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Numerical Representation of Weirs Using the Concept of Inverse Problems

Olesya I. Sarajlic[1,*], Alexandra B. Smirnova[2]

[1]Department of Physics and Astronomy, Georgia State University, Atlanta, Georgia, 30303-3083, USA
[2]Department of Mathematics and Statistics, Georgia State University, Atlanta, Georgia, 30303-3083, USA

Abstract The use of Laplace transform and other computational tools allows the study of elementary inverse problems in hydraulics, such as for weirs. Torricelli's Law provides the relationship between the notch shape of the weir and the respective flow rate. The flow rate function *r(h)* gives the rate at which the volume of water hits the notch of a particular shape *f(y)*. The essence of the direct problem here is to determine the flow rate *r(h)* from the notch shape *f(y)*. Mathematically, this comes down to a computationally stable integration process. However, the corresponding inverse problem, i.e., identification of the notch shape *f(y)* from the flow rate measurements, is ill-posed in a sense that even a small error in the input data can result in a substantial inaccuracy in the computed solution. Numerical modelling offers the opportunity to test design parameters over large ranges and varieties of shapes. It may detect significant flow patterns and capture the amount of noise that a given function can tolerate. The goal of the study is to interpret the produced notch shape functions of the weirs, to discuss advantages and disadvantages of weirs' structure, and to present a regularized numerical algorithm for getting a less noisy and a more stable outcome of the inversion procedure.

Keywords Weirs, Inverse Problems, Laplace Transform, Torricelli's Law, Lavrentiev's Regularization

1. Introduction

Weirs are the overflow structures of defined shape that are built across open channels or streams to measure the flow rate[1-3]. There are several standard equations that are developed to describe the relations between the shape and design of the weir, rates of flow, head, and hydraulic conductivity[3]. Every design of the weir can be controlled by the depth of water, which in turn can predict how the crest elevation that is relative to the upstream head changes with discharge[4].

There are two types of weirs, sharp-crested and broad-crested. Sharp crested weirs can take variety of shapes. The most common weir structures that are used for measuring irrigation water include rectangular, triangular (V-notch), trapezoidal (Cipolletti design), and parabolic weirs[4,5]. The designs of some weirs have certain advantages over the others. For example, the V-notch is designed to have small changes in discharge which results in a large change in depth of water through the notch allowing for a more accurate head measurement[6]. While V-notch weirs are built to monitor low flow conditions, Cipolletti weirs are designed to measure higher flows[1].

There exist certain requirements to all sharp-crested weirs to assure accuracy of flow measurement. The most important ones are: the upstream face and head of the weir plates should be normal to the axis of the channel, the entire weir panel should have the same thickness, the edges of the weir opening should be straight and sharp, and other more technical conditions[4,7,8]. Once the weir is constructed, it can be tested and resulting calibrations can be compared with standard results. However, depending on the dimensions of the design, the weirs can cease to provide accurate measurements of the total stream flow during the events of prolonged heavy rainfall where the weirs are bypassed and the stream over-topped the banks[1].

In the work that was done by Hively et.al.[1], the authors carried out the in-situ measurements of the range of flows through the V-notch and Cipolletti weirs, which dimensions were known. Unfortunately, several factors such as heavy rainfall, affected the accuracy of the measurements. In this work, the flow rate will be derived from the notch shape and vice versa. This method of computational analysis portends the outcome of the design, in particular Cipolletti and parabolic weirs, given the amount of relative error that can potentially disrupt the measurements.

2. Mathematical Background

The principles of hydraulics serve as one of the branches of the fluid dynamics. Torricelli's theorem, which follows

* Corresponding author:
osarajlic@gmail.com (Olesya I. Sarajlic)
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immediately from the law of conservation of energy, relates the velocity of the fluid flowing out of an elevated irrigation canal, fitted with a weir notch, under the force of gravity to the height of the water[9,10]. In what follows, we consider an elevated irrigation canal that is much wider in comparison to the weir notch; thus, there is an inconsequential change in water level when the sluice gate in the notch is removed. It is assumed that the weir notch is symmetric with respect to a central vertical axis so that for the non-negative values of x it can be characterized by the shape function $x=f(y)$, see Figure 1.

