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On a Long-Wave Instability of Steady-State Plane-Parallel Flows of an Ideal Fluid with Free Boundary in the Gravity Field

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Abstract We consider the problem of linear stability of steady-state plane-parallel flows of inviscid incompressible fluid of uniform density with free surface in the gravity field. Using the method of coupling the integrals of motion, we prove that sufficient conditions for the stability of these flows to small plane long-wave perturbations are absent. We construct an analytic example of a steady-state plane-parallel flow with small plane long-wave perturbations superimposed as normal modes

Keywords Ideal Fluid, Free Boundary, Gravity Field, Steady-State Plane-Parallel Flows, Stability, Small Plane Long-Wave Perturbations, Condition For Hyperbolicity, Analytic Solutions, Instability

1. Introduction

The wave motions in fluids undoubtedly belong to those natural phenomena which the human civilization has been dealing with essentially throughout its entire history. Thus, it is quite natural that people have constantly tried, and keep trying, to learn how to use waves in fluids beneficially. However, this requires a thorough understanding of wave motions in fluids and their properties.

It is known that mathematical modelling is one of the main methods for studying waves in fluids. Unfortunately, often mathematical models of wave motion in fluids turn out quite complicated, and their analytic treatment is difficult. Then we can simplify the mathematical models using the long-wave approximation[1].

Mathematical models of long-wave fluid motions are distinguished by a combination of relative simplicity and acceptable accounting for the physical characteristics of interest. Nevertheless, while applying mathematical models of long waves in a fluid, we should pay particular attention to the adequacy of scientific results to the fluid wave motions being described.

In this article we study precisely the problem of adequacy of mathematical modelling of waves in fluids on an example of a basic mathematical model of long-wave fluid motions: the model of propagation of long waves on the free boundary of a horizontal layer of a whirling ideal incompressible fluid of uniform density in the gravity field [2].

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2. Statement of the Exact Problem

We study plane long-wave flows of inviscid incompressible fluid of uniform density in an infinitely long thin layer above horizontal bottom in the gravity field. We neglect surface tension on the free boundary of the fluid layer.

Under these assumptions, the Benney system of equations [2, 3] becomes

$$u_t + uu_x + vu_y = -gh_x, \quad u_x + v_y = 0, \tag{1}$$

where $u(x, y, t)$ and $v(x, y, t)$ are the horizontal and vertical components of the velocity field of the fluid; $g \equiv \text{const} > 0$ is the free fall acceleration; $h(x, t)$ is the thickness of the fluid layer; t is time; x and y are Cartesian coordinates. Henceforth independent variables appearing in the lower indices indicate the corresponding partial derivatives of the functions in question.

In addition to (1), we impose the following boundary conditions: (a) the impermeability condition

$$v = 0 \tag{2}$$

at the bottom (for $y = 0$); (b) the kinematic condition

$$h_t + uh_x - v = 0 \tag{3}$$

on the free boundary of the fluid layer (for $y = h$).

We take

$$u(x, y, 0) = u_0(x, y), \quad h(x, 0) = h_0(x) \tag{4}$$

as the initial data for the first equation in (1) and for (3), requiring that u_0 and h_0 satisfy the second equation in (1), as well as (2).

Then in the initial-boundary value problem (1)-(4) it is convenient to pass from the Euler independent variables t, x, and y to the mixed Euler-Lagrange variables t', x', and λ [4]. This transition enables us to replace the initial-boundary value problem in a fluid layer with free boundary with a simpler mixed problem in a channel with fixed boundaries.

The change of variables is given by[4]

$$t = t', \quad x = x', \quad y = \Phi(t', x', \lambda); \quad \lambda \in [0, 1]. \tag{5}$$

Here λ is the Lagrangian coordinate enumerating the trajectories of fluid particles in the layer; the function Φ is assumed to satisfy

$$\Phi_{t'} + u\Phi_{x'} = v. \tag{6}$$

It is important that the validity of the boundary conditions (2) and (3) follows automatically from (5) and (6).

As a result, the mixed problem (1)-(4) becomes

$$u_t + uu_x + gH_x = 0, \quad \rho_t + (u\rho)_x = 0;$$

$$H \equiv \int_0^1 \rho d\lambda, \quad \rho \equiv \Phi_\lambda(t, x, \lambda); \tag{7}$$

$$u(0, x, \lambda) = u_0(x, \lambda), \quad \rho(0, x, \lambda) = \rho_0(x, \lambda),$$

where we omit the primes on the independent variables t' and x' in order to avoid making the subsequent relations too bulky.

