

Traveling Wave Solutions for Foam Drainage Equation by Modified F-Expansion Method

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Abstract In this paper, using modified *F*-expansion method, we present some explicit formulas of exact traveling wave solutions for the foam drainage equation. A modified *F*-expansion method is proposed by taking full advantages of *F*-expansion method and Riccati equation in seeking exact solutions of non-linear partial differential equations.

Keywords Foam Drainage Equation, Exact Solution, Modified *F*-expansion Method

1. Introduction

Most scientific problems and physical phenomena occur nonlinearly. Except in a limited number of these problems, finding the exact analytical solutions of such problems are rather difficult. Recently, many kinds of powerful methods have been proposed to find exact solutions of nonlinear partial differential equations, e.g., the homogeneous balance method[1], homotopy analysis method[2, 3], three-wave method[4, 5, 6], extended homoclinic test approach[7, 8, 9], the $(\frac{G'}{G})$ -expansion method[10, 11] and the exp-function method[12, 13, 14].

In this paper, we consider the following foam drainage equation

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(\psi^2 - \frac{\sqrt{\psi}}{2} \frac{\partial \psi}{\partial x}) = 0, \quad (1)$$

where ψ is the cross section of a channel formed where three films meet, usually indicated as *Plateau border* and x and t are scaled position and time coordinates, respectively. Foams are of great importance in many technological processes and applications, and their properties are subject of intensive studies from both practical and scientific points of view. Liquid foam is an example of soft matter (or complex fluid) with a very well-defined structure, first clearly described by Joseph plateau in the 19th century. Foam is important in a number of everyday activities, both natural and industrial. This is why foam has been of great interest for academic research. Because of the everyday occurrence of foams, they are very well known to scientists as well as to common people[15, 16]. Foams are common in foods and

personal care products such as lotions and creams and foams often occur during cleaning of clothes and scrubbing (see, e.g.[17]). They have important applications in food and chemical industries, mineral processing, fire fighting and structural material sciences (see, for example[18]). Everyday experiences put us in direct contact with foams. Shampooing hair, washing dishes, eating chocolate bars and chocolate mousse desserts are only a few examples. History connects foams with a number of famous scientists and foam continue to excite imaginations[19]. There are now many applications of polymeric foam[20] and more recently metallic foams which are foams made out of metals such as aluminum[21]. Some popular mentioned applications include the use of foams for reducing the impact of explosions and for cleaning up oil spills. Uniformity of the foam is important for the designer interested in these applications. Gravitational drainage of the liquid is one mechanism leading to non uniformity. Polymeric foams are used in cushions, packing and structural materials[20]. Glass, ceramic and metal foams[22] can also be made. Recent research in foams has centered on three topics which are often treated separately, but are, in fact, interdependent: drainage, coarsening and rheology. We concentrate on a quantitative description of the coupling of drainage. The flow of liquid through Plateau borders (the liquid-filled channels) and intersections of four channels between the bubbles, driven by gravity and capillarity, is called foam drainage. Foams' drainage plays a very important role in foam stability. In fact, when a foam dries, its structure becomes fragile (see, for example[23]). In spite of many applications and numerous scientific investigations of properties and mechanics of foams, dynamics of foam drainage have only recently been examined in detail. Verbist studied the main features of both free drainage[24, 25]. In force drainage, a solitary wave of constant velocity is generated when liquid is added at a constant rate[26]. Hellal and Mehanna[27] used a

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semi-analytical method, that is the Adomian decomposition and the tanh method to handle the foam drainage equation (1). Also, equation (1) was studied by another authors using different methods, such as the exp-function method[28], self-similar technology[29], homotopy analysis method[30] and the variational approach[31]. In this paper, we will apply the F -expansion method on foam drainage equation (1).

2. The Modified F -Expansion Method

We simply describe the modified F-expansion method. To do this we follow descriptions which has presented in[32]. Consider a given nonlinear partial differential equation with independent variables $x = (x_1, x_2, \dots, x_l, t)$ and dependent variable u as

$$P(u, u_t, u_{x_1}, u_{x_2}, \dots) = 0. \quad (2)$$

Generally speaking, the left-hand side of (2) is a polynomial in u and its various partial derivatives. We seek its traveling wave solutions to (2) by taking

$$u(x_1, x_2, \dots, x_l, t) = u(\xi), \quad (3)$$

$$\xi = k_1(x_1 + k_2 x_2 + \dots + k_l x_l + w t)$$

where k_1, k_2, \dots, k_l and w are constants to be determined. Inserting (3) into (2) yields an ODE for $u(\xi)$