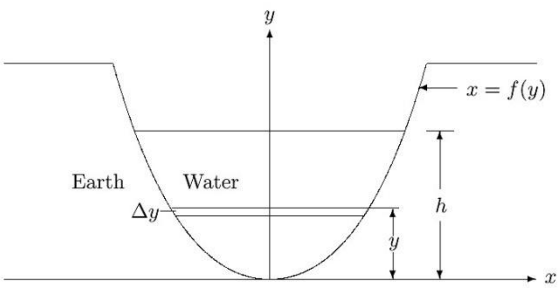

Figure 1. A weir notch

To understand the relationship between the notch shape $f(y)$, the height h of the water, and the flow rate $r(h)$ using Torricelli's law, one takes into consideration the energy losses due to viscosity and turbulence of water as it exits the weir[9]. The illustration of this concept can be done by considering a thin horizontal slab of water of thickness Δy positioned at height y in the notch. In accordance with Torricelli's law, the velocity of this slab as it leaves the notch is $\sqrt{2g(h-y)}$, which in other words can be described as subtracting a constant from the kinetic energy term due to pre-existing viscosity and turbulence conditions. Therefore, the volume of water, ΔV, as the water crosses the notch in small time intervals, Δt, through a cross-sectional area of the slab, $2f(y)\Delta y$, can be approximated as

$$\Delta V \approx 2f(y)\sqrt{2g(h-y)}\,\Delta y\,\Delta t. \qquad (1)$$

Furthermore, the flow rate function, $r(h)$, which is the rate at which the volume of water strikes the notch can be obtained by summing $\Delta V/\Delta t$ over all horizontal slabs. Invoking the usual limit argument, we arrive at the following expression for the flow rate:

$$r(h) = 2\int_0^h \sqrt{2g(h-y)}\, f(y)\, dy. \qquad (2)$$

Given that the acceleration due to gravity, g, is 32 ft/sec^2, the basic equation that links the notch shape and the flow rate attains the form:

$$r(h) = 16\int_0^h \sqrt{h-y}\, f(y)\, dy. \qquad (3)$$

Consequently, the direct problem of determining the flow rate function, $r(h)$, from the notch shape, $f(y)$, is computationally stable and amounts to fairly straightforward integration using equation (3). On the other hand, as seen from equation (3), the inverse problem is modelled by Volterra's integral equation of the first kind. This problem is known to be severely unstable[10] in a sense that even a small mistake in the input data has a potential to cause a significant error in the corresponding solution. Following this fact, the regularized numerical algorithm for getting a less noisy and a more stable solution to the weir design inverse problem will be incorporated.

3. Numerical Simulations: Weir Design

Figure 2 shows the field constructed weirs that are typically used in agriculture with Figure 2(a) illustrating the Cipolletti design and Figure 2(b) being the parabolic weir. The shape functions for the Cipolletti and parabolic sharp-crested weirs are numerically fit to determine the corresponding flow rates by numerical integration carried out for equation (3).

Figure 3 demonstrates stability of the direct problem where the Cipolletti weir with the shape function, $f(y)$

$$f(y) = \begin{cases} 0 & y < H/2 \\ 1 & y \geq H/2 \end{cases}$$

generates the flow rate $r(h)$ of

$$r(h) = \begin{cases} 0 & h < H/2 \\ \frac{32}{3}\left(h-\frac{H}{2}\right)^{3/2} & h \geq H/2 \end{cases},$$

and H is the height of the weir. The original shape data $f(y)$ is perturbed with random noise of various amplitudes to show the range of fault the computed flow rate $r(h)$ can withstand before deviating from the model solution. Figure 3(a) illustrates the numerical simulation for the Cipolletti weir with height of 60 feet. As one can see, the model and the noise-free computed solutions of the direct problem are almost identical. If the shape data $f(y)$ is perturbed with random noise of about 100%, which can be caused by the friction or the capillary effects of the water as it passes through the weir, the flow rate $r(h)$ does not deviate from the exact solution significantly. Even when the relative noise increases to about 300%, the shape of the flow rate function can still be reconstructed. Figure 3(b) shows numerical outcomes for the Cipolletti shaped weir that is 100 feet high. The increase in noise levels produces similar results for the taller weir as for the shorter one. Hence the process of solving the direct problem is computationally stable, though the error in the computed flow rate does increase as the weir takes on higher value of height.

