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Fundamental Concepts of Geometry

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Abstract **Euclidean Geometry** is a mathematical system attributed to Alexandrian Greek mathematician Euclid, which he described in his textbook on geometry: the *Elements*. Euclid's method consists in assuming a small set of intuitively appealing axioms, and deducing many other propositions (theorems) from these. Although many of Euclid's results had been stated by earlier mathematicians, [1] Euclid was the first to show how these propositions could fit into a comprehensive deductive and logical system. [2] The *Elements* begins with plane geometry, still taught in secondary school (high school) as the first axiomatic system and the first examples of formal proof. It goes on to the solid geometry of three dimensions. Much of the *Elements* states results of what are now called algebra and number theory, explained in geometrical language. [1] For more than two thousand years, the adjective "Euclidean" was unnecessary because no other sort of geometry had been conceived. Euclid's axioms seemed so intuitively obvious (with the possible exception of the parallel postulate) that any theorem proved from them was deemed true in an absolute, often metaphysical, sense. Today, however, many other self-consistent non-Euclidean geometries are known, the first ones having been discovered in the early 19th century. An implication of Albert Einstein's theory of general relativity is that physical space itself is not Euclidean, and Euclidean space is a good approximation for it only over short distances (relative to the strength of the gravitational field). [3] Euclidean geometry is an example of synthetic geometry, in that it proceeds logically from axioms describing basic properties of geometric objects such as points and lines, to propositions about those objects, all without the use of coordinates to specify those objects. This is in contrast to analytic geometry, which uses coordinates to translate geometric propositions into algebraic formulas.

Keywords Point, Line, Plane, Euclidean Geometry, Hyperbolic Geometry, The Consistency of Euclidean Geometry

1. Absolute Geometry

1.1. Introduction

We shall be concerned mainly with Geometry, like arithmetic, requires for its logical development only a small number of simple, and fundamental principles. These fundamental principles are called the Axioms Of Geometry. The choice of the axioms and the investigation of their relation to one another is a problem which, since the time of Euclid, has been discussed in numerous excellent memories to be found in the mathematical literature.

Geometry is a science of shape, size and symmetry. While arithmetic dealt with numerical structures, geometry deals with metric structures. Geometry is one of the oldest mathematical disciplines and early geometry has relations with arithmetic. Geometry was also a place, where the axiomatic method was brought to mathematics: Theorems are proved from a few statements which are called axioms.

Absolute Geometry is a geometry which depends only on the first four of Euclid’s postulates and not on the parallel postulates. It is sometimes referred to as neutral geometry, as it is neutral with respect to the parallel postulates.

Let us consider three distinct systems of things. The things composing the first system, well will call points and designate them by the letter A, B, C, those of the second, we will call straight lines designate them by the letters a, b, c, and those of the third system and we will call planes and designate them by the Greek letters $\alpha, \beta, \gamma ...$. The points are called the elements of linear geometry; the points and straight lines, the elements of plane geometry; and the points, lines, and planes, the elements of the geometry of space or the elements of space.

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We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by mean of such word as “are situated”, “between”, “parallel”, “congruent”, “continuous”, etc. the complete and exact description of these relations follows as a consequence of the axioms of geometry. These axioms may be arranged in five groups.

Each of these groups expressed, by itself, certain related fundamental fact of our intuition.

I. Axioms of incidence (connection)
II. Axioms of betweenness (order)
III. Axioms of parallel (Euclid’s axiom)
IV. Axioms of congruence
V. Axioms of continuity

Although point, line and plane etc. do not have normal definition, we can describe them intuitively as follows.

Point: We represent points by dot and designate them by capital letters. (See figure 1.1.1).

Line: We represent lines by the indefinitely thin and long mark. Lines are designated by small letters. We regard lines as a set of points that can be extended as far as desired in either direction (See figure 1.1.2).

Plane: We think of a plane as a flat surface that has no depth (or thickness). We designate planes by Greek letters $\alpha, \beta, \gamma ...$ and represent it by some appropriate figure in space. (See figure 1.1.3)

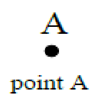


Figure 1.1.1. Point

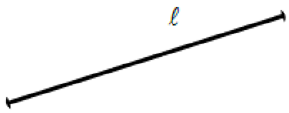


Figure 1.1.2. Line

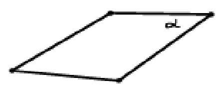


Figure 1.1.3. Line on a plane

In this subsection the axioms completely and exactly describe the properties or characteristics that the undefined elements should possess. They also state the relationships that hold among the undefined terms and the existence of some of these elements. We shall see them one by one.

1.2. Axioms of Incidence

The axioms of incidence determine the properties of mutual disposition of points, lines and planes by the term “incident”. Statement such as “a point is incident with a line”, “a point lies on the line”, a line passes through a point” and a line contains a point” are assumed to be equivalent. Thus we can use them interchangeably.

So if a point is incident with two lines then we say that they intersect at the point or the point is their common point. Analogues statement will be used for a point and a plane, and for a line and a plane.

Convention: When numbers like “two”, “three” and “four” and so on are used in any statement of this material, they will describe distinct objects. For instance by “two planes”, “three lines”, “four points” we mean “two distinct planes”, “three distinct lines”, “four distinct points”, respectively. But by line $\boldsymbol{m}$ and $\boldsymbol{n}$, we mean $\boldsymbol{m}$ and $\boldsymbol{n}$ may represent different or the same line. The same true for points and planes. The group of the axioms of incidence includes the following:

AI$_1$: If A and B are two points, then there is one and only one line ℓ that passes through them.

Figure 1.2.1. A Line through two points A & B

At least two points on any line (exactly one line through two points). This axiom asserts the existence and uniqueness of a line ℓ passing through any two given distinct points A and B.

Here ℓ can be described as a line determined by the two points A and B. We denote the line passing through A and B by $\overleftrightarrow{AB}$.

AI$_2$: Given any three different no collinear points, there is exactly one plane containing them.

For every plane there exists a point which it contains.

It follows from AI$_2$ that any three given distinct points not all on the same line determine a plane passing through the three points and there is no other plane different from this containing all the three given points.

AI$_3$: If two points A and B lie in a plane α then the line containing them lies in the plane α

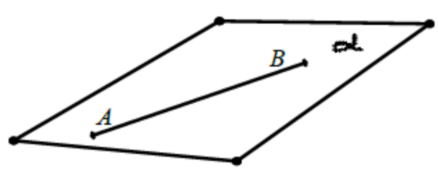


Figure 1.2.2. Two Points A and B on a plane

AI_4: If two planes intersect, then their intersection is a line.
AI_5: a) There exists at least two points which lie on a given line.
b) There exists at least three points which do not lie on a line
(Every plane contains at least three non collinear points).

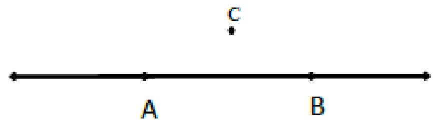


Figure 1.2.3. At least three non-collinear points

c) There exists at least four points which do not lie on a plane.

Now let us see some of the immediate consequences of this group of axioms.

Theorem 1.2.1: If m_1 and m_2 are two lines, then they have at most one point in common.

Proof: Suppose m_1 and m_2 have two points in common. Let these points be A and B. Thus both m_1 and m_2 passes through A and B.

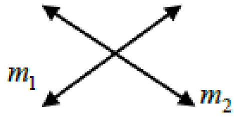


Figure 1.2.4. Two Lines having a common point

But this is impossible by AI_1. Hence they cannot have two or more points in common. Therefore they have at most one point in common.

Remark: From the above theorem it follows that two distinct lines either intersect only at one point or do not intersect.

Figure 1.2.5. Two Intersecting and Parallel Lines

Theorem 1.2.2: Two planes meet in a line or they do not meet at all.

Proof: Suppose two planes π_1 and π_2 have a point P in common. Then π_1 and π_2 have one more point Q in common by AI_4. Thus P and Q lie on π_1 and P and Q lie on π_2. But P and Q determine a unique line, say h, by AI_1. So line h lies on both π_1 and π_2 (why?), that is every point on line h is common to both π_1 and π_2.

Further they cannot have any other point not on h in common. (What will happen if they have such a point in common?). Therefore they meet in a line if they have a point in common otherwise they do not meet.

Definition 1.2.1: Three or more points are called **collinear** if and only if they lie on the same line.

Definition 1.2.2:

a) Points that lie in the same plane are called **coplanar points**.
b) Lines that lie on the same plane are called **coplanar lines.**

Notation: If P, Q and R are three non collinear points on a plane π then we denote π as PQR.

Theorem 1.2.3 Two intersect lines determine one and only one plane.

Proof: Left for students as an exercise.

Activity: Answer the following questions and the give a formal proof of theorem 1.2.3
i. How many lines are given?
ii. Are they assumed to be intersecting?
iii. What do we need to show?
iv. How many points do two intersecting lines have in common? Why?
v. Is there a point on each of these lines different from their common point? Why?
vi. How many points do we need to determine a unique plane?

Example 1.2.1

a) Prove that a line and a point not lying on it determine one and only one plane.
b) Show that there are at least four planes.

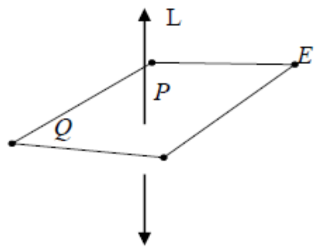
Solution:

a) Given a line ℓ and a point P not lying on ℓ. We need to show that there is one and only one plane containing P and ℓ. By AI5 (a), ℓ contains at least two points say X and Y. Points X, Y, and P are no collinear (Why?). Thus they determine a unique plane α containing them. Moreover ℓ lies completely on α (why?). so, we can conclude that P and ℓ determine plane α uniquely.

b) By AI_5 (c), there are at least four points that are not coplanar. Let us designate these points by X, Y, Z and W. Any three of them are not collinear (why?). Thus we have four planes namely X Y Z, X Y W, X Z W and YZW. This completes the proof.

Example 1.2.2:

Prove that if the line intersects a plane not containing it, then the intersection is a single point.



Proof: Let L be a line intersecting a plane E. We have given that $L \cap E$ contains at least one point P; and we need to prove that $L \cap E$ contains no other point Q. Suppose that there is a second point Q in $L \cap E$. Then $L = \overleftrightarrow{PQ}$ by theorem 1.2.1 and also $\overleftrightarrow{PQ}$ by AI_3. Therefore, L lies in E which is contradicts the hypothesis for L.

Activity: prove that a plane and a line not lying on it cannot have more than one point in common.

1. Show that there are at least 6 lines.

2. Develop model for the system described by the axioms of incidence.

(An interpretation satisfying all the five axioms)

1.3. Distance Functions and the Ruler Postulate

For most common day-to-day measurement of length, we use rulers, meter stick, or tape measures. The distance and ruler postulates formulize our basic assumptions of these items into a general geometric axiomatic system. The Ruler postulate defines a correspondence between the points on a line marking on a meter stick and the real numbers (units of measurement) in such a manner that the absolute value of the difference between the real numbers is equal to the distance between the points (measurement of the length of an object by the meter stick matches our usual Euclidean distance)

The Ruler placement postulate basically says that it does not matter how we place a meter stick to measure the distance between two points; that is, the origin (end of the meter stick) does not need to be at one of the two given points.

The Ruler Postulate

The points of a line can be placed in a correspondence with the real numbers such that:

i. To every point of the line there corresponds exactly one real number.

ii. To every real number there corresponds exactly one point of the line.

iii. The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

Note that the first and second conditions of the Ruler Postulate imply that there exist a one-to-one and onto function. As a reminder, we write the definitions for one-to-one and onto function.

Definition 1.3.1 A function f from A to B is onto B if for any b in B there is at least one such that $f(a) = b$.

Definition 1.3.2 A function f from A to B is one-to-one (1-1) if $f(x) = f(y)$ then $x = y$ for any x and y in A. (note that the contra positive of this definition can be used in writing proofs.)

Definition 1.3.3 the **line segment** determined by A and B, denoted by $\overline{AB}$, is the set of points P such that P is between A and B and the end points A and B. In other words the (line) segment (joining A and B) is $\overline{AB} = \{A, B\} \cup \{P: P \text{ is between } A \text{ and } B\}$

Definition 1.3.4 The length of segment $\overline{AB}$ denoted by AB is the distance from A to B. call the points A and B end points of $\overline{AB}$.,

Definition 1.3.5 for two segments $\overline{AB}$ and $\overline{AC}$ $AB = BC \Leftrightarrow \overline{AB} \cong \overline{BC}$.

(I.e. $\overline{AB}$ is congruent to $\overline{AC}$)

Axiom 1.3.1 (Ruler Postulate)

For every pair of points P, Q there is a number PQ called distance from P to Q. for each line ℓ there is one to one mapping

$f:\ell \mapsto R$ such that $f(P)=x$ and $f(Q)=y$ then $PQ=|x-y|$ is the value of the distance.

Definition1.3.6 Let Þ be the collection of all points. A distance (coordinate) function d is mapping from Þ × Þ into $\mathcal{R}$satisfying the following conditions:

1. The mapping d is a function, I.e. each pair of points in Þ × Þ assigned one and only one negative real number.
2. $d(P,Q)=d(Q,P)$ (for all $P,Q \in$ Þ)
3. $d(P,Q)=0$ if and only if P=Q

Theorem 1.3.1 (Triangle Inequality)

$d(P,Q)\le d(P,R)+d(R,Q)$

Existence postulate 1.3.1: The collection of all points from a non-empty set with more than (i.e at least two) points.

Lemma 1.3.1 Given any two points $P,Q \in$ Þ then there exist line containing both P and Q.

Proof: We have two cases (either $P=Q$ or $P\neq Q$)

If $P\neq Q$, then there is exactly on line $\ell=\overline{PQ}$ such that P and Q both line ℓ (incidence postulate)

If $P=Q$, then by the existence postulate there must be a second point $R\neq P$ and by incidence postulate there is a unique line ℓ that contains both P and R. Since, $P=Q$, then $Q\in\ell$. Hence there is a line ℓ that contains both P and Q.

Definition 1.3.7 A metric is a function d: Þ × Þ $\mapsto \Re$ (where Þ is the set of all points) that satisfies:

1. $d(P,Q)=d(Q,P)$ (for all $P,Q \in$ Þ)
2. $d(P,Q)\ge 0$ (for all $P,Q \in$ Þ)
3. $d(P,Q)=0$ if and only if P=Q

Theorem 1.3.2: Distance is a metric.