$$P(u, u', u'', u''', \dots) = 0 \quad (4)$$

conversely, we suppose that $u(\xi)$ can be expressed as

$$u(\xi) = a_0 + \sum_{i=-N}^N a_i F^i(\xi) \quad (a_N \neq 0) \quad (5)$$

where a_0 and a_i s are constants to be determined. $F(\xi)$ satisfies Riccati equation

$$F'(\xi) = A + B F(\xi) + C F^2(\xi) \quad (6)$$

where A, B, C are constants to be determined. Integer N can be determined by considering the homogeneous balance between the governing nonlinear term(s) and highest order derivatives of $u(\xi)$ in (4). Substituting (5) into (4), and using (6), then the left-hand side of (4) can be converted into a finite series in $F^p(\xi)$, ($p = -N, \dots, -1, 0, 1, \dots, N$). Equating each coefficient of $F^p(\xi)$ to zero yields a system of algebraic equations for a_i , ($i = -N, \dots, -1, 0, 1, \dots, N$) and k_j , ($j = 1, \dots, l$) and w . Then solving the system of algebraic equations, probably with the aid of Maple, a_i, k_j and w can be expressed by A, B, C (or the coefficients of ODE (4)). Upon substituting these results into (5), we can obtain the general form of traveling wave solutions to (4). From the general form of traveling wave solutions of equation (6) listed in Table 1, we can give a series of soliton-like solutions, trigonometric function solutions, and exponential function solutions to (2).

Table 1. Solutions to eq. (6) given in[32]

A	B	C	$F(\xi)$
0	1	-1	$\frac{1}{2} + \frac{1}{2} \tanh(\frac{\xi}{2})$
0	-1	1	$\frac{1}{2} - \frac{1}{2} \coth(\frac{\xi}{2})$
$\frac{1}{2}$	0	$-\frac{1}{2}$	$\coth(\xi) \pm \operatorname{csch}(\xi), \tanh(\xi) \pm \operatorname{sech}(\xi)$
1	0	-1	$\tanh(\xi), \coth(\xi)$
$\frac{1}{2}$	0	$\frac{1}{2}$	$\sec(\xi) + \tan(\xi), \csc(\xi) - \cot(\xi)$
$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\sec(\xi) - \tan(\xi), \csc(\xi) + \cot(\xi)$
$1(-1)$	0	$1(-1)$	$\tan(\xi), \cot(\xi)$
0	0	$\neq 0$	$-\frac{1}{C\xi + m}$ (m is an arbitrary constant)
arbitrary constant	0	0	$A\xi$
arbitrary constant	$\neq 0$	0	$\frac{\exp(B) - A}{B}$

3 New Exact Solutions to the Foam Drainage Equation

In this section, we apply the F -expansion method to construct the traveling wave solutions for foam drainage equation

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x} (\psi^2 - \frac{\sqrt{\psi}}{2} \frac{\partial \psi}{\partial x}) = 0. \quad (7)$$

Upon using the transformation

$$\psi = \phi(\xi), \quad \xi = k(x + wt) \quad (8)$$

where k and w are constants, eq. (7) is transferred to

$$kw \frac{\partial \phi}{\partial \xi} + k \frac{\partial}{\partial \xi} (\phi^2 - \frac{k}{2} \sqrt{\phi} \frac{\partial \phi}{\partial \xi}) = 0. \quad (9)$$

Integrating eq. (9) with respect to ξ and considering the zero constants for integration we have

$$kw \phi + k (\phi^2 - \frac{k}{2} \sqrt{\phi} \frac{\partial \phi}{\partial \xi}) = 0. \quad (10)$$

Afterwards we use the transformation

$$\phi(\xi) = u^2(\xi), \quad (11)$$

to convert eq. (10) to

$$kw u^2 + k u^4 - k^2 u^2 u' = 0, \quad (12)$$

or equivalently

$$w + u^2 - k u' = 0, \quad (13)$$

where the prime denotes differentiation with respect to ξ . By balancing the orders of u' and u^2 in eq. (13), we have $2N = N + 1$ then $N = 1$. So we can write

$$u(\xi) = a_0 + a_{-1} F^{-1}(\xi) + a_1 F(\xi) \quad (14)$$

a_0, a_1, a_{-1} are constants to be determined later. Substituting (14) into (13), and using (6), the left-hand side of (13) can be converted into a finite series in $F^p(\xi)$, ($p = -5, \dots, -1, 0, 1, \dots, 5$). Equating each coefficient of $F^p(\xi)$ to zero yields a system of algebraic equations for a_{-1}, a_0, a_1, k, w .

