot 10^6} + 6,5 \cdot 10^{-16} \frac{\partial}{\partial t} W_0 e^{-6 \cdot 10^6} + 76 \frac{\partial^2}{\partial t^2} W_0 \gg 0;$$

$$W_0 e^{-6 \cdot 10^6} + 6,5 \cdot 10^{-16} \frac{\partial}{\partial t} W_0 e^{-6 \cdot 10^6} + 76 \frac{\partial^2}{\partial t^2} J_0 \gg 0;$$

$$K_W = K_J = \frac{208,59}{10^{24}}; \hat{K}_W = \hat{K}_J = 8 \cdot 10^{-11}.$$

Let the initial approximation be

$$W^{(0)}(t) = \begin{cases} W_0 \sin w_0 t, & t \in [0, T] \\ 0, & t \in [T, \infty) \end{cases}; J^{(0)}(t) = \begin{cases} J_0 \sin w_0 t, & t \in [0, T] \\ 0, & t \in [T, T+T_0] \end{cases}$$

$$w_0 = \frac{2\pi}{T_0}, E_0(t) = E_1(t) = E_0 \sin w_0 t \text{ and } t_{k+1} - t_k = T_0.$$

Then we have

$$W^{(n+1)}(t) = B_u(W^{(n)}, J^{(n)}), J^{(n+1)}(t) = B_i(W^{(n)}, J^{(n)}) \\ (n = 0, 1, 2, \dots).$$

Since

$$\left| 2\sqrt{L} \frac{\partial}{\partial t} \frac{R_1(W(s-T) + J(s))/(2\sqrt{L})}{\tilde{L}_1(s-T)} ds \right| \\ \ll \frac{2\sqrt{L}}{\hat{L}_1} \left( |r_1^{(1)}| + |r_3^{(1)}| \right) \frac{\partial}{\partial t} e^{m(s-t_k)} ds \ll \frac{1,6\sqrt{0,45 \cdot 10^{-6}}}{17 \cdot 10^{12}} \gg 0;$$

$$2\sqrt{L} \left| \frac{\partial}{\partial t} \frac{1}{\tilde{L}_0(s-T)} \frac{\partial}{\partial t} \frac{J_0}{2\sqrt{L}} \sin w_0 q \frac{\partial^2}{\partial t^2} ds \right|$$

$$\ll W_0 \frac{2\sqrt{L}}{\hat{L}_0} \frac{\partial}{\partial t} e^{m(s-t_k)} ds \ll 10^{-17} \frac{3(e-1)}{17} \gg 0;$$

$$2\sqrt{L} \left| \frac{\partial}{\partial t} \frac{1}{\tilde{L}_1(s-T)} \frac{\partial}{\partial t} \frac{W(q-T) + J(q)}{2\sqrt{L}} \frac{\partial^2}{\partial t^2} ds \right| \\ \gg \frac{3 \cdot 10^{-6} \sqrt{0,45}}{17} \frac{e-1}{10^{12}} \gg 0;$$

$$\left| \frac{\partial}{\partial t} \frac{Z_0 J_0 \sin w_0 s}{\tilde{L}_0(s-T)} ds \right| \ll \frac{Z_0 J_0}{\hat{L}_0} \frac{\partial}{\partial t} e^{m(s-t_k)} ds \gg \frac{5,7}{10^{15}} \gg 0;$$

$$\left| \frac{\partial}{\partial t} \frac{Z_0 W_0 \sin w_0 s}{\tilde{L}_1(s-T)} ds \right| \gg \frac{0,04}{10^{18}} \gg 0;$$

then the first approximations become

$$W^{(1)}(t) = B_u^{(k)}(W^{(0)}, J^{(0)})(t) = J_0 \sin w_0 t$$

$$+ 2E_0 \sqrt{L} \frac{\partial}{\partial t} \frac{\sin w_0 s}{\tilde{L}_0(s-T)} ds - \frac{t-t_k}{T_0} \frac{\partial}{\partial t} \frac{\sin w_0 s}{\tilde{L}_0(s-T)} ds;$$

$$J^{(1)}(t) = B_i(W^{(0)}, J^{(0)})(t) = -W_0 \sin w_0 t$$

$$- 2E_1 \sqrt{L} \frac{\partial}{\partial t} \frac{\sin w_0 s}{\tilde{L}_1(s-T)} ds - \frac{t-t_k}{T_0} \frac{\partial}{\partial t} \frac{\sin w_0 s}{\tilde{L}_1(s-T)} ds;$$

$$\hat{W}^{(1)}(t) = w_0 J_0 \cos w_0 t + \frac{2E_0 \sqrt{L} \sin w_0 t}{\tilde{L}_0(t-T)}$$

$$- \frac{2E_0 \sqrt{L}}{T_0} \frac{\partial}{\partial t} \frac{\sin w_0 s}{\tilde{L}_0(s-T)} ds;$$

$$\hat{J}^{(1)}(t) = -w_0 W_0 \cos w_0 t + \frac{2E_1 \sqrt{L} \sin w_0 t}{\tilde{L}_1(t-T)}$$

$$- \frac{2E_1 \sqrt{L}}{T_0} \frac{\partial}{\partial t} \frac{\sin w_0 s}{\tilde{L}_1(s-T)} ds.$$

