Applied Mathematics 2020, 10(2): 28-33
DOI: 10.5923/j.am.20201002.02

Solving Fourth Order Boundary Value Problem by Using Extended Quantic B-spline Interpolation

Intidhar Zamil Mushtt*, Saad Shakir Mahmood, Dunya Mohamed Hamed

Mustansiriyah University, College of Education, Baghdad, Iraq

Abstract The aim of this paper is to solve the fourth order boundary value problem by using quantic b-spline where the rectangular system be solved by using singular value decomposition technique (SVD), and because the solution is not unique we used the least square optimization to optimal the numerical solution of the BVP. Numerical results are reported where we make a comparison between the exact solution and the approximate solution using the new technique.

Keywords Quantic b-spline, Fourth order BVP, Approximate solution, Exact solution, SVD

1. Introduction

Differential equations are an important tool in constructing mathematical models for physical phenomena. This modeling allows for a much clearer understanding and interpretation of the particular event. Finding the analytical and approximate solution of these models with boundary conditions thus becomes essential. Many analytical and approximate methods were developed solution of ordinary differential equation with boundary value conditions and a many these method as [1,2,3,4].

In this paper, the approximate solution of fourth order boundary value problem will be determine via Quantic B-spline, and comparisons with current studie will be made in the literature where the extended are given by the continuous least square optimization.

Since the fourth order B.V.P which is in the from:

$$z^{(4)}(t)+\alpha(t)z^{(3)}(t)+\beta(t)z''(t)+\gamma(t)z'(t)+\delta(t)z(t)=\omega(t) \tag{1}$$

Such that $\boldsymbol{x \le t \le y}$, and $\boldsymbol{t(x)=\tau_1, t(y)=\tau_2}$

Where x, y, τ_1, τ_2 are all constants, and $\alpha(t), \beta(t), \gamma(t), \delta(t), \omega(t)$ are all a continuous functions defined on interval $[x, y]$.

Hence there has been much research activity concerning B-spline for solving boundary value problem we refer the reader [5,6,7,8,9,10].

Quantic B-spline interpolation method [11]

The interval $[x, y]$ of domain has been subdivided as $x=t_0<t_1<\cdots<t_N=y$.

To provide the support for the quantic B-spline near the end boundaries, ten additional knots have been introduced as

$$t_{-5}<t_{-4}<\cdots<t_{-1}<0 \text{ and } t_N<t_{N+1}<\cdots<t_{N+5}.$$

The basis function $B_j(x), j=-2,-1,\dots,N+2$ of quantic B-spline are as:

$$B_{5,j}(t)=\frac{1}{h^5}\begin{cases}(z-z_{m-3})^5 & [z_{m-3},z_{m-2}) \\ (z-z_{m-3})^5-6(z-z_{m-2})^5 & [z_{m-2},z_{m-1}) \\ (z-z_{m-3})^5-6(z-z_{m-2})^5+15(z-z_{m-1})^5 & [z_{m-1},z_m) \\ (z_{m+3}-z)^5-6(z_{m+2}-z)^5+15(z_{m+1}-z)^5 & [z_m,z_{m+1}) \\ (z_{m+3}-z)^5-6(z_{m+3}-z)^5 & [z_{m+1},z_{m+2}) \\ (z_{m+3}-z)^5 & [z_{m+2},z_{m+3}) \\ 0 & otherwise\end{cases} \tag{2}$$

* Corresponding author:
Intidhar.Z.Mushtt@uomustansiriyah.edu.iq (Intidhar Zamil Mushtt)
Received: Sep. 16, 2020; Accepted: Oct. 12, 2020; Published: Oct. 26, 2020
Published online at http://journal.sapub.org/am

The set of quantic B-spline $B_{5,j}(t)$ form a basis over the region $x \le t \le y$, the global approximation defined using quantic B-spline:

$$U_N(t,m) = \sum_{i=-2}^{N+2} R_i B_i(t) \tag{3}$$

By [12], The nodal value of U and it's derivatives of U to fourth order are given in terms of the parameters R_i from the use of spline (2) and the trial solution (3).