Proof: let P and Q be points. Then we need to show that each of the following hold:

- $PQ=QP$
- $PQ\ge 0$
- $PQ=0 \Leftrightarrow P=Q$

By lemma 1.3.1 there is line ℓ that contains both P and Q. By the ruler postulate; there is a one to one function $f:\ell\mapsto R$. Let $x=f(P)$ and $y=f(Q)$ such that the distance is given by

$$PQ=|f(P)-f(Q)|=|x-y|$$

To see (a),

$$PQ=|x-y|=|y-x|=QP$$

To see (b)

$$PQ=|x-y|\ge 0$$

To see (c)

First suppose that PQ=0. Then

$$0=PQ=|x-y|$$
$$\Rightarrow x=y$$
$$\Rightarrow P=Q$$

Where the last step follows because f is one-to-one. To verify the converse of (c) suppose that $P=Q$. Then $x=f(P)=f(Q)=y$ so that $PQ=|x-y|=0$ which verifies the converse of (c).

Example1.3.1: Let $P=(x_1,y_1)$, $Q=(x_2,y_2)$, Then $d(P,Q)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. To show that this is a metric, calculate

$$d(P,Q)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=d(P,Q)$$

This Verifies Property (1).

To get property (2), observe that the value of the square root is a non-negative number, hence the square root is defined and positive or zero.

For property (3), first assume P=Q. then $d(P,Q)=d(P,P)=\sqrt{(x_1-x_1)^2+(y_1-y_1)^2}$.

This shows that (P=Q) implies $d(P,Q)=0$. Then

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=0$$

$$(x_1-x_2)^2+(y_1-y_2)^2=0$$

If either $x_1-x_2\neq 0$ or $y_1-y_2\neq 0$ then the right hand side of equation 1.1 is non-zero. Hence $x_1=x_2$ and $y_1=y_2$, which means P=Q. Thus $d(P,Q)=0$ implies P=Q.

Example 1.3.2: In the Cartesian plane any (non-vertical) line ℓ can be described by some equation $y=mx+b$ and any vertical line $x=a$. Show that an arbitrary line in the Euclidean plane satisfies the ruler postulate.

Solution:

Let $f(x,y)=x\sqrt{1+m^2}$ if ℓ is non-vertical, and set $f(x,y)-f(a,y)=y$ if ℓ is vertical.

To see that f is a distance (coordinate) function and that this works, we need to consider each case (vertical and non-vertical) separately and to show that f is 1-1, onto, and satisfies

$PQ=|f(P)-f(Q)|$

In each case, suppose first that ℓ is non-vertical, and define f.

(a) To show that f is one-to-one, let $P=(x_1,y_1)$, $Q=(x_2,y_2)$ and suppose that $f(P)=f(Q)$

Then $x_1\sqrt{1+m^2}=x_2\sqrt{1+m^2}$

Since $\sqrt{1+m^2}\neq 0$ it can be cancelled out, giving $x_1=x_2$ Thus

$y_1=mx_1+b=mx_2+b=y_2$

Hence, f is one-to-one $(P=Q\Rightarrow f(P)=f(Q))$

(b) To show that f is onto, pick and $z\in R$ define $x=\dfrac{z}{\sqrt{1+m^2}}$ and $y=mx+b$. Then $P(x,y)\,\epsilon\,\ell$ and $f(P)=f(x,y)=x\sqrt{1+m^2}=z$. Thus f is onto.

(c) To verify the distance formula, let $P=(x,y)\,\epsilon\,\ell$ and $Q=(x,y)\,\epsilon\,\ell$. Then $y_1=mx_1+b$ and $y_2=mx_2+b$.

Hence, $PQ=d(P,Q)$

$$=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$=\sqrt{(x_2-x_1)^2+(mx_2+b-mx_1-b)^2}$$

$$=\sqrt{(x_2-x_1)^2+(m(x_2-x_1))^2}$$

$$=\sqrt{1+m^2}\,|x_2-x_1|$$

$$=\left|x_2\sqrt{1+m^2}-x_1\sqrt{1+m^2}\right|$$

$$=|y_2-y_1|$$

$$=|f(P)-f(Q)|$$

Thus, if ℓ is not a vertical line, f is a coordinate function.

Now suppose that ℓ is a vertical line with equation $x=a$ and define $f:\ell\mapsto R$ by $f(a,y)=y$.

(a) To show that f is one-to-one, let $P=(a,y_1)\in\ell$ and $Q=(a,y_2)\in\ell$, where $P\neq Q$, hence $y_1\neq y_2$ and $f(P)=y_1\neq y_2=f(Q)$. Which show that f is one-to-one $P\neq Q\Rightarrow f(P)\neq f(Q)$

(b) To show that f is onto, let $y\in R$ be any number. Then $P=(a,y)\in\ell$ and $f(P)=y$. Hence, ℓ is onto.

To verify the distance formula, $P=(a,y_1)$ and $Q=(a,y_2)$.

Then, $PQ=d(P,Q)$

$$=\sqrt{(a-a)^2+(y_2-y_1)^2}$$

$$=\sqrt{(y_2-y_1)^2}$$

$= |y_2 - y_1|$

$= |f(P) - f(Q)|$

The following theorem tells us that we can place the origin of the ruler at any place we want, and orient the ruler in any direction we want.

Theorem 1.3.3: (Ruler Placement Postulate)

For every pair of distinct points P, Q there is a coordinate function $f: \overrightarrow{PQ} \mapsto R$ such that $f(P) = 0$ and $f(Q) \geq 0$.

Lemma 1.3.2: Let $f: \ell \mapsto R$ be a coordinate function for ℓ and let $c \in R$. Then $g: \ell \mapsto R$ given by $g(p) = f(P) + c$ is also a coordinate function for ℓ

Proof: We need to show three things: g is one-to-one, onto and $PQ = |g(P) - g(Q)|$

a) Suppose $g(p) = g(Q)$. Then $f(p) + c = f(Q) + c$. So $f(p) = f(Q)$. Since, f is one-to-one, P=Q. Thus, $g(p) = g(Q) \Rightarrow P = Q$. So g is one-to-one.
b) Let $x \in R$. Since, f is onto, there exist $P \in \ell$ such that $f(p) = x - c$. So that $g(P) = f(p) + c = x$. Hence for all $x \in R$, there exist $P \in \ell$ such that $g(P) = x$. Thus, g is onto.

Activity: Verify the distance formula for the above lemma, i.e. show that $PQ = |g(P) - g(Q)|$

Lemma 1.3.3: Let $f: \ell \mapsto R$ be a coordinate function. Then, $g(x) = -f(x)$, is a coordinate function.

Proof: We need to show three things: g is one-to-one, onto and $PQ = |g(P) - g(Q)|$

a. Let $g(P) = -f(P)$. Suppose that $g(P) = g(Q)$. Then $-f(Q) = -f(P)$ hence P=Q. hence g is one-to-one.
b. Let $x \in R$. Since, f is onto, there is some point $P \in \ell$ such that $f(p) = -x$ hence, there exist $P \in \ell$ such that $g(P) = -x$ there is some point $P \in \ell$ such that $g(p) = x$.

Hence, g is onto.

The last property is left as an activity. Thus, g is a coordinate function.

Activity:

1. In lemma 1.3.3 shows the last property.
2. Show that the Euclidean distance function d satisfies the triangle inequality.

Now let us prove theorem 1.3.3

Proof: (theorem 1.3.3)

Pick any two distinct points $P \neq Q$. By the incidence postulate there is a line $\ell = \overrightarrow{PQ}$. By the ruler postulate there exists a coordinate function $g: \ell \mapsto R$. Define $c = -g(P)$. And define $h: \ell \mapsto R$ by $h(x) = g(x) + c$. Then h is a coordinate function by lemma 1.3.1. Since $h(P) = 0$, it must be the case that $h(Q) \neq 0$ because h is one-to-one. We have two cases to consider. $h(Q) > 0$ Or $h(Q) < 0$. If $h(Q) > 0$, then set $g(P) = h(P)$ and the theorem is proven.

If $h(Q) < 0$, define $g: \ell \mapsto R$ by $g(R) = -h(R)$, which is a coordinate function by lemma 1.3.2. Since, $g(P) = -h(P) = 0$ and $g(Q) = -h(Q) > 0$, we see that g has a desired properties. Fig. 1.3.1 circles that intersect in the real plane do not necessarily intersect in the rational plane.

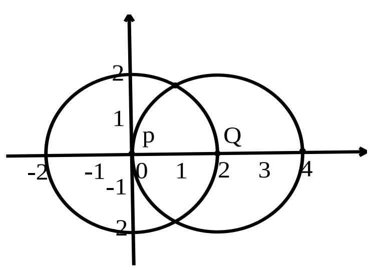


Figure 1.3.1. Two intersecting lines

The following examples illustrate why rulers (hence distance) requires real numbers and no rational numbers.

Example 1.3.3: The distance between the point (1, 0) and (0, 1) in Þ is $\sqrt{2}$

Example 1.3.4: Find the intersection of the line $y = x$ and the unit circle using whatever knowledge you may already have of circles and triangles.

Example 1.3.5: Let $P=(0,0)$ and $Q=(2,0)$. The circle of radius 2 centered at P and Q do not intersect in Q^2 (Q is a rational numbers). Their intersection in $\Re^2$ is $(1,\pm\sqrt{3})$ (fig. 1.3.1)

Remark! The Euclidean distance on $\Re^2$ where $\Re$ is the set of real numbers is given by $d(P,Q)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ for $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ for x_1,x_2,y_1,y_2 real numbers.

Activity:

Let L be a vertical line L_a. Then $P\square L_a$ implies that $P=(a,y)$ for some y. Define the standard ruler $f:L_a\to R$ by $f(a,y)=y$. Let L be a line $L_{m,b}$. Then $P\square L_{m,b}$ with $P=(x,y)$ implies that $y=mx+b$. Define the standard ruler $f:L_{m,b}\to R$ by $f(P)=f(x,y)=x\sqrt{1+m^2}$.

In the Cartesian plane:

a. Find the Euclidean distance between P=(2,5) and Q=(0,1).

b. Find the coordinate of (2, 3) with respect to the line x=2.

c. Find the coordinate of (2, 3) with respect to the line y=-4x=11.

Remark! The distance function d satisfies the triangle inequality if

$d(A,C)\le d(A,B)+d(B,C)$ for all A, B and C.

Example 1.3.6: consider the simplest non-vertical line $y=x$. The points $(0,0),(1,1),(2,2)$ *and* $(3,3)$ are on the line. What is the distance from $(0,0)$ *to* $(1,1)$, From $(1,1)$ *to* $(2,2)$?, From $(1,1)$ *to* $(3,3)$? Note the standard ruler for this line is $f(x,y)=x\sqrt{2}$. The coordinate for the four points determined by the standard ruler are $0,\sqrt{2},2\sqrt{2}$ *and* $3\sqrt{2}$ respectively.

Activity: By subtracting the appropriate coordinates of the ruler, can you obtain the distance between the points?

Example 1.3.7: Let L be the line $L_{2,3}$ (i.e. a line with slope 2 containing the point (0, 3)) in the Cartesian plane with distance function d. show that if for an arbitrary point $Q=(x,y)$, $f(Q)=5x$, then show that f is a ruler for L. also, find the coordinate of R=(1,5). We first show f satisfies the ruler equation.

Let $P=(x_1,y_1)$ and $Q=(x,y)$ be points in $L_{2,3}$.

$$
\begin{aligned}
d(P,Q)&=\sqrt{(x_1-x)^2+(y_1-y)^2} && \text{why?}\\
&=\sqrt{(x_1-x)^2+((2x_1+3)-(2x+3))^2} && \text{why?}\\
&=\sqrt{(x_1-x)^2+(2x_1-2x)^2} && \text{why?}\\
&=\sqrt{(x_1-x)^2+4(x_1-x)^2} && \text{why?}\\
&=\sqrt{5(x_1-x)^2} && \text{why?}\\
&=\sqrt{5}\,|x_1-x| && \text{why?}\\
&=\left|\sqrt{5}x_1-\sqrt{5}x\right| && \text{why?}\\
&=|f(P)-f(Q)| && \text{why?}
\end{aligned}
$$

This proves the ruler equation.

Activity:

A. Why is f bijective?

B. What is that inverse function?

C. Given a real number r, set x=5r. then find y using the equation y=2x+3

D. Find the coordinate of R

1.4. The Axiom of Betweenness

One of the simplest ideas in geometry is that of betweenness for points on the line. Here we use the undefined term "between" to establish some properties of an order relation among points on a line and plane.

Definition 1.4.1: Let A, B and C is three collinear points. If $AB+BC=AC$, then B is between A and C.

Notation: Point B is between points A and C will be denoted as A-B-C.

AB$_1$: If point B is between points A and C, then A, B, C are three distinct points on a line and B is also between C and A.

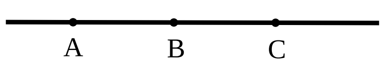


Figure 1.4.1

This axiom implies that the term ‘between’ is used only for points on a line and state that the relative position of points A and C does not affect B’s property of lying between A and C.

AB_2: If A and B are any two different points on a line h then there exist at least one point C on h such that A-C-B.

This axiom guarantees the existence of at least three points on a line.

AB_3: If A, B and C are three collinear points then one and only one of them is between the others. **AB_3** states that for any three collinear points A, B and C, exactly one of the following is true:

AB_4: Any four point on a line can be labeled in an order A, B, C and D in such a way that

A-B-C-D. As a result of **AB_4**, we have:

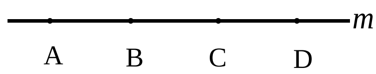


Figure 1.4.2. Four Points on a line

Definition 1.4.2: Let A and B be two points. The set of points on the line $\overleftrightarrow{AB}$ that consists of points A and B, and all points between A and B is called a line segment determined by A and B. we denote it by $\overleftrightarrow{AB}$. Points A and B are called end points of the line segment $\overleftrightarrow{AB}$. Using set notation we write $\overleftrightarrow{AB} = \{X : A - X - B\} \cup \{A, B\}$

This means that for distinct points A, B, C; B is between A and C, and write A-B-C, if $C \in \overleftrightarrow{AB}$ and AC+BC=AC

Definition 1.4.3: Let O be a point on a line ℓ. A set of points containing of point O and all points which are on one and the same side of O is called a ray. Point O is called end point of the ray. We use point O and any other points say A, on the ray to name it.

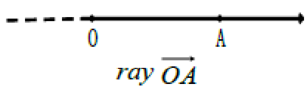


Figure 1.4.3

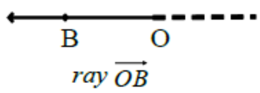


Figure 4.4.4

In short the ray (from A in the direction of B) is $\overrightarrow{AB} = \overline{AB} \cup \{P : A - B - P\}$

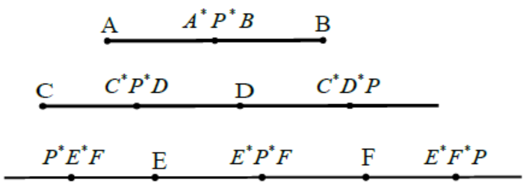


Figure 1.4.5. Betweenness on a line segment (top), ray (middle), and line (bottom)

Activity:

1. From the discussion we made so far, what do you conclude about the number of points on a line? Give justification for your answer?
2. In how many ways we can label four point P, Q, R and S on a line if P-Q-R is given.