$$\begin{aligned} F^2: & a_1^2 - k a_1 C \\ F^1: & a_1^2 - k a_1 C \\ F^0: & a_0^2 + 2a_{-1} a_1 - k a_1 A + w + k a_{-1} C \\ F^{-1}: & 2a_0 a_{-1} + k a_{-1} B \\ F^{-2}: & a_{-1}^2 + k a_{-1} A. \end{aligned} \quad (15)$$

Solving the algebraic equations above by symbolic computations, we have the following solutions:

Case 1: when $A = 0$, we have

$$a_0 = \frac{kB}{2}, a_{-1} = 0, a_1 = Ck, k = k, w = -\frac{k^2 B^2}{4}. \quad (16)$$

Case 2: when $B = 0$, we have

$$a_0 = 0, a_{-1} = -kA, a_1 = kC, k = k, w = 4k^2 CA \quad (17)$$

$$a_0 = 0, a_{-1} = -2kA, a_1 = 0, k = k, w = k^2 CA$$

Case 3: when $A = C = 1$, we have

$$a_0 = \frac{kB}{2}, a_{-1} = 0, a_1 = k, \quad (18)$$

$$m = m, n = n, w = -\frac{k^2 B^2}{4} + k^2.$$

So we can list the solutions of u as follows:

(1) When $A = 0, B = 1, C = -1$, from Table 1 and Case 1, we have

$$u_1 = -\frac{k}{2} \tanh[\frac{1}{2} k (x - \frac{k^2}{4} t)] \quad (19)$$

setting $k = 2\sqrt{C}$ yields the same solution (20) of [28] which is also in agreement with the selection for the solutions in [27, 29].

(2) When $A = 0, B = -1, C = 1$, from Table 1 and Case 1, we have

$$u_2 = -\frac{k}{2} \coth[\frac{1}{2} k (x - \frac{k^2}{4} t)]. \quad (20)$$

For $k = \lambda$, solution (19) of [28] will obtain.

(3) When $A = \frac{1}{2}, B = 0, C = -\frac{1}{2}$, from Table 1 and Case 2, we have

$$u_3 = -\frac{k (\coth[k(x + -k^2 t)] \mp \operatorname{csch}[k(x - k^2 t)])}{2} - \frac{k}{2(\coth[k(x + -k^2 t)] \mp k \operatorname{csch}[k(x - k^2 t)])} \quad (21)$$

$$u_4 = -\frac{k (\tanh[k(x + -k^2 t)] \mp \operatorname{sech}[k(x - k^2 t)])}{2} - \frac{k}{2(\tanh[k(x + -k^2 t)] \mp \operatorname{sech}[k(x - k^2 t)])}.$$

(4) When $A = 1, B = 0, C = 1$, from Table 1 and Case 2, we have

$$u_5 = k \tan[k(x + 4k^2 t)] + k \cot[k(x + 4k^2 t)] \quad (22)$$

$$u_6 = -2k \tan[k(x + k^2 t)]$$

$$u_7 = -2k \cot[k(x + k^2 t)].$$

(5) When $A = C = \frac{1}{2}, B = 0$, from Table 1 and Case 3, we have

$$u_8 = k \sec[k(x + k^2 t)] + k \tan[k(x + k^2 t)] \quad (23)$$

$$u_9 = k \csc[k(x + k^2 t)] - k \cot[k(x + k^2 t)].$$

(6) When $A = C = -\frac{1}{2}, B = 0$, from Table 1 and Case 2, we have

Case 2, we have

$$u_{10} = k \sec[k(x + k^2 t)] - k \tan[k(x + k^2 t)] \quad (24)$$

$$u_{11} = k \csc[k(x + k^2 t)] + k \cot[k(x + k^2 t)].$$

(7) When $A = 1, b = 0, C = -1$, from Table 1 and Case 2, we have

$$u_{12} = k \tanh[k(x - 4k^2 t)] - k \coth[k(x - 4k^2 t)] \quad (25)$$

$$u_{13} = -2k \tanh[k(x - k^2 t)]$$

$$u_{14} = -2k \coth[k(x - k^2 t)].$$

4. Conclusions

Foaming occurs in many distillation and absorption processes. The drainage of liquid foams involves the interplay of gravity, surface tension and viscous forces. In this work, we have applied the F -expansion method to handle the foam drainage equation. In fact, we have presented fourteen different solutions for the nonlinear foam drainage equation, which some of them are new and interesting. Any solution has physical aspect and can be useful in industry. The current work illustrates that the F -expansion method is indeed powerful analytical technique for most types of nonlinear problems and several such problems in scientific studies and engineering may be solved by this method. In comparison of this method with exp-function method and homotopy analysis method, the F -expansion method is more concise and easier than the others. Also, we could recover the obtained solutions by exp-function and homotopy analysis methods by our solutions which have obtained in this paper.

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