The water flow through a parabolic sharp-crested weir can also be determined numerically by using equation (3). If the shape of the weir is defined as $f(y)=\sqrt{y}$, the corresponding solution for the flow rate that the weir is generating is $r(h) = 2\pi h^2$. Figure 4 shows how stable the solution to the direct problem is even under a very high perturbation of the data. Similar to the case of Cipolletti weir, the noise-free computed flow rate for the parabolic weir notch has the same outcome as the model solution for both 60 and 100 feet in height. Once a random noise higher than 100% or even 300% is added to the system, the flow rate can still be distinguished fairly easy, as Figure 4(a) and Figure 4(b) suggest. Analogous to the previous example, parabolic weir experiences great stability at short ranges of height. However, the flow function starts to deviate (slowly) from the model solution as the height becomes greater.

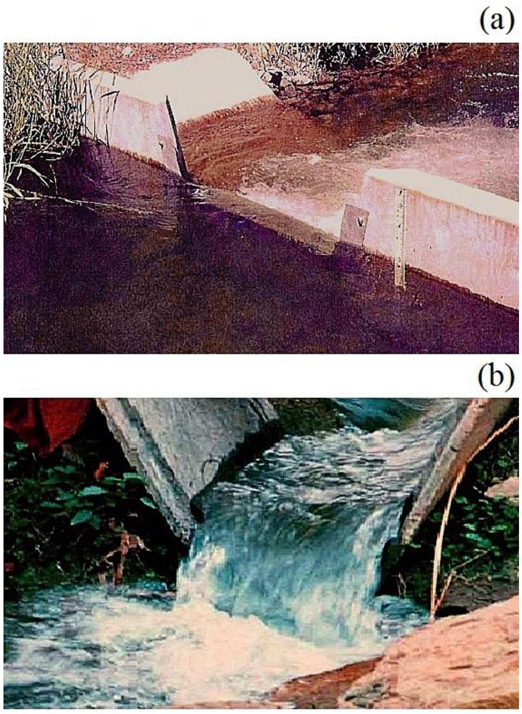

Figure 2. Weir types - field constructed - that are commonly used in agriculture: (a) Cipolletti design that corresponds to the computational results in Figure 3, (b) parabolic weir corresponding to the numerical demonstration displayed in Figure 4

From theoretical standpoint, these results are fully justified. Indeed, allow f and g to be notch shapes satisfying $|f(y) - g(y)| < \varepsilon$ for all $y \in [0, H]$. Suppose that r_f and r_g are the flow rate functions corresponding to f and g, respectively. According to equation (3), the upper bound for the difference in the flow rates is $|r_f(h) - r_g(h)| \le (32/3)\, H^{3/2}\varepsilon$. This estimate demonstrates that the forward solver is stable, especially for a small H, length of the interval on which the flow rate function is to be approximated. At the same time, the error grows as H increases, and the computed solution moves away from exact solution at the right end of the interval.

Considering the fact that in some cases the shape of the weir might be unknown, there are several computational techniques that can be used to solve for the shape given the flow rate of the water in the canal or stream. And while the direct problem can be solved by a straightforward integration, the inverse problem is not that trivial. One way to solve it is by applying the Laplace transform, which allows to reduce an integral equation for the unknown function to the algebraic equation for its Laplace transform. This technique is particularly useful in solving convolution type linear integral equations, such as equation (3) for finding the weir notch shape that accounts for a certain flow rate function. The transformation of equation (3) with the use of Laplace transform and the convolution theorem gives

$$\Im\{r\} = 16\,\Im\{f \star \sqrt{h}\} = 16\,\Im\{f\}\Im\{\sqrt{h}\}$$

and

$$\Im\{f\} = \frac{\Im\{r\}s^{3/2}}{8\sqrt{\pi}}. \quad (4)$$

If one takes, for example, the parabolic flow rate $r(h) = 2\pi h^2$ (as shown in Figure 4), then it follows from the above that the corresponding weir notch shape is, indeed, $f(y) = \sqrt{y}$.