Exercises

1. Explain why collinear is necessary in the definition of betweenness.
2. Prove that a segment has a unique midpoint.

Theorem 1.4.1: If A-B-C, then C-B-A.

Proof: We must show that CB+BA=CA

CB+BA = BA+CA	why?
=AB+AC	why?
=AC	why?
=CA	why?

Activity:

1. Suppose the intersection of $\overline{AB}$ and $\overline{CD}$ is $\overline{CB}$. Is A-C-B-D? Explain your answer.
2. In the Euclidean plane A-B-C if and only if there is a number t with $0 < t < 1$ and $B = A + t(C - A)$.

A line ℓ lying in plane π, divides the remaining points of the plane in two parts (called half planes), so that the line segment determined by two points in the same half planes doesn’t intersect ℓ, whereas the line segment determined by two points in different half planes intersect ℓ.

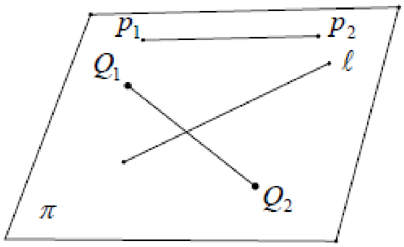


Figure 1.4.6

In fig.1.4.6, P_1 and P_2 are on the same half plane. But Q_1 and Q_2 are on different half planes.

Theorem 1.4.2: Every segment contains at least one point different form its end points.

Proof: Let A and B be end points of a segment $\overline{AB}$. From AI_5b, we have a point C not on line $\overleftrightarrow{AB}$. Now taking A and C there exist a point D on the line through A and C such that A-C-D (AI_2). Again by using AI_2, we have a point E on line $\overleftrightarrow{BD}$ such that D-B-E. Now consider line $\overleftrightarrow{EC}$. It divides a plane into two half planes thus points A and D are on different half planes and points B and D are on the same half planes. Hence, A and B are on different half planes. So $\overline{AB}$ intersects $\overleftrightarrow{EC}$ at some point, say X. point X different from A and B (why?) and X is on $\overline{AB}$. Consequently $\overline{AB}$ contains at least one point.

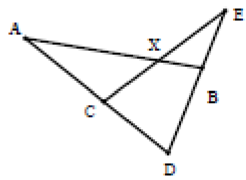


Figure 1.4.7

Activity:

1. Prove that if m is a line then there exist at least three points in the plane containing m, which does not lie on m.
2. Let A and B be two points. Does there exist a point X on the line through A and B such that A-X-B?

Theorem 1.4.3: (Betweenness theorem for points)

Let A, B, C be distinct points on the line. Let $f : \ell \mapsto R$ be a coordinate function for ℓ.

Then A-C-B if and only if either $f(A) < f(C) < f(B)$ or $f(A) > f(C) > f(B)$

Proof: Suppose that $f(A) < f(C) < f(B)$. Then

$$\begin{aligned} AC + CB &= |f(C) - f(A)| + |f(B) - f(C)| \\ &= f(C) - f(A) + f(B) - f(C) \\ &= f(B) - f(A) \\ &= AB \end{aligned}$$

So that A-C-B. A similar argument holds in $f(A) > f(C) > f(B)$.

Now consider the converse. Assume that A-C-B so that AC+CB=AB, i.e. $|f(C) - f(A)| + |f(B) - f(C)| = |f(B) - f(A)|$. But by algebra, we also have $f(C) - f(A) + f(B) - f(C) = f(B) - f(A)$. Hence, $|f(C) - f(A) + f(B) - f(C)| = |f(B) - f(A)|$. Now let $u = f(C) - f(A)$ and $v = f(B) - f(C)$. Then $|u| + |v| = |u + v|$. From algebra we know that this means that either u or v is both positive and both negative. Assume the converse. If u>0 and v<0, then this gives u-v=u+v which implies v=0 or $f(B) = f(C)$: But C and B are distinct points so $f(B) \neq f(C)$; if u<0 and v>0, then u+v=u+v which implies u=0 or $f(A) = f(C)$ which is impossible because A and C are distinct points. Since u and v have the same sign, then both $f(C) - f(A)$ and $f(B) - f(C)$ have the same sign. If both $f(C) - f(A) > 0$ and $f(B) - f(C) > 0$, then $f(C) > f(A)$ and $f(B) > f(C)$ so that $f(B) > f(C) > f(A)$ If both $f(C) - f(A) < 0$ and $f(B) - f(C) < 0$, then $f(C) < f(A)$ and $f(B) < f(C)$ so that $f(B) < f(C) < f(A)$

Corollary1.4.1: If A, B, C is distinct collinear points then exactly one of them lies between the other two.

Proof: Since A, B, C are distinct then they correspond to real numbers x, y, z. Then this is properties of real numbers, exactly one of x, y, and z lies between the other two.

Corollary 1.4.2: Let A, B, C be points such that $B \in \overrightarrow{AC}$. Then $A - B - C \Leftrightarrow AB < AC$

Proof: By theorem 1.4.5 one of the following holds

$$f(A) < f(B) < f(C) \tag{1.1}$$

$$f(A) > f(B) > f(C) \tag{1.2}$$

If 1.1 holds, then $AB = f(B) - f(A) < f(C) - f(A) = AC$

If 1.2 holds, then $AB = f(A) - f(B) < f(A) - f(C) = AC$

To prove the converse, suppose that $AB < AC$. By the corollary one of A, B, C lies between the other two. We have three possibilities A-B-C, B-A-C or A-C-B

But B-A-C is not possible $B \in \overrightarrow{AC}$ and B is distinct from A. so suppose A-C-B. Then either $f(A) < f(C) < f(B)$ or $f(A) > f(C) > f(B)$. If $f(A) > f(C) > f(B)$, then $-AC = f(C) - f(A) > f(B) - f(A) = -AB$. So $AB > AC$ this is contradiction.

If $f(A) < f(C) < f(B)$, then $AC = f(C) - f(A) < f(B) - f(A) = AB$ this is also contradiction. Hence A-C-B is not also possible. All that is left is A-B-C.

Definition 1.4.4: The point M is the midpoint of the segment $\overline{AB}$ if A-M-B and AM=MB.

Theorem 1.4.4: If A and B are distinct points then there exist a unique point M that is a midpoint of AB.

Proof: To prove existence, let f be a coordinate function for the line $\overleftrightarrow{AB}$, and define

$$x = \frac{f(A) + f(B)}{2}$$

Since, f is onto, there exist some point $M \in \overleftrightarrow{AB}$ such that $f(M) = x$. Hence, $2f(M) = f(A) + f(B)$ or $f(M) - f(B) = f(A) + f(M)$. Thus, AM=MB. To see that A-M-B, let $a = \min\{f(A), f(B)\}$ and $b = \max\{f(A), f(B)\}$. Since A and B are distinct then $a \neq b$ and we have $x = \frac{a+b}{2}$ with a<b. hence $x < \frac{2b}{2} = b$ and $x > \frac{2b}{2} = a$ giving a<x<b. hence, either $f(A) < f(M) < f(B)$ or $f(A) > f(M) > f(B)$. By theorem 1.4.3 A-M-B. To verify the uniqueness, let $M' \in \overleftrightarrow{AB}$, where $M' \neq M$ and $AM' = M'B$. Suppose that $f(A) < f(B)$.

Then both the following holds: $f(A) < f(M) < f(B)$ and $f(A) < f(M') < f(B)$. Furthermore, since M and M' are midpoints. $|f(A) - f(M)| = AM = \frac{1}{2}AB = AM' = |f(A) - f(M')|$. Since $f(A) < f(M)$ and $f(A) < f(M')$, this gives $f(M) - f(A) = f(M') - f(A)$ or $f(M) = f(M')$. Since f is one to one then $M = M'$, which proves uniqueness when $f(A) > f(B)$. If $f(A) > f(B)$, then the inequalities are reversed and we get $f(A) > f(M)$ and $f(A) > f(M')$ which leads to $f(A) - f(M) = f(A) - f(M')$. Hence, $M = M'$ by the same argument. Thus, the midpoint is unique under all cases.

Definition 1.4.5: The union of three line segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are formed by three non collinear point A, B and C is called a triangle. The points A, B and C are called vertices and segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are called sides.

We denote triangle with vertices A, B, C as ΔABC

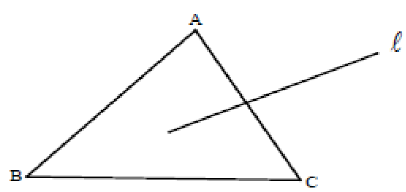


Figure 1.4.8

Theorem 1.4.5: If a line in the same plane of a triangle does not pass through any vertex of a triangle and intersects one of its sides then it intersects one and only one of the other two sides.

Proof: Let ABC be a triangle and ℓ be a line lying in the plane determined by A, B and C. suppose ℓ does not pass through any of A, B,C and intersects side AB. Then A and B are in different half-planes with respect to ℓ, Since ℓ does not pass through C, point C is in one of the two half planes. If C is in the same half plane with A then ℓ does not intersect $\overline{AC}$, but intersect $\overline{BC}$ by AII_5 (as B and C are in different half planes in this case). If C is in the same half plane with, B and ℓ

does not intersect $\overline{BC}$, but intersect $\overline{AC}$ by AII_5 (as A and C are in different half planes in this case). Consequently, in both cases ℓ intersect one and only one of the sides AC and BC of the triangle.

Activity:

1. Restate theorem 1.4.4 using the undefined term "between"
2. Prove: If A and B are two points on a line m then there exist at least three points which lie on m and are between A and B.
3. Prove: If A, B, C are three non collinear points and D, E are points such that A-B-D, B-E-C, then the line through D and E has a point in common with $\overline{AC}$

Remark:

1. We know that a line contains at least two points by AI_5. Now by using AB_2 and theorem 1.4.4 repeatedly we get the following result: a line contains infinitely many points.
2. As a line lying in a plane divides the plane into two parts called half planes, any point of a line divides the line into two parts. We call them half lines.

Let O be any point on line h. then we say that point A and B of h are on different sides of O if A-O-B, otherwise we say that they are on the same side of O.

Activity:

1. Examine possible cases in which two different rays can intersect.
2. Give your own definition for an angle. After having done this, compare your definition with that given below.

1.5. The Plane Separation Postulation

Intuitively, we know that a line divides a plane into halves. These two halves are called half-planes. We will take this observation as an axiom.

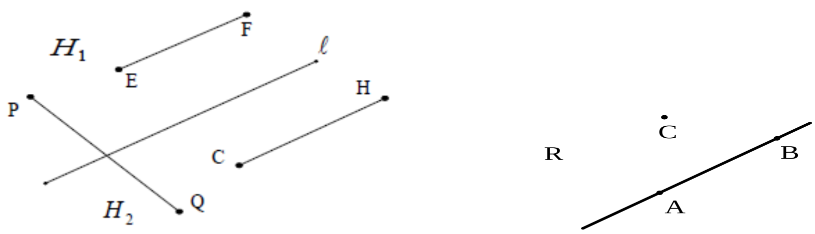
Definition 1.5.1: A set of points S is convex if for every $P, Q \in S$, the entire segment $PQ \in S$.

Convex Not convex

Figure 1.5.1

Axiom 1.5.1 (Plane separation postulation)

For every line ℓ the points that do not lie on ℓ from two disjoint convex non-empty sets H_1 and H_2 called half-planes bounded by ℓ such that if $P \in H_1$ and $Q \in H_2$ then $\overline{PQ}$ intersects ℓ.



$\overline{PQ}$ intersects the line because P and Q are in different half planes.

The shaded region represents half plane denoted by $H_{R,\ell}$ that is half plane formed by line ℓ and point R.

Figure 1.5.2

We can see that this postulate gives rise to the following notion. If both E and F lie in the same set (i.e. in the same half-plane determined by 1), then the line segment EF does not intersect 1.

In this case we say that E and F lie on the same side of 1. More specifically, the plane separation postulate tells us the following (see fig. 1.5.2).

$H_1 \cup H_2$ =the whole plane minus ℓ

$H_1 \cap H_2 = \varphi$

$(E, F \in H_1) \Rightarrow (\overline{EF} \subseteq H_1)$ and $(\overline{EF} \cap \ell) = \varphi$

$(C.H \in H_2) \Rightarrow (\overline{CH} \subseteq H_2)$ and $(\overline{CH} \cap \ell) = \varphi$

Thus, $P \in H_1$ and $Q \in H_2 \Rightarrow \overline{PQ} \cap \ell \neq \varphi$

Definition 1.5.2: Let ℓ be a line and A a point not on ℓ. Then we use $H_{A\ell}$ to denote the half-plane of ℓ that contains A. When the line is clear from the context we will just use notation H_A.

Definition 1.5.3: Two points A, B are said to be on the same side of the line ℓ if they are both in the same have-plane. They are said to be on opposite sides of the line if they are in different half planes.

In figure 1.5.2 points P and Q are on opposite sides of ℓ, while point P and F are on the same sides of ℓ. In terms of this notation, we can restate the plane separation postulate as follows.

Axiom 1.5.2: (Plane Separation Postulate)

Let ℓ be a line and A, B be points not on ℓ. Then A and B are on the same sides of ℓ if and only if $\overline{AB} \cap \ell = \varphi$ and are on opposite sides of ℓ if and only if $\overline{AB} \cap \ell \neq \varphi$

Definition 1.5.4: Two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ having the same endpoint A are opposite rays if $\overrightarrow{AB} \neq \overrightarrow{AC}$ and $\overleftrightarrow{BC} = \overrightarrow{AB} \cup \overrightarrow{AC}$

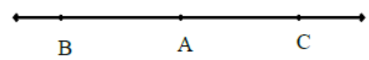


Figure 1.5.3

Definition 1.5.5: An angle is the union of two non-opposite rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ having the same endpoint, and is denoted by $\angle BAC$ or $\angle CAB$. The point A is called the vertex of the angle and the two rays are called the sides of the angle.

Definition 1.5.6: Let A, B, C be points such that the rays $\overrightarrow{AB} \neq \overrightarrow{AC}$ are not opposite. The interior of $\angle BAC$ is $H_{H,\overleftrightarrow{AC}} \cap H_{C,\overleftrightarrow{AB}}$ (i.e. the intersection of the two half-planes)

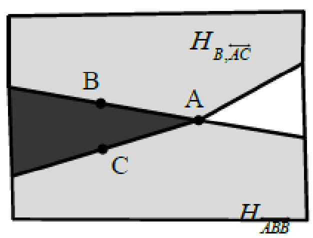


Figure 1.5.4. The interior of the angle $\angle BAC$ is the intersection of the two half planes and shaded darker

Definition 1.5.7: Three points A, B, C are collinear if there exists a single line ℓ such that A, B and C all lie on ℓ. If no such line exists, then the points are non-collinear.