The solutions to the initial-boundary value problem (7) characterize the plane long-wave flows of ideal incompressible fluid of uniform density in the gravity field in an infinitely long horizontal channel of unit width, whose upper wall $\lambda = 1$, because of the change (5), (6) of independent variables, corresponds to the free surface $y = h$ of the fluid layer, while the bottom wall $\lambda = 0$, to its bottom $y = 0$.

The mixed problem (7) possesses the energy integral

$$E \equiv \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^1 \rho u^2 d\lambda dx + \frac{g}{2}\int_{-\infty}^{+\infty} H^2 dx = \text{const} \tag{8}$$

provided that its solutions are either periodic or localized along the x-axis.

Moreover, it is not difficult to show that the initial-boundary value problem (7) also has the integral of motion [5]

$$C \equiv \int_{-\infty}^{+\infty}\int_0^1 u_\lambda F(\kappa) d\lambda dx = \text{const}. \tag{9}$$

Here

$$\kappa \equiv \frac{\rho}{u_\lambda}: \quad \kappa_t + u\kappa_x = 0$$

and $F(\kappa)$ is an arbitrary function of its argument.

Finally, the mixed problem (7) admits exact stationary solutions

$$u = u^0(\lambda), \quad \rho = \rho^0(\lambda), \quad H = H^0 \equiv \text{const}, \tag{10}$$

where u^0 is an arbitrary function of the independent variable λ, while ρ^0 is a non-decreasing function of λ. It is not difficult to show that u^0, ρ^0, and H^0 indeed make all equations of the initial-boundary value problem (7) into identities.

The goal of our further study is to find out whether the exact stationary solutions (10) are stable to small plane long-wave perturbations $u'(t, x, \lambda)$, $\rho'(t, x, \lambda)$, and $H'(t, x)$.

3. Statement of the Linearized Problem

To this end, we linearize the mixed problem (7) in the vicinity of the exact stationary solutions (10), which leads to the initial-boundary value problem

$$u'_t + u^0 u'_x + gH'_x = 0, \quad \rho'_t + u^0\rho'_x + \rho^0 u'_x = 0;$$

$$H' = \int_0^1 \rho' d\lambda; \tag{11}$$

$$u'(0, x, \lambda) = u'_0(x, \lambda), \quad \rho'(0, x, \lambda) = \rho'_0(x, \lambda),$$

on whose evolutionary solutions, in turn, there is a time-invariant functional

$$E_1 \equiv \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^1 \left[\rho^0 u'^2 + 2u^0 u'\rho' + \frac{du^0}{d\lambda}\frac{d^2F}{d\kappa^2}(\kappa^0)\kappa'^2\right] d\lambda dx +$$

$$+ \frac{g}{2}\int_{-\infty}^{+\infty} H'^2 dx, \tag{12}$$

a linear analog of the energy integral. Here

$$\kappa^0 \equiv \rho^0 \left(\frac{du^0}{d\lambda}\right)^{-1};$$

$$\kappa' \equiv \left(\rho'\frac{du^0}{d\lambda} - \rho^0 u'_\lambda\right)\left(\frac{du^0}{d\lambda}\right)^{-2}: \quad \kappa'_t + u^0\kappa'_x = 0.$$

We found this expression for the functional E_1 using the method of coupling the integrals of motion [6-10].

Indeed, it is easy to verify that the first variation δJ of the functional $J \equiv E + C = \text{const}$ (8), (9) vanishes on the exact stationary solutions (10) when

$$\frac{dF}{d\kappa}(\kappa^0) = -\frac{u^{02}}{2} - gH^0,$$

$$\int_{-\infty}^{+\infty}\left[F(\kappa^0) + \kappa^0\left(\frac{u^{02}}{2} + gH^0\right)\right]\delta u\bigg|_{\lambda=1} dx = \tag{13}$$

$$= \int_{-\infty}^{+\infty}\left[F(\kappa^0) + \kappa^0\left(\frac{u^{02}}{2} + gH^0\right)\right]\delta u\bigg|_{\lambda=0} dx,$$

where δu is the first variation of the horizontal component u of the velocity field of the fluid. The second variation $\delta^2 J$ of the integral J, calculated using the first relation of the system (13) and expressed in terms of the small plane long-wave perturbations (11), is of the same form as the functional E_1 of (12) in the case that the constraint

$$\int_{-\infty}^{+\infty}\left[F(\kappa^0) + \kappa^0\left(\frac{u^{02}}{2} + gH^0\right)\right]\delta^2 u\bigg|_{\lambda=1} dx =$$

$$= \int_{-\infty}^{+\infty}\left[F(\kappa^0) + \kappa^0\left(\frac{u^{02}}{2} + gH^0\right)\right]\delta^2 u\bigg|_{\lambda=0} dx \tag{14}$$

is satisfied (here $\delta^2 u$ is the second variation of the horizontal component u of the velocity field of the fluid).