$$U(t_i) = R_{i-2} + 26R_{i-1} + 60R_i + 26R_{i+1} + R_{i+2}$$

$$U'(t_i) = \frac{5}{h}[-R_{i-2} - 10R_{i-1} + 10R_{i+1} + R_{i+2}]$$

$$U''(t_i) = \frac{20}{h^2}[R_{i-2} + 2R_{i-1} - 6R_i + 2R_{i+1} + R_{i+2}]$$

$$U'''(t_i) = \frac{60}{h^3}[-R_{i-2} + 2R_{i-1} - 2R_{i+1} + R_{i+2}]$$

$$U''''(t_i) = \frac{120}{h^4}[R_{i-2} - 4R_{i-1} + 6R_i - 4R_{i+1} + R_{i+2}]$$

Definition: [13]

Any real number $r \times n$ matrix C can be decomposed as $C = ESV^T$ where E is $r \times r$ and column orthogonal (it's column are eigenvectors of $C.C^T$)

V is $n \times n$ and orthogonal (it's column are eigenvectors of $C^T.C$)

S is $r \times n$ diagonal (non-negative real values called singular values)

$S =$ diagonal $(s_1, s_2, s_3, \ldots\ldots, s_n)$ order so that $s_1 \ge s_2 \ge \cdots \ge s_n$ (if s is a singular value of C it's square is an eigenvalue of $C^T.C$).

Extended Quantic B-spline by using SVD and continuous least square error

Consider the Fourth order BVP:

$$z^{(4)}(t) + \alpha(t)z^{(3)}(t) + \beta(t)z''(t) + \gamma(t)z'(t) + \delta(t)z(t) = \omega(t)$$

On $[x,y]$ such that $z(x) = \tau_1, z(y) = \tau_2$, where $\alpha(t), \beta(t), \gamma(t), \delta(t), \omega(t)$ are a continuous function defined on $[x,y]$, and τ_1, τ_2, x, y are all constants.

The quantic B-spline is defined in equation (2)

Then

$$B'_{5,j}(x) = \frac{1}{h^5}\begin{cases} 5(z-z_{m-3})^4 & [z_{m-3}, z_{m-2}) \\ 5(z-z_{m-3})^4 - 30(z-z_{m-2})^4 & [z_{m-2}, z_{m-1}) \\ 5(z-z_{m-3})^4 - 30(z-z_{m-2})^4 + 75(z-z_{m-1})^4 & [z_{m-1}, z_m) \\ -5(z_{m+3}-z)^4 + 30(z_{m+2}-z)^4 - 75(z_{m+1}-z)^4 & [z_m, z_{m+1}) \\ -5(z_{m+3}-z)^4 + 30(z_{m+3}-z)^4 & [z_{m+1}, z_{m+2}) \\ -5(z_{m+3}-z)^4 & [z_{m+2}, z_{m+3}) \\ 0 & otherwise \end{cases}$$

And

$$B''_{5,j}(t) = \frac{1}{h^5}\begin{cases} 20(z-z_{m-3})^3 & [z_{m-3}, z_{m-2}) \\ 20(z-z_{m-3})^3 - 120(z-z_{m-2})^3 & [z_{m-2}, z_{m-1}) \\ 20(z-z_{m-3})^3 - 120(z-z_{m-2})^3 + 300(z-z_{m-1})^3 & [z_{m-1}, z_m) \\ 20(z_{m+3}-z)^3 - 120(z_{m+2}-z)^3 + 300(z_{m+1}-z)^3 & [z_m, z_{m+1}) \\ 20(z_{m+3}-z)^3 - 120(z_{m+2}-z)^3 & [z_{m+1}, z_{m+2}) \\ 20(z_{m+3}-z)^3 & [z_{m+2}, z_{m+3}) \\ 0 & otherwise \end{cases}$$

$$B'''_{5,j}(t) = \frac{1}{h^5}\begin{cases} 60(z-z_{m-3})^2 & [z_{m-3}, z_{m-2}) \\ 60(z-z_{m-3})^2 - 360(z-z_{m-2})^2 & [z_{m-2}, z_{m-1}) \\ 60(z-z_{m-3})^2 - 360(z-z_{m-2})^2 + 900(z-z_{m-1})^2 & [z_{m-1}, z_m) \\ -60(z_{m+3}-z)^2 + 360(z_{m+2}-z)^2 - 900(z_{m+1}-z)^2 & [z_m, z_{m+1}) \\ -60(z_{m+3}-z)^2 + 360(z_{m+2}-z)^2 & [z_{m+1}, z_{m+2}) \\ -60(z_{m+3}-z)^2 & [z_{m+2}, z_{m+3}) \\ 0 & otherwise \end{cases}$$