Corollary 1.5.1: If A, B and C are non-collinear, then the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are neither opposite nor equal.

Definition 1.5.8: Let A, B and C is non-collinear points. Then the triangle ΔABC is the union of the three segments

$$\Delta ABC = \overline{AB} + \overline{BC} + \overline{CA}$$

The points A, B and C are called the vertices of the triangle, and the segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are called the sides of the triangle.

Theorem 1.5.1: (Pasch's theorem)

Let ΔABC be a triangle and suppose that ℓ is a line that does not include A, B or C. then if ℓ intersects $\overline{AB}$ then it also intersects either $\overline{BC}$ or $\overline{AC}$.

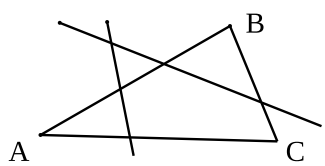


Figure 1.5.5. Any line that intersects $\overline{AB}$ must intersect either $\overline{CA}$ or $\overline{BC}$

Proof: Suppose that ℓ intersects $\overline{AB}$ and does not include any of the vertices A, B or C. let H_1 and H_2 be the two half planes determined by ℓ. Then the points A and B are in opposite half planes by the plane separation postulate and by hypothesis.

Suppose $A \in H_1$ and $B \in H_2$ (this is just notation; we could have made the alternative assignment without any loss of generality) then either $C \in H_1$ or $C \in H_2$.

If $C \in H_1$, then B and C are in opposite half- planes. So $\overline{BC}$ intersects ℓ by the plane separation postulate.

Alternatively, if $B \in H_2$, then A and C are in opposite half- planes. So $\overline{BC}$ intersects ℓ by the plane separation postulate. $\overline{AC}$

Activity: What needs to be added so that we can define the interior of triangle ABC?

1.6. Angular Measures

Recall that an angle is the union of two rays with common end point. The common end point is called the vertex; the two rays are called sides of the angle. (See fig.1.6.1)

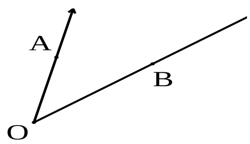


Figure 1.6.1

Notation: The angle which is the union of the two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is denoted as $A\hat{O}B$ or $\angle AOB$.

Definition 1.6.1:

i. The interior of an angle $A\hat{O}B$ is the intersection of

a. The half plane determined by the line $\overleftrightarrow{OA}$ which contains B and

b. The half plane determined by the line $\overleftrightarrow{OB}$ which contains A.(see figure 1.6.2)

The interior of an angle $A\hat{O}B$ will be denoted by int($A\hat{O}B$)

ii. The exterior of an angle $A\hat{O}B$ is the set of all points which are neither on $A\hat{O}B$ nor int ($A\hat{O}B$). The exterior of an angle $A\hat{O}B$ will be denoted by ext($A\hat{O}B$). (see figure 1.6.3)

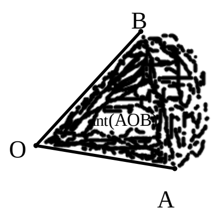


Figure 1.6.2

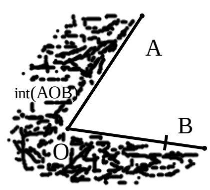


Figure 1.6.3

Remarks:

1. A line segment formed by any two points in the $\text{int}(A\ddot{O}B)$ does not intersect AOB
   (That is it lies completely in this region). A line segment whose one end point lies in the $\text{int}(A\ddot{O}B)$
2. In a triangle, an angle will be referred to as being included between two sides when its sides contain those sides of the triangle. A side will be referred to as being included between two angles when its end points are the vertices of those angles of the triangle.

In view of this, in ΔABC, $\overline{AB}$, $\hat{A}$ and $\overline{AC}$ are two sides and the included angle, while $\overline{AB}$, $\hat{A}$ and $\hat{B}$ are two angles and the included sides. Can you mention two more triplets of

i. Two sides and the included angle

ii. Two angles and the include side.

Theorem 1.6.1: (Angle addition theorem)

If $A\hat{B}C$ and $D\hat{E}F$ are angles such that G and H are in the $\text{int}(A\hat{B}C)$ and $\text{int}(D\hat{E}F)$ respectively. $A\hat{B}G \equiv D\hat{E}H$ and $G\hat{B}C \equiv H\hat{E}F$ then $A\hat{B}C \equiv D\hat{E}F$.

Proof: Suppose $A\hat{B}G \equiv D\hat{E}H$ And $G\hat{B}C \equiv H\hat{E}F$ with G in the $\text{int}(A\hat{B}C)$. And H in the $\text{int}(D\hat{E}F)$

To show that $A\hat{B}C \equiv D\hat{E}F$

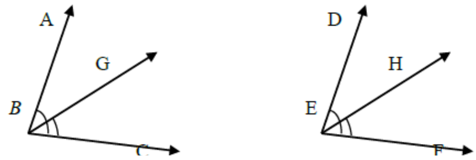


Figure 1.6.4

Consider $A\hat{B}C$ and ray $\overrightarrow{ED}$. Then by axiom of angle construction, there exists a point I on the half plane determined by line $\overleftrightarrow{ED}$ containing H and F such that $A\hat{B}C \equiv D\hat{E}I$. Moreover, we have only one and only one ray $\overrightarrow{EI}$ satisfying this condition.

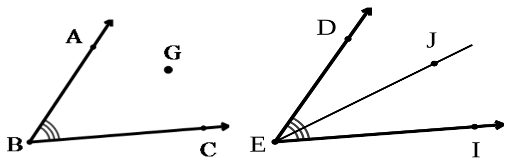


Figure 1.6.5

Now since G is in the $\text{int}(A\hat{B}C)$ and $A\hat{B}C \equiv D\hat{E}I$, there exists a unique ray $\overrightarrow{EJ}$ with J in the $\text{int}(D\hat{E}I)$ such that $A\hat{B}G \equiv D\hat{E}J$ and $G\hat{B}C \equiv J\hat{E}I$. Why?

So, $G\hat{B}C \equiv D\hat{E}H$ (hypothesis of the theorem) and $A\hat{B}G \equiv D\hat{E}J$. Thus, $\overleftrightarrow{EH}$ and $\overrightarrow{EJ}$ cannot be two distinct rays by axiom of angle construction, as H and J are on the same half plane determined by $\overleftrightarrow{ED}$ and $D\hat{E}H \equiv A\hat{B}G \equiv D\hat{E}J$. That is $\overleftrightarrow{EH}$ and $\overrightarrow{EJ}$ represents the same ray. Points F and I are on one and the same half plane determined by $\overleftrightarrow{EH}$ and $\overrightarrow{EJ}$. (Why?). Again from $G\hat{B}C \equiv H\hat{E}F$, $G\hat{B}C \equiv J\hat{E}I$ and $\overleftrightarrow{EH}$ is the same as that of $\overrightarrow{EJ}$, it follows that $\overrightarrow{EI}$ and $\overrightarrow{EF}$ represent the same ray by (why?). Hence, $D\hat{E}I \equiv D\hat{E}F$. Therefore, $A\hat{B}C \equiv D\hat{E}F$ by AC4 (as $A\hat{B}C \equiv D\hat{E}I$ and $D\hat{E}I \equiv D\hat{E}F$)

Theorem 1.6.2: (Angle subtraction theorem)

If $A\hat{B}C$ and $D\hat{E}F$ are angles such that point G in the $\text{int}(A\hat{B}C)$, point H in the $\text{int}(D\hat{E}F)$, $A\hat{B}G \equiv D\hat{E}H$ and $A\hat{B}C \equiv D\hat{E}F$, then $G\hat{B}C \equiv H\hat{E}F$.

Proof: Left as an exercise.

Activity:

1. In fig. 1.6.6, $AE \equiv DE$, $BE \equiv CE$ and $AEB \equiv DEC$. Prove that $ABD \equiv DCA$. Use SAS theorem, angle addition and subtraction theorems.

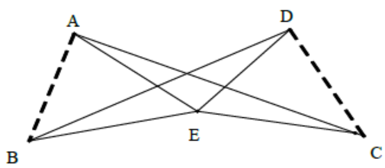


Figure 1.6.6

2. You are familiar with certain pairs of angles like adjacent angles, supplementary angles, vertically opposite angles and so on using the undefined terms, axioms and theorem so far discussed give your own definition for each of them.

Definition 1.6.2: Two angles are said to be adjacent if and only if they have the same vertex, one side in common and neither contains an interior parts of the other.

Definition 1.6.3: Two angles which are congruent, respectively, to two adjacent angles whose non-common sides form a straight line are called supplementary angles. Each of a pair of supplementary angles is called the supplement of the other.

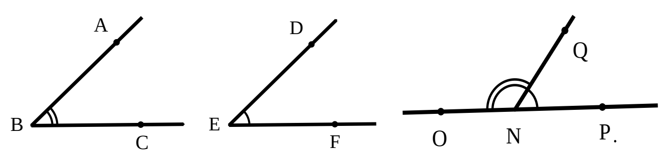


Figure 1.6.7

$$\left.\begin{array}{l} A\hat{B}C \equiv O\hat{N}Q \\ D\hat{E}F \equiv P\hat{N}Q \\ P, N, O \text{collinear} \end{array}\right\} \Rightarrow A\hat{B}C \text{ and } D\hat{E}F \text{ are supplementary angles.}$$

When do you say that two adjacent angles are supplementary?

Definition 1.6.4: Non adjacent angles formed by two intersecting lines are called vertical angles.

Definition 1.6.5: An angle is said to be a right angle if and only if it is congruent to its supplementary angle. An angle whose two sides form a straight line is called straight angle.

Illustration: In fig. 1.6.8, if lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at O, then $U\hat{O}Q$ and $Q\hat{O}T$ are adjacent angles, $Q\hat{O}T$ and $T\hat{O}P$ are supplementary angles, $P\hat{O}R$ and $Q\hat{O}S$ are vertical angles and $P\hat{O}Q$ is a straight angle. Can you list some more pairs of adjacent, supplementary and vertical angles?

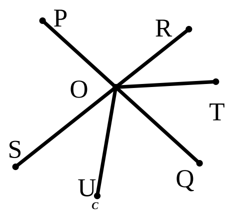


Figure 1.6.8

So far we examined different relationships that exist between line segments and between angles. Some of these relationships are expressed in terms of the undefined terms "between" and "congruence".

In the following groups of axioms we will investigate further relationships between angles and between line segments in terms of the notion of equality. For this we first raise the following question: what is wrong if we say

i. $\overline{AB} \equiv \overline{CD}$

ii. $X\hat{Y}Z \equiv R\hat{S}T$

1.7. Axioms of Congruence

So far we have seen two groups of axioms. These are axioms of incidence, order axiom. Each consists of a number of axioms that characterize the undefined terms (e.g. point, line and plane) using the undefined relations 'incident', "between" and so on. By using these axioms, we have stated and proved some properties concerning the undefined terms and the relations that exist among them. In the same manner we continue our discussion on Euclidean geometry with a study of the ideas of congruence. The undefined term congruence will be examined relative to segments, angles and triangles.

Notations: We use the symbol "$\equiv$" to mean is congruent to and "$\not\equiv$" not congruent.

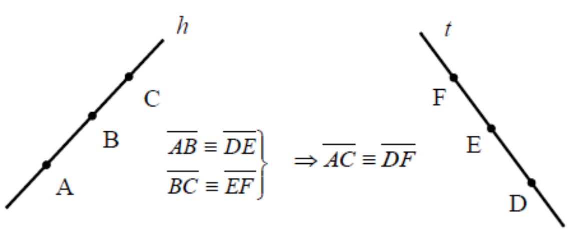
AC$_1$:

a) If $\overline{AB}$ is a line segment then $\overline{AB} \equiv \overline{AB}$ (Reflexivity)

b) If $\overline{AB}$ and $\overline{CD}$, are line segment such that $\overline{AB} \equiv \overline{CD}$, then $\overline{CD} \equiv \overline{AB}$ (symmetry)

c) If $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$ are line segments such that $\overline{AB} \equiv \overline{CD}$ and $\overline{CD} \equiv \overline{EF}$, then $\overline{AB} \equiv \overline{EF}$ (Transitivity)

AC$_2$: If A, B, C, D, E, F are points such that A-B-C, D-E-F, $\overline{AB} \equiv \overline{DE}$ and $\overline{BC} \equiv \overline{EF}$ then $\overline{AC} \equiv \overline{DF}$.
(Axiom of addition of segment)



$$\left.\begin{array}{l} \overline{AB} \equiv \overline{DE} \\ \overline{BC} \equiv \overline{EF} \end{array}\right\} \Rightarrow \overline{AC} \equiv \overline{DF}$$

Figure 1.7.1

AC$_3$: If $\overline{AB}$ is a line segment and C is a point on a line ℓ then there exists on ℓ on one side of C exactly one point D such that $\overline{AB} \equiv \overline{CD}$. (Axiom of segment construction). Whenever we have given a line segment $\overline{XY}$ and a point W on a line ℓ, AC_3 enables us to conclude the existence of a unique line segment $\overline{WZ}$ on either of the two sides of W on ℓ such that $\overline{XY} \equiv \overline{WZ}$. That is if U and V are on the same side of W on ℓ such that $\overline{XY} \equiv \overline{WU}$ and $\overline{XY} \equiv \overline{WV}$, then U and V must represent the same point i.e. U=V.

AC_4:

a) If $A\hat{B}C$ is an angle then $A\hat{B}C \equiv A\hat{B}C$ (Reflexivity)

b) If $A\hat{B}C$ and $D\hat{E}F$ are angles such that $A\hat{B}C \equiv D\hat{E}F$ then $D\hat{E}F \equiv A\hat{B}C$ (symmetry)

c) If $A\hat{B}C$, $D\hat{E}F$, $G\hat{H}I$ are angles such that $A\hat{B}C \equiv D\hat{E}F$ then $D\hat{E}F \equiv G\hat{H}I$, then $A\hat{B}C \equiv D\hat{H}I$ (Transitivity)

AC_5: If $A\hat{B}C$ is an angle and ℓ is a line on any plane and $\overrightarrow{ED}$ is a ray on ℓ the there is one and only one ray $\overrightarrow{EF}$ whose all points except E lie on one of the two half-planes determined by ℓ such that $A\hat{B}C \equiv D\hat{E}F$. (Axiom of angle construction)

Given $A\hat{B}C$ and ray $\overrightarrow{ED}$ on line ℓ (see fig. 1.7.2). There exists exactly one angle on each side of ℓ congruent to $A\hat{B}C$. That is it is not possible for $A\hat{B}C$ to be congruent to $D\hat{E}F$ and $D\hat{E}G$, where F and G are on the same half-plane determined by ℓ, unless F=G. (see fig. 1.7.3)

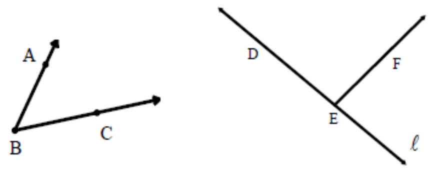


Figure 1.7.2

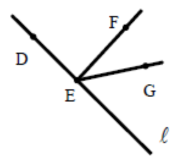


Figure 1.7.3

AC_6: If ΔABC and ΔDEF, $\overline{AB} \equiv \overline{DE}$, $A\hat{B}C \equiv D\hat{E}F$ and $\overline{BC} \equiv \overline{EF}$, then $B\hat{A}C \equiv E\hat{D}F$ and. That is then $\hat{A} \equiv \hat{D}$ and $\hat{C} \equiv \hat{F}$

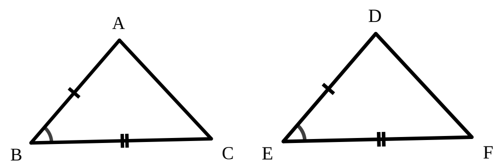


Figure 1.7.4

By using the axioms of congruence we will prove several theorems concerning congruence of segments, angles and triangles. We first prove two theorems about congruence of segments.