It is worth observing that the second expression in (13) and the relation (14) are not dynamically contradictory and do not overdetermine the mixed problem (11) since its equations form a system of order 2, while the walls $\lambda = 0$ and $\lambda = 1$ of the channel in which the fluid is flowing are free of boundary conditions. Incidentally, we can satisfy the second expression in (13) and the relation (14) by restricting not the class of small plane long-wave perturbations (11), but the class of exact stationary solutions (10) by imposing on the function $F(\kappa)$, in addition to the first expression in (13) as a requirement, the restriction

$$F\left(\kappa^0\right) = -\kappa^0 \int\limits_{\kappa^0(0)}^{\kappa^0} \frac{F_1\left(\kappa_1^0\right)}{\kappa_1^{02}} d\kappa_1^0,$$

where $F_1\left(\kappa^0\right)$ is a function of its argument such that

$$F_1\left(\kappa^0\right)\Big|_{\kappa^0=\kappa^0(0)} = 0, \quad F_1\left(\kappa^0\right)\Big|_{\kappa^0=\kappa^0(1)} = 0;$$

here $\kappa^0(0)$ and $\kappa^0(1)$ are the values of $\kappa^0(\lambda)$ for $\lambda = 0$ and $\lambda = 1$ respectively, and κ_1^0 is the variable of integration.

In accordance with the method of coupling the integrals of motion, the exact stationary solutions (10) to the initial-boundary value problem (7) are stable to small plane long-wave perturbations (11) if and only if the functional E_1 of (12) is sign definite or at least semidefinite.

In order to find out whether E_1 is sign (semi)definite, it is convenient to express it as

$$E_1 = \int\limits_{-\infty}^{+\infty} \int\limits_0^1 \left(A\mathbf{f}, \mathbf{f}\right) d\lambda dx; \quad \mathbf{f} \equiv \left(u', \rho', \kappa', H'\right)^{\mathrm{T}}. \qquad (15)$$

Here $A = \|a_{ik}\|$ is the 4×4 matrix with nonzero entries

$$a_{11} = \frac{\rho^0}{2}, \quad a_{12} = a_{21} = \frac{u^0}{2}, \quad a_{24} = a_{42} = \frac{g}{4},$$

$$a_{33} = \frac{1}{2}\frac{du^0}{d\lambda}\frac{d^2F}{d\kappa^2}\left(\kappa^0\right).$$

According to Sylvester's criterion[11], the integrand in the functional E_1 of (15) is positive definite if and only if all principal minors of A are positive. At the same time, this expression is negative definite if and only if the principal minors of A have the signs $(-1)^m$, where m is the size of the principal minor.

It is easy to verify that the principal minors of A fail the requirements for sign definiteness. Indeed, for both positive and negative definiteness of the integrand in E_1 of (15) the size two principal minor Δ_2 of A must be positive. However, $\Delta_2 = -u^{02}/4 < 0$, which prevents the integral E_1 from being sign (semi)definite.

This implies that no sufficient conditions exist for the stability of the exact stationary solutions (10) to the mixed problem (7) to small plane long-wave perturbations $u'(t, x, \lambda)$, $\rho'(t, x, \lambda)$, and $H'(t, x)$.

However, for the initial-boundary value problem (7) a condition for the hyperbolicity of its equations was obtained in[1,3] by generalizing the method of characteristics to eigenvalue problems for operators. Whenever those articles discuss the mixed problem (11), this condition is treated no less than as a sufficient condition for the stability of the exact stationary solutions (10) to the initial-boundary value problem (7) to small plane long-wave perturbations.

The result on the absence of sufficient conditions for linear stability, established above using the method of coupling the integrals of motion[6-10], clearly contradicts this result of[1,3].

Since the very fact of presence or absence of sufficient conditions for stability must be independent of the choice of a method for obtaining them, in order to resolve this contradiction, below we construct, jointly with my former student E. Yu. Knyazeva, an analytic example of an exact stationary solution (10) to the mixed problem (7) with small plane long-wave perturbations (11) superimposed as normal modes.