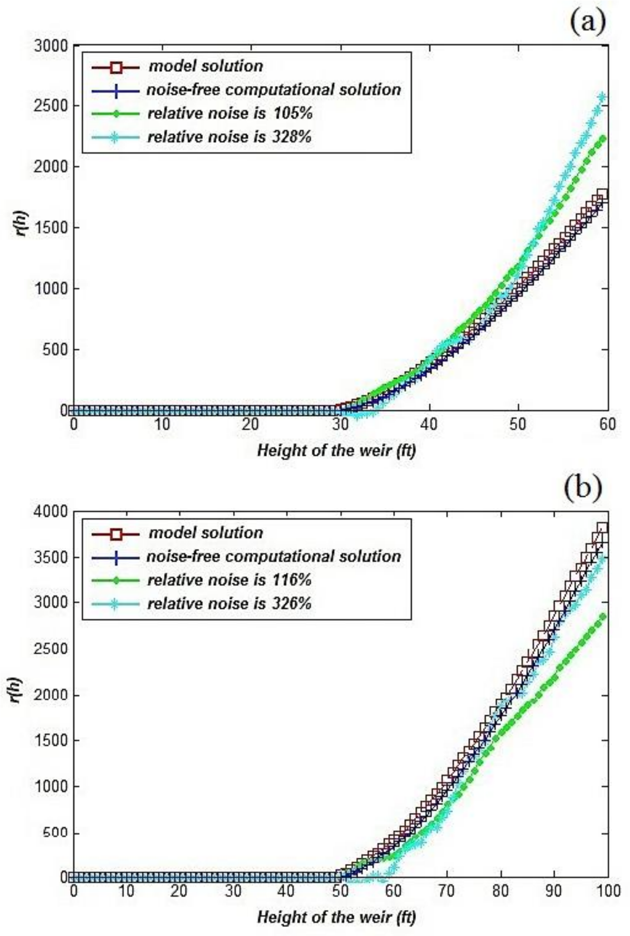

Figure 3. Numerical demonstration of stability of the direct problem where the flow rate is $r(h) = \begin{cases} 0 & h < H/2 \\ \frac{32}{3}\left(h - \frac{H}{2}\right)^{3/2} & h \ge H/2 \end{cases}$ reconstructed from the shape $f(y) = \begin{cases} 0 & y < H/2 \\ 1 & y \ge H/2 \end{cases}$. Both exact shape data and shape data that is perturbed with random noise of various amplitudes were considered

Further applications of Laplace transform along with the convolution theorem allow to obtain an explicit formula for the weir notch shape[10]. The second identity in equation (4) yields

$$s\Im\{r\} = 8\sqrt{\pi}\,s^{-1/2}\Im\{f\}$$

and

$$\Im\{f\} = \frac{s^{-1/2}}{8\sqrt{\pi}}\left[s^2\Im\{r\}\right]. \qquad (5)$$

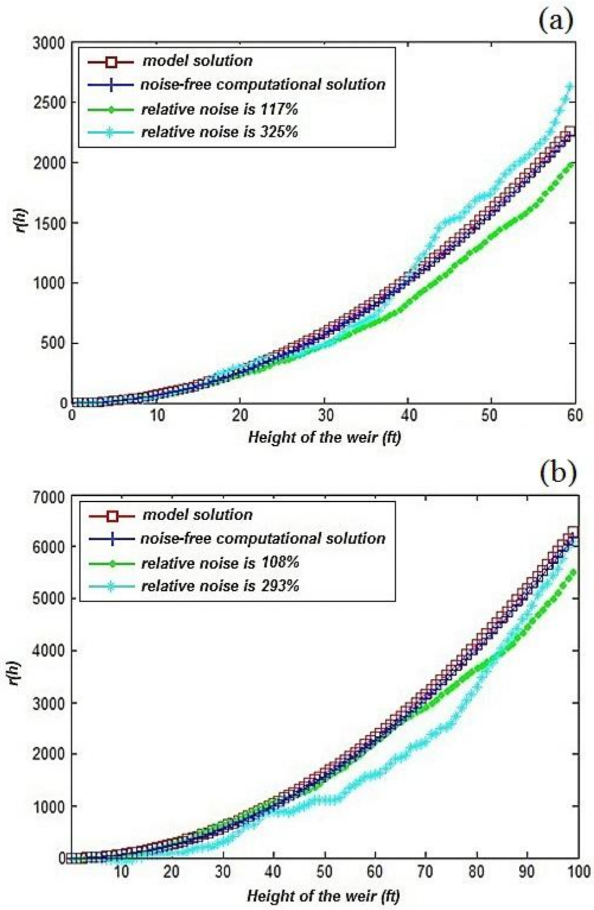

Figure 4. Numerical demonstration of stability of the direct problem where the flow function $r(h)= 2\pi h^2$ is reconstructed from the weir notch shape $f(y)=\sqrt{y}$. Both exact shape data and shape data that is perturbed with random noise of various amplitudes were considered

Since $r(0)$ is obviously 0, the first equality in (5) implies that $r'(h) = 8\int_0^h \frac{f(y)}{\sqrt{h-y}}dy$, and, therefore, $r'(0)=0$. From the second equality in (5) along with the property $r(0)=r'(0)=0$, one concludes

$$f(y) = \frac{1}{8\pi}\int_0^y \frac{r''(h)}{\sqrt{y-h}}\,dh. \qquad (6)$$

Explicit formula (6) for the notch shape $f(y)$ hints to ill-posedness of the inverse problem and it won't give an accurate outcome due to a well-known fact that differentiation of noisy functions is an unstable procedure.