Consequently

$$|W^{(1)}(t) - W^{(0)}(t)| \ll$$

$$\ll J_0 + W_0 + \frac{4E_0 \sqrt{L}}{\hat{L}_0} \frac{e^{mT_0} - 1}{m} \gg J_0 + W_0$$

and

$$\left| J^{(1)}(t) - J^{(0)}(t) \right| \leq J_0 + W_0 + \frac{4E_1\sqrt{L}}{\hat{L}_1} \frac{e^{m_0} - 1}{m} \gg J_0 + W_0.$$

For the derivative we have

$$\begin{aligned} & \left| \dot{W}^{(1)}(t) - \dot{W}^{(0)}(t) \right| \leq \left| \dot{W}_0 J_0 \cos w_0 t + \frac{2E_0\sqrt{L} \sin w_0 t}{\tilde{L}_0(t-T)} \right. \\ & \quad \left. - \frac{2E_0\sqrt{L}}{T_0} \dot{\tilde{L}}_0 \frac{\sin w_0 s}{\tilde{L}_0(s-T)} ds - w_0 W_0 \cos w_0 s \right| \\ & \leq \dot{W}_0 (W_0 + J_0) + \frac{2E_0\sqrt{L}}{\tilde{L}_0} + \frac{2E_0\sqrt{L}}{\tilde{L}_0} \frac{e^{m_0} - 1}{m} \gg p \cdot 10^9 + 0,65. \end{aligned}$$

In the same way we can obtain estimates for the second component of the operator  $B$

$$\begin{aligned} & \left| \dot{J}^{(1)}(t) - \dot{J}^{(0)}(t) \right| \leq \left| \dot{J}_0 W_0 \cos w_0 t + \frac{2E_1\sqrt{L} \sin w_0 t}{\tilde{L}_1(t-T)} \right. \\ & \quad \left. - \frac{2E_1\sqrt{L}}{T_0} \dot{\tilde{L}}_1 \frac{\sin w_0 s}{\tilde{L}_1(s-T)} ds - w_0 J_0 \cos w_0 s \right| \\ & \leq \dot{J}_0 (W_0 + J_0) + \frac{2E_1\sqrt{L}}{\tilde{L}_1} + \frac{2E_1\sqrt{L}}{T_0\tilde{L}_1} \frac{e^{m_0} - 1}{m} \gg p \cdot 10^9 + 0,65. \end{aligned}$$

Consequently  $r_m^{(k)}((W, J), (\bar{W}, \bar{J})) \leq p \cdot 10^9 + 0,65$  and then

$$\hat{r}_m^{(k)}((W^{(n+1)}, J^{(n+1)}), (W^{(n)}, J^{(n)})) \leq \frac{(8 \cdot 10^{-11})^n}{1 - 8 \cdot 10^{-11}} (p \cdot 10^9 + 0,65) \\ (n = 0, 1, \dots).$$

## 5. Conclusions

• We consider transmission lines taking into account the losses. This means there is attenuation in time of the signals. This natural physical fact is confirmed by the mathematical method we apply. Namely, the transformation (we have used to reduce the mixed problem for hyperbolic system to a problem for neutral system on the boundary) contains an exponential function  $e^{-(R/L)t}$  which implies that signals (current and voltage) vanish exponentially. It reminds us that natural global solutions are not periodic ones. That is why we formulate the problem of existence-uniqueness of an oscillatory solution.

• In order to prove an existence-uniqueness theorem we introduce an operator (unknown in the literature up to now) whose fixed points are oscillatory solution of the problem stated.

• It turns out that the space of oscillating functions does not form a metric space but a uniform one. This requires applying fixed point theorems of operators acting on uniform spaces.

• We would like to point out that by means of this fixed point method we solve nonlinear equations with various nonlinearities as polynomial, exponential and transcendental ones.

• By virtue of the theorems obtained in this paper we show

that attenuating oscillating modes are natural for the lossy transmission lines terminated by such configuration of the nonlinear loads.

• The numerical example demonstrates a frame of applicability of the theory exposed (for instance to design of circuits) and shows that the method could be applied checking few simple inequalities between the basic specific parameter of the lines and loads.

• Finally we note that a lot of papers have been done where numerical (or other) methods are applied without uniqueness is assured. Then it is not clear to which solution is approaching. Our fixed point method guarantees a uniqueness of solution.

• The calculation of the successive approximations and the estimations of some terms (leading to their disregarding) simplify the calculation of the next approximations. It is extremely important for any program implementing the method.

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