Theorem 1.7.1: If A, B, C, D, E, and F are points such that A-B-C, D-E-F, $\overline{AB} \equiv \overline{DE}$ and $\overline{AC} \equiv \overline{DF}$ then $\overline{BC} \equiv \overline{EF}$.

Proof: Suppose $\overline{BC}$ is not congruent to $\overline{EF}$. Then by AC_3 there exist a point G on ray $\overrightarrow{EF}$ different from E and F such that $\overline{BC} \equiv \overline{EG}$. Hence either E-G-F or E-F-G. in both cases we have D-E-G. Now from $\overline{AB} \equiv \overline{DE}$, $\overline{BC} \equiv \overline{EF}$, A-B-C and D-E-G it follows that $\overline{AC} \equiv \overline{DG}$ by AC_2. But $\overline{AC} \equiv \overline{DF}$ by assumption. So G=F by AC_3. Thus we have $F \neq G$ and $F = G$ (contrary). Therefore, the supposition $\overline{BC}$ is not congruent to $\overline{EF}$ is false. Consequently, $\overline{BC} \equiv \overline{EF}$.

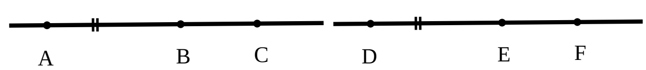


Figure 1.7.5

Theorem 1.7.2: If A, B, C, D, E are points such that A-B-C and $\overline{AC} \equiv \overline{DE}$, then there exists exactly one point X such that $\overline{AB} \equiv \overline{DX}$ and D-X- E.

Proof: Suppose A, B, C, D, E are points such that A-B-C and $\overline{AC} \equiv \overline{DE}$. Then $\overline{AB}$ is not congruent to $\overline{DE}$ by AC_3 as $\overline{DE} \equiv \overline{AC}$ and B, C are on the same side of A on line $\overleftrightarrow{AC}$.

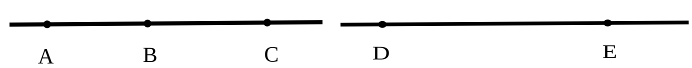


Figure 1.7.6

Thus there exists a unique point X on ray $\overrightarrow{DE}$ such that $\overline{AB} \equiv \overline{DX}$ by AC_3. Again by using AC_3 there exists a point G on the line through D and E such that D-X-G and $\overline{BC} \equiv \overline{XG}$. But A-B-C, D-X-G, $\overline{AB} \equiv \overline{DX}$, $\overline{BC} \equiv \overline{XG}$ implies $\overline{AC} \equiv \overline{DG}$. From $\overline{AC} \equiv \overline{DE}$, $\overline{AC} \equiv \overline{DG}$ and G, E are on the line through D and E on the same side of D it follows that E=G by AC_3. Therefore, there exists exactly one point X such that $\overline{AB} \equiv \overline{DX}$ and D-X-E. one can prove the following statement by using theorem 1.7.1 and 1.7.2. we state it as a corollary, as it is an immediate consequence of the two theorems.

Corollary 1.7.1: Given two congruent segments $\overline{XZ}$ and $\overline{PR}$. If Y is any point on $\overline{XZ}$ different from X and Z then there exists a unique point Q on $\overline{PR}$ different from P and R such that $\overline{XY} \equiv \overline{PQ}$ and $\overline{YZ} \equiv \overline{QR}$

Proof: Left for students.

Activity:

1. Explain why the following statements are not necessarily true.
   a. If $\overline{PQ} \equiv \overline{ST}$ and $\overline{QR} \equiv \overline{TU}$, then $\overline{PR} \equiv \overline{SU}$
   b. AIV_3 asserts that there is exactly one line segment on a given line that is congruent to a given line segment.
   c. Given angle $D\hat{E}F$ and a line h containing point O. then we have at most two angles whose vertex is O and congruent to $D\hat{E}F$.
2. Complete the proof of the above corollary
   1. $\overline{XZ} \equiv \overline{PR}$ ………………..hypothesis
   2. X-Y-Z…………………….hypothesis
   3. ………………..by theorem 1.7.2 and steps 1 and 2
   4. ……………….. ………………

   Therefore, …………………

Now let us deal with some basic points about congruent triangles. We shall discuss more about triangles in chapter two.

Recall that a triangle is defined as a set of points that lie on three segments which are formed by three non collinear points. That is if A, B and C are three non collinear points then ΔABC is the union of the segments $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$. So every triangle has three vertices, three sides and three angles. Thus we can establish a one-to-one correspondence among the vertices, sides and angles of any two given triangles ABC and DEF. we denote this by $\Delta ABC \leftrightarrow \Delta DEF$ and we have the following correspondence

1.8. Congruence between Triangles

So far, we have proved a theorem called side angle side (SAS) congruence theorem. In this section we will prove theorems, like SAS, that are concerned with conditions which cause one triangle to be congruent to another triangle. These theorems, which deal with conditions for triangle congruence, will be used to establish several theorems in this material. First let us restate SAS theorem.

Restatement: (Side angle side theorem)

If two sides and the including angle of one triangle are congruent, respectively, to two sides and the including angle of another triangle, then the triangles are congruent.

Theorem (ASA)

If two angles and the including side of one triangle are congruent, respectively, to two angles and the including angle of another triangle, then the triangles are congruent.

Proof: let ΔABC and ΔDEF be triangles such that $\angle CAB \equiv \angle FDE$, $\angle ABC \equiv \angle DEF$, and $\overline{AB} \equiv \overline{DE}$. We need to show $\Delta ABC \equiv \Delta DEF$

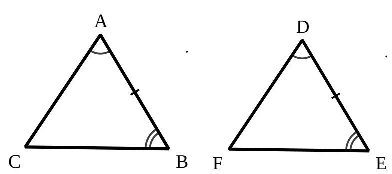


Figure 1.8.1

For this suffices to show that $\overline{BC} \equiv \overline{EF}$. (Why?). Now consider $\overline{EF}$. Then there exists a unique point G on the ray $\overrightarrow{BC}$ such that $\overline{BG} = \overline{EF}$ by AC_2. Thus in ΔABG and ΔDEF, we have $\overline{AB} \equiv \overline{DE}$, $A\hat{B}G \equiv D\hat{E}F$ and $\overline{BG} \equiv \overline{EF}$. So $\Delta ABG \equiv \Delta DEF$ by SAS, which in turn implies, $B\hat{A}G \equiv E\hat{D}F$, by definition of congruence of triangles. Since $B\hat{A}G \equiv E\hat{D}F$, $B\hat{A}C \equiv E\hat{D}F$ and $B\hat{A}G$ and $B\hat{A}C$ are on the same half plane determined by $\overleftrightarrow{AB}$, ray $\overrightarrow{AC}$ must be the same as $\overrightarrow{AG}$ by AC_5 and hence G=C as they are on the same line on the same side of B. therefore we have $\overline{BC} \equiv \overline{EF}$ (as $\overline{BG} \equiv \overline{EF}$ and G=C), $A\hat{B}C \equiv D\hat{E}F$ and $\overline{AB} \equiv \overline{DE}$ (hypothesis). Consequently, $\Delta ABC \equiv \Delta DEF$, by SAS.

Theorem 1.8.2: (SSS)

If the three sides of one triangle are congruent, respectively, to the three sides of another triangle then the triangles are congruent.

Proof: Let ABC and DEF be triangles such that, $\overline{AB} \equiv \overline{DE}$, $\overline{BC} \equiv \overline{EF}$ and $\overline{AC} \equiv \overline{DF}$. To show that $\Delta ABC \equiv \Delta DEF$

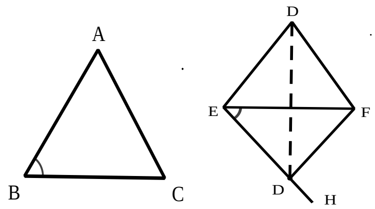


Figure 1.8.2

On the half plane determine by not containing D, there exists a point H such that $C\hat{B}A \equiv F\hat{E}H$ by AC_5. Mark point D' on $\overrightarrow{EH}$ so that $\overline{BA} \equiv \overline{ED'}$, this is possible by axiom of segment construction (AC_2) Further $C\hat{B}A \equiv F\hat{E}D'$ why? Thus in ΔABC and $\Delta D'EF$ we have $\overline{AB} \equiv \overline{D'E}$, $A\hat{B}C \equiv D'\hat{E}F$ and $\overline{BC} \equiv \overline{EF}$ which implies $\Delta ABC \equiv \Delta D'EF$ by SAS. It then follows from the definition of congruence of triangle that $\overline{AC} \equiv \overline{D'F}$, $B\hat{A}C \equiv E\hat{D}'F$ and $A\hat{C}B \equiv D\hat{F}E$. Now since $\overline{DE} \equiv \overline{D'E}$ and $\overline{DF} \equiv \overline{D'F}$, $\Delta DED'$ and $\Delta DFD'$ are isosceles. Hence $E\hat{D}D' \equiv E\hat{D}'D$ and $F\hat{D}D' \equiv F\hat{D}'D$ (base angle of isosceles triangles are congruent). So, $E\hat{D}F \equiv E\hat{D}'F$, by angle addition theorem. But, $B\hat{A}C \equiv E\hat{D}'F$ and hence $B\hat{A}C \equiv E\hat{D}F$ by AC4. Thus from, $\overline{AB} \equiv \overline{DE}$, $B\hat{A}C \equiv E\hat{D}F$ and $\overline{AC} \equiv \overline{DF}$ it follow that $\Delta ABC \equiv \Delta DEF$ by SAS.

Theorem 1.8.3: If two angles of a triangle are congruent then the sides opposite these angles are congruent.

Activity:

1. Let ΔABC be isosceles such that $\overline{AB} \equiv \overline{AC}$ and D is the midpoint of $\overline{BC}$. Use SSS congruence theorem to show that $\overline{AD} \perp \overline{BC}$ and $\overline{AD}$ is the bisector of $B\hat{A}C$
2. Prove theorem 1.8.3 by using ASA theorem.

The proof of the following theorem follows identically the same pattern as that used for proof of the SSS theorem.

Theorem 1.8.4: (RHS)

If the hypotenuse and a leg of a right triangle are respectively congruent to the hypotenuse and a leg of another right triangle then the two triangles are congruent.

Proof: Exercise

Illustration: In fig. 1.8.3 D is the midpoint of $\overline{BC}$, $\overline{DE} \equiv \overline{DF}$, $\overleftrightarrow{DE} \perp \overleftrightarrow{AC}$ and $\overleftrightarrow{DF} \perp \overleftrightarrow{AB}$. Prove that $\overline{AB} \equiv \overline{AC}$.

Proof: $D\hat{F}B$ is a right angle as $\overleftrightarrow{DF} \perp \overleftrightarrow{AB}$ and $D\hat{E}C$ is a right angle as $\overleftrightarrow{DE} \perp \overleftrightarrow{AC}$. Thus ΔDFB and ΔDEC are right angle with right angles at F and E. moreover, $\overline{DF} \equiv \overline{DE}$ by assumption and $\overline{DB} \equiv \overline{DC}$ as D is the midpoint of $\overline{BC}$. Therefore, $\Delta DFB \equiv \Delta DEC$, by, RHS and hence $D\hat{B}F \equiv D\hat{C}E$ by definition of congruence of triangles. That is $C\hat{B}E \equiv B\hat{C}A$.

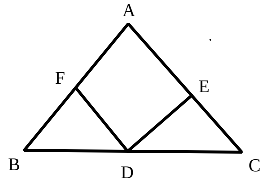


Figure 1.8.3

Activity: Prove that the perpendicular line segment from the vertex to the base of an isosceles triangle

a. Bisect the vertex angle

b. Divides the base in to two congruent segments

We are left with one theorem on congruence of triangles. Before dealing with this theorem, we state and prove a theorem about a line perpendicular to a given line through a given point not on the given line. Suppose ℓ is a line and P is a point not on ℓ. By parallel axiom there exists a unique line through P parallel to ℓ. What about a line through P perpendicular to ℓ?

Theorem 1.8.5: If m is a line and A is a point not on m, then there exists exactly one line which contains A and is perpendicular to m.

Proof: Suppose m is a line and A is a point not on m. First let us show that there exists at least one line through A perpendicular to m. Since a line contains at least two points, there exists a point B and C on m. Since A is not on m, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are two different rays. Thus by angle construction axiom there exist a point E on the half plane determined by m not containing A such that $A\hat{B}C \equiv E\hat{B}C$.

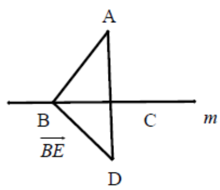


Figure 1.8.4

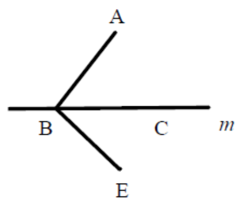
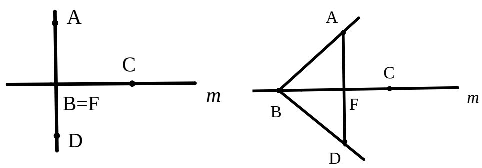


Figure 1.8.5

By segment construction axiom there exist point D on $\overrightarrow{BE}$ such that $\overline{BA} \equiv \overline{BD}$. $A\hat{B}C \equiv D\hat{B}C$ as rays $\overline{BD}$ and $\overrightarrow{BE}$ are identical. Since A and D are on different half planes determined by m, $\overline{AD}$ intersects m at some point F. now there are two possibilities: $F = B$ or $F \neq B$

Case 1: If F=B, then $\overleftrightarrow{AF} \perp m$ Case 2: If $F \neq B$, then $\overleftrightarrow{AF} \perp m$



Case 1 *Case 2*

Figure 1.8.6

(You will be asked to prove case 1 and 2, as an activity.)