4. Example

We study the exact stationary solution (10) to the initial-boundary value problem (7) as

$$u = u^0(\lambda) \equiv 3\sqrt{I}\,\mathrm{th}\left[100(\lambda - 0{,}5)\right],$$

$$\rho = \rho^0(\lambda) \equiv 1, \quad H = H^0 \equiv 1; \quad g \equiv 1, \qquad (16)$$

$$I \equiv \int\limits_0^1 \frac{9\mathrm{th}^2\left[100(\lambda-0{,}5)\right]-1}{\left(9\mathrm{th}^2\left[100(\lambda-0{,}5)\right]+1\right)^2} d\lambda \approx 0{,}0767 > 0.$$

We superimpose on this solution small plane long-wave perturbations $u'(t, x, \lambda)$, $\rho'(t, x, \lambda)$, and $H'(t, x)$ (11) as normal modes. Namely,

$$u'(t, x, \lambda) = u_1(\lambda)\exp(\alpha t + i\beta x), \quad \rho'(t, x, \lambda) = \rho_1(\lambda)\exp(\alpha t + i\beta x),$$

$$H'(t, x) = \int\limits_0^1 \rho_1(\lambda)\exp(\alpha t + i\beta x)d\lambda; \qquad (17)$$

$$u_1(\lambda) = \frac{B}{\alpha + i\beta u^0}, \quad \rho_1(\lambda) = -\frac{i\beta B}{\left(\alpha + i\beta u^0\right)^2},$$

where $\alpha \equiv \alpha_1 + i\alpha_2$ and B are arbitrary complex constants, while β, α_1, and α_2 are arbitrary real constants.

The functions (16) and (17) make the first two relations of the mixed problem (11) into identities provided that the characteristic equation[1,3]

$$\chi(k) \equiv 1 - \int\limits_0^1 \frac{d\lambda}{\left(u^0 - k\right)^2}, \quad k \equiv \frac{i\alpha}{\beta} \qquad (18)$$

is satisfied.

It is appropriate to note that the characteristic equation of type (18) was studied in detail in[1,3] only in the following two cases: (1) k is a real constant, χ is a real characteristic function; (2) k is a complex constant, χ is a complex characteristic function. Furthermore, in the second case it is established that if for definiteness, and without loss of generality, we assume that $u_\lambda > 0$ then the characteristic equation (18) lacks complex roots if and only if a hyperbolicity condition is fulfilled; this is the limiting relation following from the condition of vanishing of the argument of the analytic function χ of the complex variable k as we go in the positive direction around a contour in the meromorphy domain of χ surrounding its zeroes and poles, but not passing through them (see, for instance, (1.11) on p. 252 of [1]).

However, the case that k is a complex constant and χ is a real characteristic function is not covered in[1,3]. Below we fill this gap.

More concretely, it is not difficult to observe, and this is also the convention in[1,3], that the function $u^0(\lambda)$ of (16) is strictly increasing since

$$\frac{du^0}{d\lambda}=\frac{300\sqrt{I}}{\mathrm{ch}^2\left[100(\lambda-0{,}5)\right]}>0.$$

Now take $\alpha_2\equiv 0$ and $\alpha_1\equiv\beta k_1$, where k_1 is some real constant. Then, since $u^0(\lambda)$ is an odd function, it follows that

$$\mathrm{Im}\,\chi(k)=-2k_1\int_0^1\frac{u^0 d\lambda}{\left(u^{02}+k_1^2\right)^2}\equiv 0,$$

and the characteristic function $\chi(k)$ of (18) becomes real:

$$\chi(k)=\mathrm{Re}\,\chi(k)\equiv\chi_1(k_1)=1-\int_0^1\frac{u^{02}-k_1^2}{\left(u^{02}+k_1^2\right)^2}d\lambda. \quad (19)$$

Moreover, direct calculations show that $k_1=\sqrt{I}$ is a root of (19). Indeed,

$$\chi_1\left(\sqrt{I}\right)=1-\int_0^1\frac{9I\,\mathrm{th}^2\left[100(\lambda-0{,}5)\right]-I}{\left(9I\,\mathrm{th}^2\left[100(\lambda-0{,}5)\right]+I\right)^2}d\lambda=$$

$$=1-\frac{1}{I}\int_0^1\frac{9\mathrm{th}^2\left[100(\lambda-0{,}5)\right]-1}{\left(9\mathrm{th}^2\left[100(\lambda-0{,}5)\right]+1\right)^2}d\lambda=\frac{1}{I}(I-I)=0.$$

Hence, $k=ik_1=i\sqrt{I}$ is a complex root of the characteristic equation (18).