$$B^{(4)}{}_{5,j}(t) = \frac{1}{h^5}\begin{cases} 120(z - z_{m-3}) & [z_{m-3}, z_{m-2}) \\ 120(z - z_{m-3}) \quad - 720(z - z_{m-2}) & [z_{m-2}, z_{m-1}) \\ 120(z - z_{m-3}) \quad - 720(z - z_{m-2}) \quad + 1800(z - z_{m-1}) & [z_{m-1}, z_m) \\ 120(z_{m+3} - z) \quad - 720(z_{m+2} - z) \quad + 1800(z_{m+1} - z) & [z_m, z_{m+1}) \\ 120(z_{m+3} - z) \quad - 720(z_{m+2} - z) & [z_{m+1}, z_{m+2}) \\ 120(z_{m+3} - z) & [z_{m+2}, z_{m+3}) \\ 0 & otherwise \end{cases}$$

Let $z(t) = \sum_{k=-2}^{n+2} R_k B_{5,k}(t)$, be an approximate solution of equation (1) where R_k is unknown real coefficients, let t_0, $t_1, \ldots, t_n$ are $n+1$ grid points in the interval $[x, y]$ and also $t_i = x + ih, i = 0,1,..,n, t_0 = x, t_n = y, h = (y - x)/n$ in order to get a matrix of transactions that is contrary to the matrix of transaction that is contrary to the matrix from behind to make the image on the following form:

	t_{i-3}	t_{i-2}	t_{i-1}	t_i	t_{i+1}	t_{i+2}	t_{i+3}
B_i	0	1	26	66	26	1	0
B'_i	0	$5/h$	$-50/h$	0	$-50/h$	$-5/h$	0
B''_i	0	$20/h^2$	$40/h^2$	$-120/h^2$	$40/h^2$	$20/h^2$	0
B'''_i	0	$60/h^3$	$-120/h^3$	0	$120/h^3$	$-60/h^3$	0
B''''_i	0	$120/h^4$	$-480/h^4$	$720/h^4$	$-480/h^4$	$120/h^4$	0

This system can be written in matrix form as follows $C.B = A$ such that

$$C = \begin{bmatrix} 1 & 26 & 66 & 26 & 1 & 0 & \cdots & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 & c_5 & 0 & \cdots & 0 & 0 \\ 0 & c_1 & c_2 & c_3 & c_4 & c_5 & \cdots & 0 & 0 \\ 0 & 0 & c_1 & c_2 & c_3 & c_4 & c_5 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & c_1 & c_2 & c_3 & c_4 & c_5 \\ 0 & 0 & \cdots & \cdots & 1 & 26 & 66 & 26 & 1 \end{bmatrix}_{(n+3)\times(n+5)}$$

Where

$$c_1 = \frac{120}{h^4} - \frac{60}{h^3}\alpha(t) + \frac{20}{h^2}\beta(t) - \frac{5}{h}\gamma(t) + \delta(t)$$

$$c_2 = \frac{-480}{h^4} + \frac{120}{h^3}\alpha(t) + \frac{40}{h^2}\beta(t) - \frac{50}{h}\gamma(t) + 26\delta(t)$$

$$c_3 = \frac{720}{h^4} - \frac{120}{h^2}\beta(t) + 66\delta(t)$$

$$c_4 = \frac{-480}{h^4} - \frac{120}{h^3}\alpha(t) + \frac{40}{h^2}\beta(t) - \frac{50}{h}\gamma(t) + 26\delta(t)$$

$$c_5 = \frac{120}{h^4} + \frac{60}{h^3}\alpha(t) + \frac{20}{h^2}\beta(t) + \frac{5}{h}\gamma(t) + \delta(t)$$

And $B = \begin{bmatrix} R_{-2} \\ R_{-1} \\ \vdots \\ \vdots \\ \vdots \\ R_{n+1} \\ R_{n+2} \end{bmatrix}_{(n+5)\times 1}$, $A = \begin{bmatrix} z(x) = \tau_1 \\ z(t_0) = \tau^* \\ \vdots \\ \vdots \\ \vdots \\ z(t_n) = \tau^{**} \\ z(y) = \tau_2 \end{bmatrix}_{(n+3)\times 1}$