From the above step it follows that, there exist at least one line through A perpendicular to m. Thus it remains to show that there does not exist more than one line which contains A and perpendicular to m. To do this, suppose h (whose existence is shown above) is a line through A perpendicular to m at O.

Now, let h' be any other line through A perpendicular to m at Q different from O. Then there exists a point A' on h such that A-O- A' and $\overline{AO} \equiv \overline{A'O}$ (why?). Points Q, A, A' are not collinear, (why?). $\Delta AOQ \equiv \Delta A'OQ$ by SAS and hence $A\hat{Q}O \equiv A'\hat{Q}O$. But $A\hat{Q}O$ is a right angle as $h' \perp m$ at Q. so $A'\hat{Q}O$ is also a right angle. This implies A, Q, A' lie on the same line contraction to that A, Q and A' are not collinear.

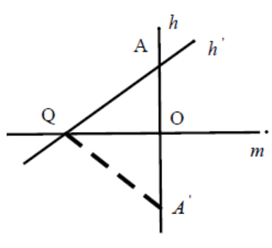


Figure 1.8.7

Therefore there does not exist a line h' through A different from h perpendicular to m. Consequently only one line exists through A perpendicular to m. In this theorem, the given point is not on the given line. What about if the given point is on the given line? In this case also we have the same result. We put it below as a theorem.

Theorem 1.8.6: Through the given point on a given line there exists one and only one line that is perpendicular to the given line.

Proof: Left as an exercise.

Activity: Complete the proof of theorem 1.8.5 (prove case 1 and 2)

Theorem 1.8.7: (RHA)

If the hypotenuse and a non right angle of one right triangle are respectively congruent to the hypotenuse and a non right angle of another right triangle then the two triangles are congruent.

Proof: Let ABC and XYZ be two right triangles with right angle at C and Z respectively such that $\overline{AB} \equiv \overline{XY}$ and $A\hat{B}C \equiv X\hat{Y}Z$.

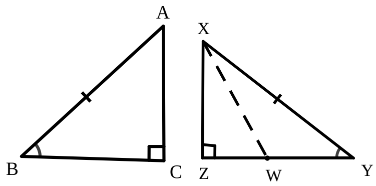


Figure 1.8.8

Exercise: Complete the proof (theorem 1.8.7)

Activity:

1. In fig 1.8.9, $\overline{AB} \equiv \overline{AC}$ and $D\hat{B}C \equiv D\hat{C}B$. Prove that $\overrightarrow{AD}$ bisects $B\hat{A}C$.

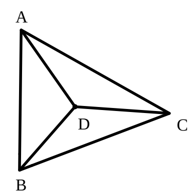


Figure 1.8.9

2. In fig 1.8.12, $\overline{CD} \perp \overline{AB}$, $\overline{BE} \perp \overline{AC}$ and $\overline{CD} \perp \overline{BE}$. Prove that $\overline{AD} \equiv \overline{AE}$.

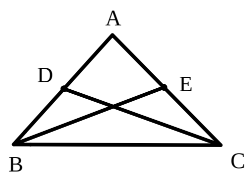


Figure 1.8.10

1.9. Geometric Inequalities

This section deals with comparison of segments and angles. The concepts of betweenness for points and congruence for segments can be combined to develop a definition which can be used for comparing segments. This definition can be used, along with some preceding theorems, to obtain several theorems pertaining to the comparison of segments. Angles can be compared in much the same manner as that of line segments. Let us see how this is possible.

Recall that, we have seen that:

Two line segments are equal in length if and only if they are congruent.

Two angles are equal measure if and only if they are congruent.

But in this section we focus on line segments of unequal length and angles of unequal measures. For this we put the following definition about inequalities of line segments and angles.

Definition 1.9.1:

a) Segment $\overline{AB}$ is said to be less than segment $\overline{CD}$ if and only if there exists a point E such that C-E-D and $\overline{AB} \equiv \overline{CE}$.

b) Angle $A\hat{B}C$ is said to be less than angle $D\hat{E}F$ if and only if there exists a ray $\overrightarrow{EG}$ such that G is in the $\text{int}(D\hat{E}F)$ and $A\hat{B}C \equiv G\hat{E}F$.

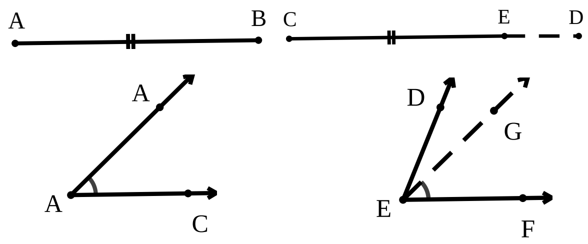


Figure 1.9.1

Notations:

- $\overline{AB}$ is less than $\overline{CD}$ is symbolized as AB<CD.
- $A\hat{B}C$ is less than $D\hat{E}F$ is symbolized as $A\hat{B}C < D\hat{E}F$.

If AB<CD we also write CD>AB and read as CD is greater than AB. The same is true for angles.

Remarks:

1. In comparing two segments $\overline{AB}$ and $\overline{CD}$, we have only the following three possibilities and exactly one of them is true.
   a. AB<CD
   b. AB=CD
   c. AB>CD
2. in comparing two angles $A\hat{B}C$ and $D\hat{E}F$, we have only the following possibilities and exactly one of them is true:
   a. $A\hat{B}C <$ $D\hat{E}F$
   b. $A\hat{B}C \equiv$ $D\hat{E}F$
   c. $A\hat{B}C > D\hat{E}F$

Now by using definition 1.9.1 and previously proved theorems let us investigate some facts about comparison of line segments and angles.

Theorem 1.9.1: an interior angle of a triangle is less than each of its remote exterior angles.

Proof: Suppose ABC is a triangle. Consider a exterior angle $B\hat{A}C$. Then by AB_2, there exists a point D and G on $\overleftrightarrow{AC}$ and $\overleftrightarrow{AB}$, respectively such that A-C-D and A-B-G. Also there exist points F and E on $\overleftrightarrow{BC}$ such that F-B-C and B-C-E (see fig.1.9.2).

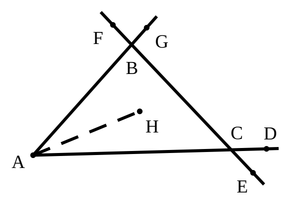


Figure 1.9.2

By definition 1.9.1 beach of $A\hat{B}F$, $C\hat{B}G$, $B\hat{C}D$ and $A\hat{C}E$ is a remote exterior angle of $B\hat{A}C$. We show that $B\hat{A}C < A\hat{B}F$, the others can be shown analogously. Now we have only three possibilities while comparing $B\hat{A}C$ and $A\hat{B}F$:

i. $B\hat{A}C \equiv A\hat{B}F$
ii. $B\hat{A}C < A\hat{B}F$
iii. $A\hat{B}F < B\hat{A}C$

Suppose $B\hat{A}C \equiv A\hat{B}F$, then $\overleftrightarrow{AC}$ is parallel to $\overleftrightarrow{BC}$ (why?). But $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ are not parallel as they intersect at C. thus the supposition is false. Therefore, $B\hat{A}C$ is not congruent to $A\hat{B}F$.

Suppose $A\hat{B}F < B\hat{A}C$, there exists a point H in the $\text{int}(B\hat{A}C)$ such that $A\hat{B}F \equiv B\hat{A}H$. Again this implies $\overleftrightarrow{AH} \| \overleftrightarrow{BC}$ (why?). But this is impossible as ray $\overrightarrow{AH}$ intersects sides $\overline{BC}$ of ΔABC at some point J different from B and C. thus the supposition is false. Therefore, $A\hat{B}F$ is not less than $B\hat{A}C$. Since $B\hat{A}C$ is not congruent to $A\hat{B}F$ and $A\hat{B}F$ is not less than $B\hat{A}C$, we have $B\hat{A}C < A\hat{B}F$ analogously, it can be shown that $B\hat{A}C < A\hat{C}E$, $B\hat{A}C < C\hat{B}G$ and $B\hat{A}C < B\hat{C}D$. Therefore, an interior angle of a triangle is less than each of its remote exterior angles.

Activity: Let O be any point inside ΔABC. Prove that $B\hat{A}C < B\hat{O}C$.

Theorem 1.9.2: If two sides of a triangle are not congruent then the angles opposite these sides are not congruent and the lesser angle is opposite the lesser sides.

Proof: Suppose ABC is a triangle with $\overline{AB} \not\equiv \overline{AC}$. Then, $A\hat{B}C \not\equiv A\hat{C}B$ otherwise h $\overline{AB} \equiv \overline{AC}$.

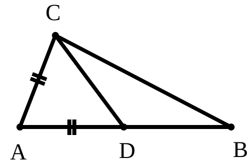


Figure 1.9.3

Now since $\overline{AB} \neq \overline{AC}$, either $AB < AC$ or $AC < AB$. Then there exist a point D on $\overline{AB}$ such that $\overline{AC} \equiv \overline{AD}$ by definition 1.9 1a. ΔACD is isosceles as $\overline{AC} \equiv \overline{AD}$ and hence $A\hat{C}D \equiv A\hat{D}C$. But, $A\hat{C}D < A\hat{C}B$ definition 1.9.1b. Moreover $A\hat{D}$ C is an exterior angle of ΔCDB and hence $A\hat{B}C < A\hat{D}C$ by theorem 1.9.1. Thus from $A\hat{C}D \equiv A\hat{D}C$, $A\hat{C}D < A\hat{C}B$ and $A\hat{B}C < A\hat{D}C$ it follows that $A\hat{B}C < A\hat{C}D < A\hat{C}B$. Therefore, $A\hat{B}C < A\hat{C}B$. Analogously, it can be shown that if AB<AC then $A\hat{C}B < A\hat{B}C$. Thus we have proved that an angle opposite to the smallest side is smallest.

Theorem 1.9.3: If two angles of a triangle are not congruent then their opposite sides are not congruent and the lasser side is opposite the lasser angle.

Proof: Suppose ABC is a triangle with $A\hat{B}C \not\equiv A\hat{C}B$. Then either $A\hat{B}C < A\hat{C}B$ or $A\hat{C}B < A\hat{B}C$.

Case 1: Suppose $A\hat{B}C < A\hat{C}B$. To prove that AC<AB.

If $\overline{AB} \equiv \overline{AC}$ then $A\hat{B}C \equiv A\hat{C}B$, contrary to the supposition $A\hat{B}C < A\hat{C}B$. Thus $\overline{AB}$ is not congruent to $\overline{AC}$.

If AB<AC then $A\hat{C}B < A\hat{B}C$ (by theorem 1.9.2), which is contrary to the supposition

$A\hat{B}C < A\hat{C}B$. Thus AB<AC is not true.

Therefore neither $\overline{AB} \equiv \overline{AC}$ nor AB<AC. Consequently AC<AB.

Case 2: Suppose $A\hat{C}B < A\hat{B}C$. To prove that AB<AC.

If $\overline{AB} \equiv \overline{AC}$ then $A\hat{C}B \equiv A\hat{B}C$, contrary to the supposition $A\hat{C}B < A\hat{B}C$. Thus $\overline{AB}$ is not congruent to $\overline{AC}$.

If AC<AB then $A\hat{B}C < A\hat{C}B$ (by theorem 1.9.2), which is contrary to the supposition

$A\hat{C}B < A\hat{B}C$. Thus AC<AB is not true.

Therefore neither $\overline{AB} \equiv \overline{AC}$ nor AC<AB. Consequently AB<AC.

Remarks:

- The angle opposite the greatset side is the greastest angle.
- The side opposite the greatset angle is the greastest side.

Thus in an obtuse triangle, the greatest side is opposite to the obtuse angle; in a right triangle the hypothenuse is the greatest side.

Definition 1.9.2 The distance of a line from a point which is not on the line is the length of the perpendicular line segment from the pont to the line.

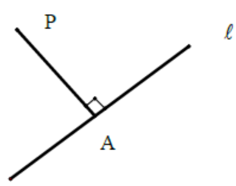


Figure 1.9.4

Theorem 1.9.3; (Triangle inequality)

The sum of the lengths of any two sides of a triangle is geater than the lenth of the third side.

Proof: Let ABC be triangle. We need to show that BA+AC>BC, BA+BC>AC and BC+AC>BA. We show only BA+AC>BC. The other can be shown in similar manner.

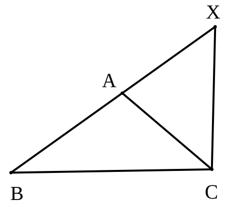


Figure 1.9.5

Extend $\overline{BA}$ to some point X on $\overleftrightarrow{AB}$ such that B-A-X and $\overline{AX} \equiv \overline{AC}$. This is possible by axiom of segment construction. Join C and X. Since $\overline{AX} \equiv \overline{AC}$ $A\hat{X}C \equiv X\hat{C}A$. $X\hat{C}A < X\hat{C}B$ by definition of angle comparison. Thus, $A\hat{X}C < X\hat{C}B$ (i.e $B\hat{X}C < X\hat{C}B$). Now, in ΔBCX we have BC<BX. But BX=BA+AX (as A, B, X are colinear and B-A-X). BX=BA+AC as $\overline{AX} \equiv \overline{AC}$. Therefore, BC<BA+AC.

Exercise

1. Prove that if two sides of one triangle are congruent to two sides of another triangle but the measures of the included angles are unequal then the lengths of the third sides are unequal in the same order.
2. Prove that if two sides of one triangle are congruent to two sides of another triangle but the lengths of the third sides are unequal then the measures of the angle included between the piars of congruent sides are unequal in the same order.
3. Prove that the difference of the lenths of any two sides of a triangle is less than the third side.

1.10. Sufficient Conditions for Parallelism

Two lines are parallel if they lie in the same plane but do not intersect. We shall use the abbreviation $L_1 // L_2$ to mean that the lines the lines L_1 and L_2 are parallel. Later, as a matter of convenience, we shall say that two segments are parallel if the lines that contain them are parallel. We shall apply the same term to a line and a segment, a segment and a ray, a ray and so on. The Euclidean parallel postulate will be introduced in the next chapter, and used thereafter, except in the chapter on non-Euclidean geometry. The postulate, in the form in which it is usually stated, say that given a line and a point not on the line, there is exactly one line which passes through the given point and is parallel to the given line.

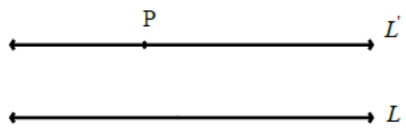


Figure 1.10.1

We shall see, however, from theorem 1.10.1 and 1.10.2, that half of this statement can be proved on the basis of the postulates that we already have.

Theorem 1.10.1: If two lines lie in the same plane, and are perpendicular to the same line, then they are parallel.