To illustrate what it means in practice, Figure 5(a) shows numerical results for the inverse problem obtained by discretising original equation (3) and solving the corresponding linear system with a triangular matrix A. While the noise-free reconstruction is still accurate, even a very small relative noise in the flow rate $r(h)$, considerably less than 1%, causes a substantial error in the computed weir shape function $f(y)$. This is hardly surprising. As one can easily verify, the relationship between the relative error on the computed solution and the relative error on the data is given by the following estimate:

$$\frac{||f_e - f_\delta||}{||f_e||} \le ||A||\,||A^{-1}||\frac{||r_e - r_\delta||}{||r_e||},$$

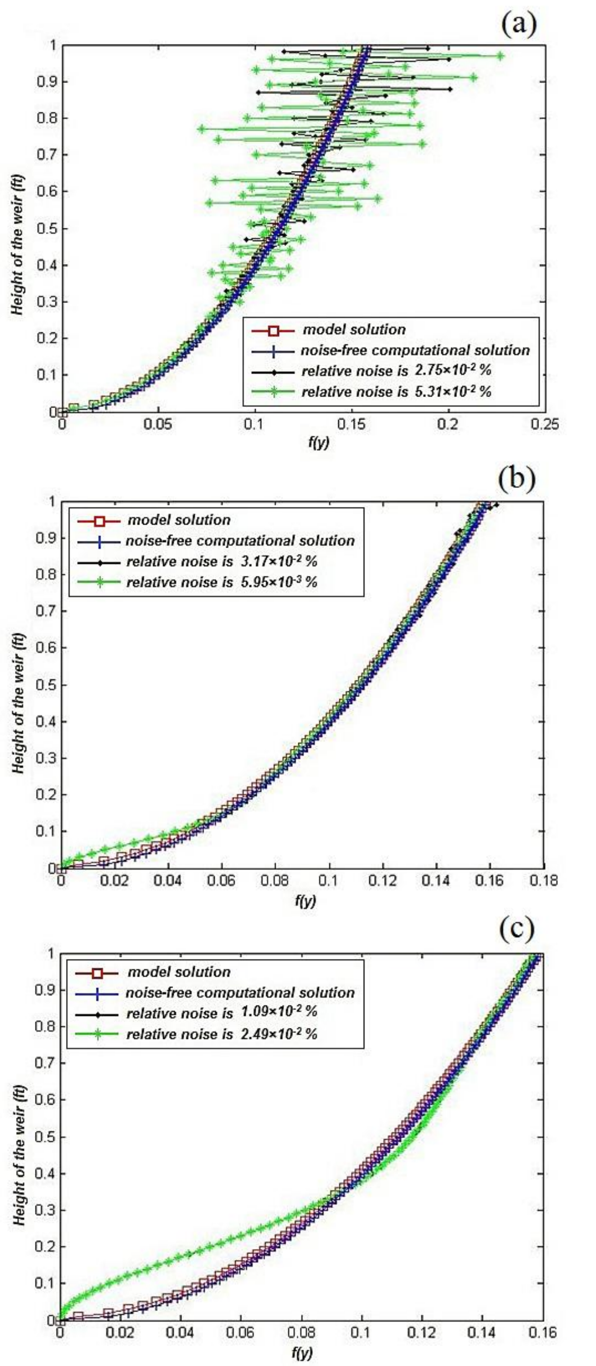

Figure 5. (a) Illustrates $f(y)= \sqrt{y}/(2\pi)$ reconstructed from $r(h)=h^2$, $cond(A)\approx 1050$. (b) Illustrates $f(y)= \sqrt{y}/(2\pi)$ reconstructed from $r(h)=h^2$. For $\alpha=0.1$, $cond(A+\alpha I)\approx 69$. (c) Illustrates $f(y)= \sqrt{y}/(2\pi)$ reconstructed from $r(h)=h^2$. For $\alpha=0.5$, $cond(A+\alpha I)\approx 16$

where the product $||A||\,||A^{-1}||$ is called the condition number of A, $cond(A)$; $r_\delta(f_\delta)$ is a perturbed right-hand side (solution), and $r_e(f_e)$ is the exact right-hand side (solution). In the experiment above, the matrix $A$is severely ill-conditioned with $cond(A) \approx 1050$. Therefore even a small amount of noise quickly perturbs the shape function even for extremely low heights of the weir. Similar (poor) reconstructions of $f(y)$ have been obtained with explicit formula (6).