The singular decomposition of C has the form $C = ESV^T$ so

$$E = \begin{bmatrix} e_1 & \cdots & e_{n+3} \\ \vdots & \ddots & \vdots \\ e_{n+3} & \cdots & e_{2(n+3)} \end{bmatrix}_{(n+3)\times(n+3)} \text{ and } V^T = \begin{bmatrix} v_1 & \cdots & v_{n+5} \\ \vdots & \ddots & \vdots \\ v_{n+5} & \cdots & v_{2(n+5)} \end{bmatrix}_{(n+5)\times(n+5)}$$

and $S = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \dots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & s_{n+3} \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix}_{(n+3)\times(n+5)}$

So $D_{(n+3)\times 1} = E^T . A,$

let $P_i = \frac{d_i}{s_i}$ then $P = \begin{bmatrix} p_1 \\ \vdots \\ \vdots \\ p_{n+3} \\ 0 \\ 0 \end{bmatrix}_{(n+5)\times 1}$

and $B = V . P$

Hence $z_i = C_{i-2}B_{i-2} + C_{i-1}B_{i-1} + C_iB_i + C_{i+1}B_{i+1} + C_{i+2}B_{i+2} + C_{i+3}B_{i+3}$

$i = 0,1,2,\dots,n$, then we have

$z_i = R_{5,i}t^5 + R_{4,i}t^4 + R_{3,i}t^3 + R_{2,i}t^2 + R_{1,i}t + R_{0,i}$ be the solution of ODE by Quintic B-spline on the interval $[x, y]$.

The last equation can be solved by using continuous least square error to obtain the values of the constants with make the difference between the left-hand side and the right –hand side of BVP (1) is minimum.

Example 1: Consider the following ODE

$$z^{(4)} - z^{(3)} + z'' - 2z' + 3z = 6t^3 - 3t^2 - 3$$

With boundary conditions $z(0) = 1, z(1) = 6$ and the exact solution is

$$z = 1 + 3t^2 + 2t^3$$

Now let $h = 0.1, n = 10$ then $t_0 = 0, t_n = 10, \alpha(t) = -1, \beta(t) = 1, \gamma(t) = -2, \delta(t) = 3, \omega(t) = 6t^3 - 3t^2 - 3$
This system can be written as follows:-

$$CB = A$$

Such that

$$C = \begin{bmatrix} 1 & 26 & 66 & 26 & 1 & 0 & \cdots & 0 & 0 \\ 1262103 & -4674922 & 7188198 & -4676922 & 1141903 & 0 & \cdots & 0 & 0 \\ 0 & 1262103 & -4674922 & 7188198 & -4676922 & 1141903 & \cdots & 0 & 0 \\ 0 & 0 & 1262103 & -4674922 & 7188198 & -4676922 & 1141903 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1262103 & -4674922 & 7188198 & -4676922 & 1141903 \\ 0 & 0 & \cdots & \cdots & 1 & 26 & 66 & 26 & 1 \end{bmatrix}_{13\times 15}$$

And

$$A = \begin{bmatrix} 1 \\ -3 \\ -3.024 \\ -3.072 \\ -3.108 \\ -3.096 \\ -3 \\ -2.784 \\ -2.412 \\ -1.848 \\ -1.056 \\ 0 \\ 6 \end{bmatrix}_{13\times 1} \qquad B = \begin{bmatrix} R_{-2} \\ R_{-1} \\ R_0 \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \\ R_{10} \\ R_{11} \\ R_{12} \end{bmatrix}_{15\times 1},$$