Restatement: Let L_1, L_2 and T be three lines, lying in a plane E, such that $L_1 \perp T$ and $L_2 \perp T$, then $L_1 // L_2$

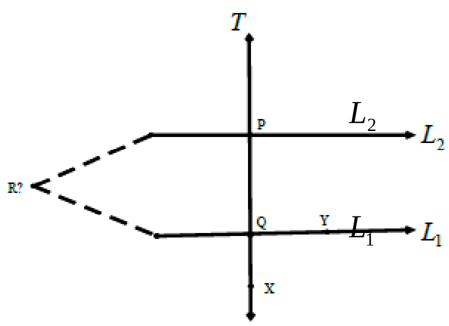


Figure 1.10.2

Proof: Suppose that L_1 and L_2 intersect T at point Q and P, respectively. Suppose that L_1 and L_2 are not parallel, and let R be the point at which they intersect. Then there are two perpendiculars to T through R; and this is a contradiction. Why?

Theorem 1.10.2: Given a line and perpendicular line, there is always at least one line which passes through the given point and is parallel to the given line.

Proof: Let L be the line, let P be the point, and let E be the plane which contains them. Then there is a line T in E which passes through P and is perpendicular to L. then there is a line L' in E which passes through P and is perpendicular to T. by the preceding theorem it follow that $L_1 // L'$, which was to be proved. There is an easy generation of theorem 1.10.1, which we shall get too presently. In the figure below, T is a transversal to the lines L_1 and L_2

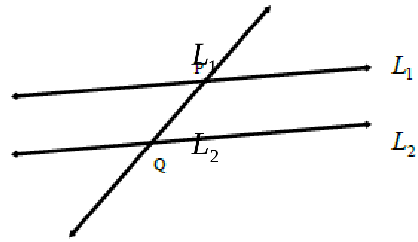


Figure 1.10.3

More precisely, if L_1, L_2 and T are three lines in the same planes, and T intersects L_1 and L_2 in two different points P and Q, respectively, then T is a transversal to L_1 and L_2

In the figure below $\angle 1$ and $\angle 2$ are alternate interior angles: and $\angle 3$ and $\angle 4$ are alternate interior angles.

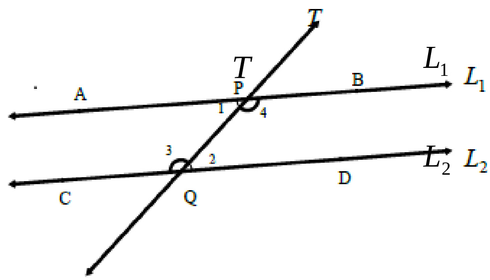


Figure 1.10.4

More precisely,

1. If T is transversal to L_1 and L_2, intersecting L_1 and L_2 in P and Q, respectively, and
2. A and D are points of L_1 and L_2 respectively, lying on opposite sides of T, then $\angle APQ$ and $\angle PQD$ are alternate interior angles.

Theorem 1.10.3: Given two lines and a transversal. If a pair of alternate interior angles is congruent, then the lines are parallel.

In the figure below, $\angle 1$ and $\angle 1'$ are corresponding angles, $\angle 2$ and $\angle 2'$ are corresponding angles and so on.

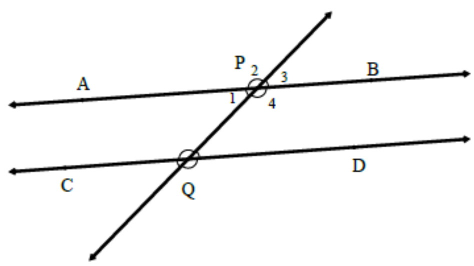


Figure 1.10.5

Definition 1.10.1: If $\angle x$ and $\angle y$ are corresponding angles, and $\angle z$ and $\angle y$ vertical angles, then $\angle x$ and $\angle z$ are corresponding angles.

Given two lines and a transversal. If a pair of angles is congruent, then the lines are parallel.

Example: prove that if $m \perp \ell$ and $n \perp \ell$, then either $m = n$ or $m // n$

Solution: let ℓ, m and n be three lines such that $m \perp \ell$ and $n \perp \ell$. We must prove that either $m = n$ or $m // n$. Let A be the point at which ℓ and m intersect and let B be the point at which ℓ and n intersect (definition of perpendicular lies). There are two cases: either A=B or $A \neq B$. If $A = B$, then $m = n$ by the uniqueness of perpendiculars

1.11. Saccheri Quadrilateral

Definition 1.11.1: Let A, B, C, D be points, no three of which are collinear, such that any two of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ either have no point in common or only have an endpoint in common. Then the point A, B, C, D determine a quadrilateral, denoted by $\square ABCD$. The points A, B, C, D are called the vertices of the quadrilateral. The segment $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are called the sides of the quadrilateral. The diagonals of $\square ABCD$ are the segment $\overline{AC}$ and $\overline{BD}$.

Two quadrilaterals are congruent if all four corresponding sides and all four corresponding angles are congruent.

Fig. 1.11.1: $\square ABCD$ is a convex quadrilateral with diagonals $\overline{AC}$ and $\overline{BD}$; $\square EFGH$ is non-convex quadrilateral with diagonals $\overline{EG}$ and $\overline{EH}$; $\square IJKL$ is not quadrilateral, although $\square IKJL$ (not shown) is a quadrilateral.

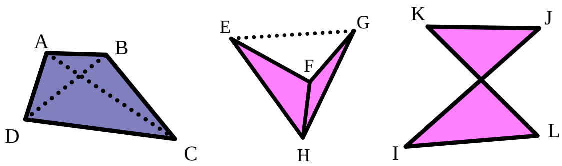


Figure 1.11.1

Definition 1.11.2: $\square ABCD$ is convex if each vertex is contained in the interior of the angle formed by the three other vertices (in their cyclic order around the quadrilateral).

Definition 1.11.3: Let $\square ABCD$ be convex. Then its angle sum is given by the sum of the measures of its interior angles:
$\sigma(BACD) = m(\angle ABC) + m(\angle BCD) + m(\angle ABC) + m(\angle BCD)$

Theorem 1.11.1: (Additively of Angle Sum)

Let $\square ABCD$ be convex quadrilateral with diagonal $\overline{BD}$. Then $\sigma(ABCD) = \sigma(\Delta ABD) + \sigma(\sigma BDC)$

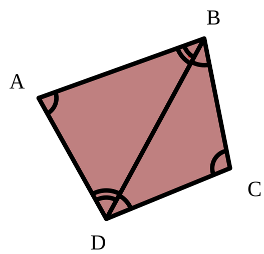


Figure 1.11.2. The angle sum of quadrilateral is equal to the sum of angle sums of the triangles defined by either diagonal

Proof: Apply the angle addition postulate to each of the angle that is split by a diagonal to get

$$\begin{aligned}\sigma(ABCD) &= \alpha + \beta + \varepsilon + \theta \\ &= \alpha + r + \delta + \in + \xi + \eta \\ &= (\alpha + r + \eta) + (\delta + \varepsilon + \varsigma) \\ &= \sigma(\Delta ABD) + \sigma(\Delta BDC)\end{aligned}$$

Definition 1.11.4: The defect of a quadrilateral is $\delta\,(\square ABCD) = 360 - \delta\,(ABCD)$

Theorem 1.11.2: (The additively of defect for convex quadrilateral)

If $\square ABCD$ is convex quadrilateral, then $\delta\,(\square ABCD) = \sigma(\Delta ABC) + (\Delta ACD)$

Proof: Apply theorem 1.11.1.

Corollary 1.11.1: If $\square ABCD$ is convex, then $\delta\,(\square ABCD) \leq 360$

Proof: Apply theorem 1.11.1.

Definition 1.11.5: $\square ABCD$ is called a parallelogram if $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$.

Theorem 1.11.3: Every parallelogram is convex.

Proof: Left for reader.

Theorem 1.11.4: let ΔABC be a triangle and D and E points such that A-D-C and A-E-C. Then $\square BCED$ is a convex quadrilateral.

Proof: Left for reader.

Theorem 1.11.5: $\square BCED$ if and only if the diagonal have an interior point in common (i.e. they intersect, but not at an endpoint)

Proof:

($\Rightarrow$) Assume $\square BCED$ is convex. Then by the definition of convexity C is the interior of $\angle DAB$. Then $\overline{BD} \cap \overline{AC} \neq \phi$; call the point of intersection E, where B-E-D, by similar argument there is a point $\overline{BD} \cap \overline{AC} = F$ where A-F-C. Since $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are distinct (they corresponds to opposite side of quadrilateral), they can meet in at most one point, we must have E=F. Hence the diagonals intersect at E. Since A-F-C and B-E-D, the intersection is not at end point.

($\Leftarrow$) Let $\square BCED$ be a quadrilateral with $E = \overline{AC} \cap \overline{BD}$ with A-E-C and B-E-D. Since A-E-C, A and E are on the same side of the line $\overleftrightarrow{CD}$. Similarly, since B-E-D, B and E are on the same side of the line $\overleftrightarrow{CD}$. Hence A and B are on the same side of $\overleftrightarrow{CD}$ (Plane separation postulate), i.e $A \in H_{B,\overline{CD}}$. Recall that $H_{B,\overline{CD}}$ is a half plane determined by line $\overleftrightarrow{CD}$ and point B not on $\overleftrightarrow{CD}$ (i.e $H_{B,\overline{CD}}$ is a half plane containing point B). By a similar argument A and D are on the same side of $\overleftrightarrow{BC}$, i.e, $A \in H_{D,\overline{DC}}$. Hence $A \in H_{B,\overline{CD}} \cap H_{D,\overline{DC}}$, and thus A is in the interior of $\angle BCD$. By a similar argument, each of the other vertices is in the interior of its opposite angle. Hence by definition of convexity, the quadrilateral is convex. Assume $\square BCED$ is a convex quadrilateral (i.e assume that R is false). Then $\overline{AC} \cap \overline{BD} \neq \varphi$, i.e the diagonals of $\square BCED$ share an internal point. Hence $\square BCED$ is not a quadrilateral.

Example 1.11.1 Show that every parallelogram is convex.

Solution: let $\square ABCD$ be a parallelogram (hypothesis). We must prove that $\square ABCD$ is a convex quadrilateral. Since $\overleftrightarrow{AD} \| \overleftrightarrow{BC}$ by definition of parallelogram, it follow that $\overleftrightarrow{AD} \cap \overleftrightarrow{BC} = \varphi$ by definition of parallel. Hence, A and D lie on the same side of $\overleftrightarrow{BC}$ (plane separation postulate).

In the same way, the fact that $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$ can be used to prove that A and B lie on the same side of $\overleftrightarrow{CD}$. Thus, is in the interior of $\angle BCD$ (definition of angle interior). The remaining conditions left as an activity. Therefore, $\square ABCD$ is convex.

Activity: In the above example:

a) Show that B is in the interior of $\angle CDA$

b) Show that C is in the interior of $\angle DAB$ and

c) Show that D is in the interior of $\angle ABC$

Theorem 1.11.6: If $\square ABCD$ is a non-convex quadrilateral then $\square ACBD$ is a quadrilateral.

Proof: Since $\square ABCD$ is a quadrilateral no three of the point A, B, C, D are collinear. Since $\square ABCD$ is a quadrilateral $\overline{BC} \cap \overline{AD} = \varphi$. Since $\square ABCD$ is not a convex then $\overline{AC}$, $\overline{BD}$ are disjoint (the diagonals do not intersect). Thus segments $\overline{AC}$, $\overline{CB}$, $\overline{BD}$ and $\overline{DA}$ share at most their endpoints. Hence $\square BCED$ is a quadrilateral.

Definition 1.11.6: $\square ACBD$ is a Saccheri quadrilateral if $\angle ABC = \angle BAD = 90^0$ and AD=BC, segment $\overline{AB}$ is called the base and segment $\overline{DC}$ is called the summit.

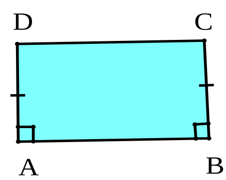


Figure 1.11.3

Theorem 1.11.7: The diagonals of the Saccherei Quadrilateral are congruent.

Proof: Consider triangle ΔABD and ΔABC. Since BC=AD, AB=AB and $\angle A = 90 = \angle B$, the triangles are congruent. Hence $\overline{BD} \cong \overline{AC}$

Theorem 1.11.8: The summit angles of a Saccheri Quadrilateral are congruent.

Proof: Repeat the argument in the previous proof, but with upper-half triangles. The triangles are congruent by SSS-they share the same top, the diagonals are congruent; and the sides are congruent. Hence the corner angles are congruent.

Definition 1.11.7: A Lambert quadrilateral is a quadrilateral in which three of the interior angles are right angles.

Corollary 1.11.2: Let $\square ABCD$ be a Lambert quadrilateral. Then it is convex.

Proof: It is a parallelogram and all parallelograms are convex

1.12. The Angle Sum Inequality for Triangles

If we only assume Euclid's first four postulates, along with the axiom of incidence, congruence, continuity and betweenness, the angle sum of a triangle is always less than or equal to 180. This geometry is called neutral (or absolute) geometry. We will also consider some important consequence of this theorem.

Theorem 1.12.1: (Exterior Angle Inequality)

The measure of an exterior angle of a triangle is greater than the measure of either remote interior angle.

Proof: Given ΔABC, extend side $\overline{BC}$ to ray $\overrightarrow{BC}$ and choose the point D on this ray so that B-C-D. We claim that $m\angle ACD > m\angle A$ and $m\angle ACD > m\angle D$. Let M be the midpoint of $\overline{AC}$ and extend the median $\overline{BM}$ so that M is the midpoint of $\overline{BE}$.

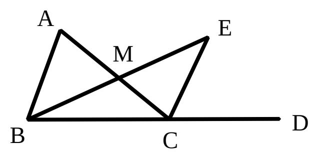


Figure 1.12.1

Then $\angle AMB$ and $\angle CME$ are congruent vertical angles and $\Delta AMB \cong \Delta CME$ by SAS. Consequently, $m\angle ACE = m\angle CAB$. Now, E lies in the half-plane of A and $\overrightarrow{CD}$, since A and E are on the same side of $\overrightarrow{CD}$. Also, E lies in the half plane of D and $\overrightarrow{AC}$ since D and E are on the same side of $\overrightarrow{AC}$. Therefore E lies in the interior of $\angle ACD$, which is the intersection of these two half-planes. Finally, $\angle ACD = \angle ACE + m\angle ECD > m\angle ACE = m\angle CAB = m\angle A$

Activity: In the above theorem (theorem 1.12.1) prove the case $m\angle ACD > m\angle B$

Corollary 1.12.1: The sum of the measures of any two interior angles of a triangle is less than 180.