This implies that since the characteristic function $\chi(k)$ of (18) and (19) is real, the hyperbolicity condition[1,3] for the equations of the initial-boundary value problem (7) is fulfilled. Nevertheless, despite this condition, we discovered that the characteristic equation (18) has the complex root $k=i\sqrt{I}$. The reason for this contradiction is that the hyperbolicity condition[1,3] for the equations of the mixed problem (7) is fulfilled not for all possible pairs of small plane long-wave perturbations (11), but only for subclass of them (which, and this is important, is not independent).

Consequently, the example constructed above shows irrefutably that the generalization[1,3] of the method of characteristics to eigenvalue problems for operators is erroneous, confirming simultaneously the validity of the method of coupling the integrals of motion[6-10]. Moreover, since much arbitrariness remains for the constant α_1, for $\alpha_1>0$ our example can be interpreted as an example of the incorrectness of the initial-boundary value problem (11) in the sense of Hadamard[12]. Finally, in the case $\alpha_1>0$ small plane long-wave perturbations $u'(t,x,\lambda)$, $\rho'(t,x,\lambda)$, and $H'(t,x)$ of (11) as normal modes (17)-(19) increase in time, which, in turn, creates the instability of the exact stationary solution (16) to the mixed problem (7).

5. Conclusions

Thus, on the example of a mathematical model of the propagation of long waves on the free surface of a thin infinitely long horizontal layer of a whirling inviscid incompressible fluid of uniform density in the gravity field we have demonstrated that the generalization[1,3] of the method of characteristics to eigenvalue problems for operators is inappropriate for adequate mathematical modelling of waves in fluids.

For this reason, it appears logical that the study of this subject must be continued in the following main directions: (1) to prove the absolute instability of the steady-state plane-parallel shear flows (10) to small plane long-wave perturbations (11); (2) to obtain sufficient conditions for the practical linear instability of these flows to the same perturbations; and (3) to find new hyperbolicity conditions which would be expressed in terms of the integrals of motion and would therefore isolate properly independent particular classes of long waves.

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REFERENCES

[1] V. Yu. Lyapidevskii and V. M. Teshukov, Mathematical Models of Propagation of Long Waves in an Inhomogeneous Fluid. Novosibirsk, Russian Federation: Publishing House of the Siberian Branch of RAS, 2000 (in Russian)

[2] Benney, D.J., 1973, Some properties of long nonlinear waves, Stud. Appl. Math., 52(1), 45–50

[3] Teshukov, V.M., 1985, On the hyperbolicity of long wave equations, Dokl. Akad. Nauk SSSR, 284(3), 555–562 (in Russian)

[4] Zakharov, V.E., 1980, Benney's equations and quasiclassical approximation in the inverse problem method, Funktsional. Anal. i Prilozhen., 14(2), 15–24 (in Russian)

[5] Gubarev, Yu.G., 1995, On an analogy between the Benney's equations and the Vlasov-Poisson's equations, Dinamika Sploshn. Sredy, Lavrentyev Institute of Hydrodynamics SB RAS, Novosibirsk, 110, 78–90 (in Russian)

[6] Fjørtoft, R., 1950, Application of integral theorems in deriving criteria of stability for laminar flows and for the baroclinic circular vortex, Geofys. Publ. Norske Vid.-Akad. Oslo, 17(6), 4–52

[7] Arnold, V.I., 1965, Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid (Ordinary differential equations theory yielding hydrodynamics conditions of stability with respect to finite perturbations for plane stationary curvilinear flow of ideal fluid), Soviet Mathematics, 6, 773–777

[8] Arnold, V.I., 1965, Variational principle for three- dimensional steady-state flows of an ideal fluid (Steady flow possessing extremal kinetic energy compared to equivortex flow

for stability analysis), PMM – Journal of Applied Mathematics and Mechanics, 29(5), 1002–1008

[9] Arnold, V.I., 1966, A priori estimate of the theory of hydrodynamic stability (Perturbation estimation theorems and stability of plane steady curvilinear flows of ideal fluid), Matematika, 5, 3–5

[10] Vladimirov, V.A., 1986, Conditions for nonlinear stability of flows of an ideal incompressible liquid, Journal of Applied Mechanics and Technical Physics, 27(3), 382–389

[11] V. A. Yakubovich and V. M. Starzhinskii, Linear Differential Equations with Periodical Coefficients and its Applications. Moscow, Russian Federation: Nauka, 1972 (in Russian)

[12] S. K. Godunov, Equations of Mathematical Physics. Moscow, Russian Federation: Nauka, 1979 (in Russian)