Since $B = V.P$ be the values of the coefficient

$R_{-2} = -0.0014, R_{-1} = 0.0085, R_0 = 0.0094, R_1 = 0.0062$

$R_2 = 0.0024, R_3 = 0, R_4 = -0.0001, R_5 = 0.0024$

$R_6 = 0.0079, R_7 = 0.0169, R_8 = 0.0294, R_9 = 0.0436$

$R_{10} = 0.0540, R_{11} = 0.0487, R_{12} = 0.0051$

Table (1). Shows the Extended B-spline with exact solution for the example

$[t_i, t_{i+1})$	Extended B-spline z_i	Extended $z_i(t_i)$	Exact $z_i(t_i)$
[0,0.1)	$z_0 = 1 - 0.164568042660572649409934869763 19e - 15m + 3m^2 + 2m^3$	1.0078	1.0077
[0.1,0.2)	$z_1 = 1 + 0.250872790441789137863116714134 90e - 15m + 3m^2 + 2m^3$	1.0742	1.0742
[0.2,0.3)	$z_2 = 1 + 0.340483969539452736355576983961 01e - 15m + 3m^2 + 2m^3$	1.2188	1.2187
[0.3,0.4)	$z_3 = 1 + 0.251660023287358299680116764161 83e - 15m + 3m^2 + 2m^3$	1.4532	1.4532
[0.4,0.5)	$z_4 = 1. +.22225336538196842050817887e\text{-}{-15}m + 3m^2 + 2m^3$	1.7898	1.7897
[0.5,0.6)	$z_5 = 1 + 3m^2 + 2m^3$	2.2403	2.2402
[0.6,0.7)	$z_6 = 1 + 3m^2 + 2m^3$	2.8167	2.8167
[0.7,0.8)	$z_7 = 1 + 3m^2 + 2m^3$	3.5313	3.5312
[0.8,0.9)	$z_8 = 1 + 3m^2 + 2m^3$	4.3957	4.3957
[0.9,1)	$z_9 = 1 + 3m^2 + 2m^3$	5.4223	5.4222

2. Conclusions

Fourth order boundary value problem solved by using the extended quantic B-spline with continuous Least Square approximation and singular value decomposition technique. The numerical results showed that the extended quantic B-spline approximations are considered very well.

REFERENCES

[1] D. D. Demir and N. Bildik, the Numerical Solution of Heat Problem Using Cubic B-Splines, Apple. Math. 2012; 2(4): 131-135.

[2] I. Dag, D. Irk and M. Sari, the Extended Cubic B-Spline Algorithm for Modified Regularized Long Wave Equation, chin. phys B. 2013; 22(4): 040207.

[3] K. R. Raslan, T. S. El-Danaf and KH. K. Ali, Collection Method with Quintic B-Spline method for Solving the Hirota Equation, Jour of Abs. and Comp. Math. 2016; 1(1): 1-12.

[4] O. Ersoy and I.Dag, Atrigonometric Cubic B-Spline Finite Element Method for Solving the Non Linear Coupled Burger Equation, Math. NA. 2016; 15: 1-18.

[5] J. chang, Q. Yang,. Langzhao, Comparison of B-Spline Method and Finite Difference Method to Solve BVP of Linea ODEs, Jour of comp, 2011(6). 10: 2011.

[6] K. N. Viswanadham and Y. S. Raju, Quintic B-Spline Collection Method for Sixth Order Boundary Value Problems, Glob. Jour., Inc. 2012; 12(1): 1-8.

[7] F. G Lang, Quintic B-Spline Collection Method for Second Order Mixed Boundary Value Problem, Comp. phys. Com. 2012; 183(4): 913-921.

[8] M. Munguia, D. Bhatta, Use of Cubic B-Spline in Approximating Solution of Boundary Value Problems, Apple. Apple. Math. 2015; 10(2): 750-771.

[9] S. M. Reddy, Collection Method for Fifth Order Boundary Value Problem by Using Quintic B-Spline, Int. Jour of Eng. and Comp. Sci. 2016; 5 (8): 7696-17701.

[10] R. K. Lodhi and H. K. Mishra, Quintic B-Spline Method for Solving Second Order Linear and nonlinear Singularly Perturbed Two Point Boundary Value Problems, Jour of Comp. and Apple. Math, 2017. 319. (1): 170-187.

[11] S. S. Siddiqi and S. Arshed, Quintic B-Spline for the numerical Solution of Fourth-Order Parabolic Partial Differential Equation, World Apple. Sci. J. 2013. 23(12): 115-122.

[12] I. Dag, B. Saka and A. Boz, Quintic B-Spline Galerkin Method for Numerical Solution of the Burgers' Equation, Dynamical system and applications proceeding. 2004: 295-309.

[13] R. L. Burden and J. D. Fairs, Numerical Analysis, Youngstown State University, 2010.