To overcome this instability, we employ a special technique called regularization, which allowsto strike a balance between accuracy and stability in a computational algorithm[11].The causal structure of the original Volterra problem suggests the use of Lavrentiev's regularization in the form $(A+ \alpha I)\ f = r$, where $\alpha> 0$is the regularization parameter introduced to bring the conditioning of A down. As Figure 5(b) demonstrates, Lavrentiev's regularization with $\alpha = 0.1$ improves the outcome of the shape of the weir. For the relative noise $5.95 \times 10^{-3}\%$ in the data, the shape function $f(y)$ is nearly perfect for the large values of the height while it diverges from the model at the bottom of the weir. For a higher noise, $3.17\times 10^{-2}\%$, the shape function follows the model solution then starts to randomly deviate from it as the height of the weir increases. Overall, we see a huge improvement compared to unregularized simulations presented in Figure 5(a). Figure 5(c), where $\alpha = 0.5$, illustrates the overregularization phenomena. Notice that for $\alpha = 0.1$, the condition number of $A+ \alpha I$ drops to 69, while for $\alpha = 0.5$ it decreases even more, i.e. to about 16.

4. Applications

The design dimensions, specifications, placement locations, and other applications of various weir structures should be taken into consideration before the construction. The weirs were developed and subsequently applied to maintain the flow through the channel or stream in order to withstand floods, to protect stream banks from erosion by redirecting the flow, to enable sediment transport and deposition along the stream banks, to control flow direction in order to provide diversion of water for agricultural needs, and to maintain fish habitat and river stability[3,8].

The numerical models of the parabolic weir (or Cross-Vane structure) and trapezoidal weir (Cipolletti design) that are illustrated on Figure 3 and Figure 4 can be manipulated to satisfy the requirements of the project before the actual construction. The Cross-Vane and Cipolletti structures have very beneficial aspects. Both Cross-Vane and Cipolletti weir designs can be used to maintain base level in the channels or streams and to reduce bank erosion by guiding the direction of the water flow to an angle orthogonal to the downstream weir face[3,8]. Another important aspect of parabolic and trapezoidal structures is the irrigation diversions. The structure is designed to create head differences at every point of the curve that enables the delivery of the water to the head gate at low flow rates. The construction of the sluice gate is considered as a part of the weir design in order to maintain the sediment delivery back to the channel which will reduce the relative error when deriving the expression for the flow rate from the notch shape function. Another application of the Cross-Vane with parabolic design is for the bridge and channel/stream stability. If there is a bridge constructed over the channel, then over time, high flow rates can cause bank erosion which can potentially occur at the support walls of the bridge. This problem can be reduced by construction a Cross-Vane in the upstream that will act as an offset[8].

Although, the weirs are fairly beneficial and have variety of applications, their structures arelimited. There are maximum and minimum head, flow rate, angle, dimension restrictions that are considered to be standards when constructing a weir. The physical parameters restrain the use of the weirs in the channels or streams with extremely high flow rates, high heads, and/or wide areas of discharge.

5. Conclusions

Several standard equations have been considered to describe the relations between the shape of the weir, rate of flow through a particular notch, head, and hydraulic conductivity. The use of Laplace transform, convolution theorem, and other computational tools allows the study of forward and inverse problems in irrigation theory. The flow rate $r(h)$ can be determined from the notch shape $f(y)$ through straightforward integration of equation (3), which is a very stable procedure and can withstand significant perturbation. On the other hand, the process of solving for the shape of the weir from known flow rate is unstable in a sense that even a small error in the input data can result in a substantial error in the computed solution. That is where the regularized numerical algorithm for getting a more stable product of the inverse problem comes into play. Thus, the method of computational analysis can predict the outcome of the constructed weir, in particular Cipolletti and parabolic weirs, given the amount of relative error that can potentially disrupt the measurements.

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