Proof: Given ΔABC, extend side $\overline{BC}$ to $\overleftrightarrow{BC}$ and choose points E and D on $\overleftrightarrow{BC}$, so that E-B-C-D (See figure 1.12.2)

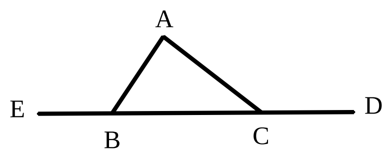


Figure 1.12.2

By theorem 1.12.1, $m\angle A < m\angle ACD$, $m\angle B < m\angle ACD$, and $m\angle A < m\angle ABE$. By adding $m\angle C < m\angle ACB$ to both sides of the first two inequalities, and by adding $m\angle B < m\angle ABC$ to both sides of the third we obtain

$$m\angle A + m\angle C < m\angle ACD + m\angle ACB = 180$$

$$m\angle B + m\angle C < m\angle ACD + m\angle ACB = 180$$

$$m\angle A + m\angle B < m\angle ABE + m\angle ABC = 180$$

Theorem 1.12.2: If two lines are cut by a transversal and pair of alternate interior angles are congruent, then the lines are parallel.

Proof: We prove the contra positive. Assume that lines ℓ and m intersect at the point R, and suppose that a transversal t cuts line ℓ at the point A and cuts line m at a point B. let $\angle 1$ and $\angle 2$ be a pair of alternate interior angles. Then either $\angle 1$ is an exterior angle of ΔABR and $\angle 2$ is a remote interior angle or vise versa.

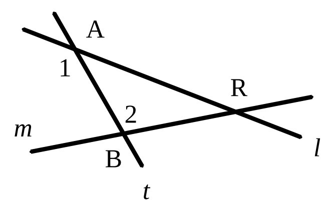


Figure 1.12.3

In either case $m\angle 1 \neq m\angle 2$ by the exterior angle inequality (theorem 1.12.1)

Theorem 1.12.3: (Saccheri-legendre theorem)

The angle sum of a triangle is less than or equal to 180.

Proof: Assume, in the contrary that the angle sum of $\Delta ABC = 180 + p$, for some $p > 0$. Construct the midpoint M of side $\overline{AC}$, then extend $\overline{BM}$ its own length to point E such that

B-M-E. Note that $\Delta ABM \cong \Delta CEM$ by SAS.
Therefore, the angle sum of ΔABC =angle sum of ΔABM +angle sum of ΔBMC
=angle sum of ΔCEM +angle sum of ΔBMC
= angle sum of ΔBEC

Furthermore, $m\angle BEC = m\angle ABE$. Therefore, either $m\angle BEC = \frac{1}{2} m\angle ABC$ or $m\angle EBC = \frac{1}{2} m\angle ABC$. Thus, we may replace ΔABC with ΔBEC, having the same angle sum as ΔABC and one angle whose measure is less than or equal to $\frac{1}{2} m\angle ABC$.

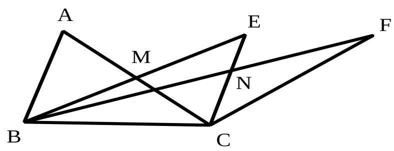
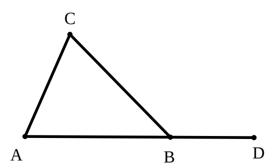


Figure 1.12.4

Now repeat this construction in ΔEBC. If $m\angle EBC \le \frac{1}{2} m\angle ABC$, construct the midpoint N of $\overline{CE}$ and extend $\overline{BN}$ its own length to point F such that B-N-F. Then ΔBEC and ΔBFC have the same angle sum and either $m\angle BFC \le \frac{1}{2} m\angle EBC$ or $m\angle FBC \le \frac{1}{2} m\angle EBC$. Replace ΔEBC with ΔFBC having the same angle sum as ΔABC and one angle whose measure is $\le \frac{1}{4} m\angle ABC$. On the other hand, if $m\angle BFC \le \frac{1}{2} m\angle ABC$, do same construction with N as the midpoint of $\overline{BC}$ and replace ΔEBC with ΔFEC. Continue this process indefinitely; the Archimedian property of real numbers guarantees that for sufficiently large n, the triangle obtained after the n$^{\text{th}}$ iteration has the same angle sum as ΔABC and one angle whose measure is $\le \frac{1}{2^n} m\angle ABC < p$, in which case the sum of its other two angles is greater than $180°$ contradicting corollary 1.12.1

Example 1.12.1: Prove that the sum of the measure of two interior angles of the triangle is less than or equal to the measure of the remote of their remote exterior angle.

Solution: Let ΔABC be a triangle and let D be appoint on $\overleftrightarrow{AB}$ such that A-B-D (hypothesis). We must prove $m(\angle BCA) + m(\angle CAB) \le m(\angle CBD)$



In the figure above, we know that $m(\angle CBA) + m(\angle ABC) + m(\angle BCA) \le 180$ (saccheri-legedre theorem). We also now that $\mu(\angle ABC) + \mu(\angle CBD) = 180$ (Linear pair theorem). Hence, from algebra
$m(\angle CAB) + m(\angle BCA) \le 180 - m(\angle ABC) = m(\angle CBD)$.

Definition 1.12.1: The defect of ΔABC is $\delta ABC = 180^0 - m\angle A - m\angle B - m\angle C$
Corollary 1.12.2: Every triangle has non-negative defect.

Proof: If $\delta ABC = 180^0 - m\angle A - m\angle B - m\angle C < 0^0$, then the angle sum of $\Delta ABC > 180^0$ contradicting theorem 1.12.3.
Theorem 1.12.4: (Additivity of defect)
Given any triangle ΔABC and any point D between A and B, $\delta ABC = \delta ACD + \delta BCD$

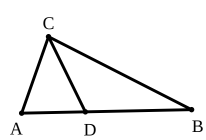


Figure 1.12.5

Proof: Since, $\angle ADC$ and $\angle BDC$ are supplementary, $m\angle CDA + m\angle CDB = 180^0$. Since, $\overline{DC}$ is in the interior of $\angle ACB$, $\angle ACB = m\angle ACD + m\angle BCD$.

Therefore, $\delta ACD + \delta BCD = 180^0 - m\angle ACD - m\angle CDA - m\angle DAC + 180^0 - m\angle BCD - m\angle CDB - m\angle DBC$

$= 360^0 - (m\angle ACD + m\angle BCD) - m\angle DBC - (m\angle CDA + m\angle CDB) - m\angle DAC$

$= 180^0 - m\angle ABC - m\angle BAC - m\angle CDA$

$= \delta ABC$.

Corollary 1.12.3: Given any triangle ΔABC and any point D between A and B, the angle sum of $\Delta ABC = 180^0$ if and only if the angle sums of ΔACD and ΔBCD both equal 180°.

Proof: If the angle sum of both ΔACD and ΔBCD equal 180^0, then $\delta ACD = \delta BCD = 0^0$. by theorem 1.12.4, $\delta ABC = 0^0$ so that the angle sum of $\Delta ABC = 180^0$, $\delta ACD = +\delta BCD = 0^0$. But, by corollary 1.12.2, $\delta ACD \geq 0^0$ and $\delta BCD \geq 0^0$. Therefore, $\delta ACD = \delta BCD = 0^0$ and both angle sums equal 180^0.

Theorem 1.12.5: If there is a triangle with angle sum 180^0, then a rectangle exists.

Proof: Consider a triangle ΔABC with angle sum180°, by corollary 1.12.1, the sum of the measures of any two interior angles is less than 180°, so at most one angle is obtuse. Suppose $\angle A$ and $\angle B$ are acute and construct the altitude $\overline{CD}$, we cliam that A-D-B. But if not, then either D-A-B or A-B-D. Suppose D-A-B and consider ΔDAC

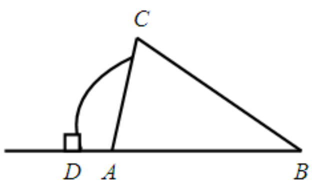


Figure 1.12.6

Then the remote interior angle $\angle CDA$ has measure 90°, which is greater than the measure of that exterior angle $\angle CAB$, contradicting the theorem 1.12.1, assuming the A-B-D leads to a similar contradiction, proving the claim. Then by corollary 1.12.3, $\delta ADC = +\delta BDC = 0^0$. Let us construct a rectangle from right triangle ΔBCD. By the congruence axioms, there is a unique ray $\overrightarrow{CX}$ with X on the opposite side of $\overleftrightarrow{BC}$ from D such that $\angle CBD \cong \angle BCX$, and there is a unique point E on $\overrightarrow{CX}$ such that $\overrightarrow{CE} \cong \overrightarrow{BD}$

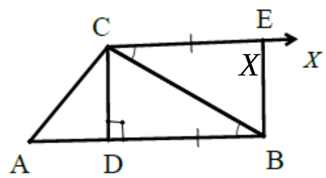


Figure 1.12.7

Then $\Delta CBD \cong \Delta BCE$ by SAS; therefore, ΔBCE is a right triangle with $\delta BCE = 0^0$ and right angle at E. also, since $m\angle DBC + m\angle BCD = 90^0$, substituting corresponding parts gives $m\angle EBC$ and $m\angle BCD = 90^0$ and $m\angle DBC + m\angle EBC = 90^0$. Furthermore, since alternate interior angles $\angle EBC$ and $\angle DBC$ are congruent $\overleftrightarrow{CE} \| \overleftrightarrow{DB}$ by theorem 1.12.2. Therefore, B is an interior point of $\angle ECD$. By the same argument $\overleftrightarrow{CD} \| \overleftrightarrow{ED}$ and C is an interior point of $\angle EBD$. Therefore, $m\angle ECD = m\angle EBD = 90^0$ and $\square CDEB$ is a rectangle.

Theorem 1.12.6: If a rectangle exists, and then the angle sum of every triangle is 180°.

Proof: We first prove that every right triangle has angle sum 180°. Given a rectangle, we can use the Archimedian property to lengthen or shorten the side and obtain a rectangle $\square AFBC$ with sides AC and BC of any prescribed length. Now given a right triangle $\Delta E'C'D'$, construct a rectangle $\square AFBC$ such that $AC > D'C'$ and $BC > E'C'$. There is a unique point D on $\overline{AC}$ and a unique point e on $\overline{BC}$ such that $\Delta ECD \cong \Delta E'C'D'$ as shown in figure 1.12.8

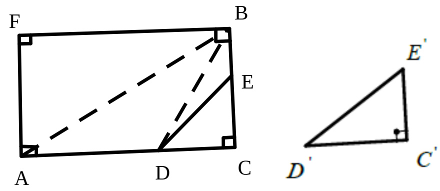


Figure 1.12.8

We claim $\delta ABC = 0^0$. If not, then $\delta ABC > 0^0$ by corollary 1.12.2 and consequently $m\angle ABC + m\angle BAC < 90^0$.

But, $m\angle CBF = m\angle ABC + m\angle ABF = 90^0$ and $m\angle CAF = m\angle BAC + m\angle BAF = 90^0$.

Therefore, $m\angle ABF = 90^0 - m\angle ABC$ $m\angle BAF = 90^0 - m\angle BAC$ so that

$$\delta ABF = 180^0 - 90^0 - m\angle ABF - m\angle BAF$$
$$= 90^0 - (90^0 - m\angle ABC) - (90^0 - m\angle BAC)$$
$$= m\angle ABC + m\angle BAC - 90^0 < 0^0$$

Contradicting the corollary 1.12.2 and proving the claim. Now by repeated application of corollary 1.12.3 we have $\delta BCD = 0^0$ and $\delta ECD = 0^0$. But $\Delta ECD \cong \Delta E'C'D'$ implies $\delta E'C'D' = 0^0$. Thus, every right triangle has zero defects. Now by the construction in theorem 1.12.5, an arbitrary triangle ΔABC can be appropriately labeled so that its altitude $\overline{CD}$ lies in the interior of ΔABC and subdivides the triangle into two right triangles (see figure 1.12.9), each having zero defect. Thus, $\delta ABC = 0^0$ by corollary 1.12.3.

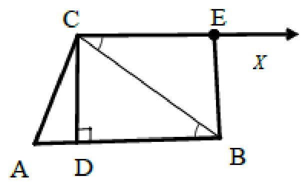


Figure 1.12.9

Corollary 1.12.4: a rectangle exists if and only if every triangle has angle sum 180°

1.13. The Critical Function

In this subunit, we shall make heavy use of the incidence and separation theorems. Convenience, we briefly restate two of them:

The postulate of pasch:

Given ΔABC and a line L (in the same plane). If L intersects $\overline{AB}$ at a point between A and B, then L also intersects either $\overline{AC}$ or $\overline{BC}$.

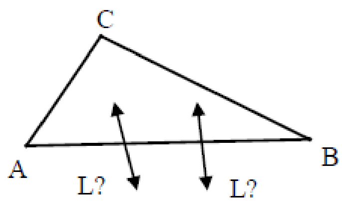


Figure 1.13.1

The Crossbar Theorem: If D is in the interior of $\angle BAC$, then $\overrightarrow{AD}$ intersects $\overline{BC}$.

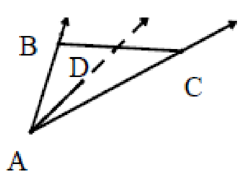


Figure 1.13.2

Given a line L and an external point P. let A be the foot of the perpendicular from P to L, and let B be any other point of L (fig. 1.13.3). For each number r between 0 and 180 there is exactly one ray $\overrightarrow{PD}$, with D on the same side of $\overleftrightarrow{AP}$ as B, such that $m\angle APD = r$

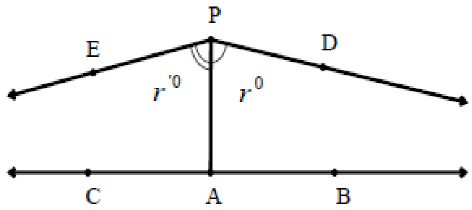


Figure 1.13.3

Obviously, for some numbers r $\overrightarrow{PD}$ will intersect $\overrightarrow{AB}$. (For example, take $r = m\angle APB$). For $r \geq 90$, $\overrightarrow{PD}$ will not intersect $\overrightarrow{AB}$. Let $K = \{r: \overrightarrow{PD}\ intersects\ \overrightarrow{AB}\}$. Then K is nonempty, and has an upper bound. Therefore K has a supremum. Let $r_0 = \sup K$. The number r_0 is called the critical number for P and $\overrightarrow{AB}$. The angle $\angle APD$ with measure equals r_0 is called the angle of parallelism of $\overrightarrow{AB}$ and P.

Theorem 1.13.1: If $m\angle APD = r_0$, then $\overrightarrow{PD}$ does not intersect $\overrightarrow{AB}$.

Proof: Suppose that $\overrightarrow{PD}$ intersect $\overrightarrow{AB}$ at Q.

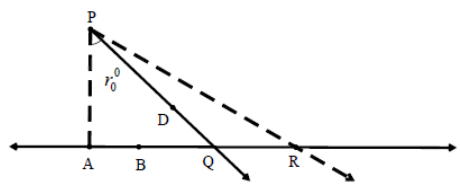


Figure 1.13.4

If R is any point such that A-Q-R, then $m\angle APD > r_0$ so that r_0 is not an upper bound.

Theorem 1.13.2: If $m\angle APD < r_0$ then $\overrightarrow{PD}$ intersects $\overrightarrow{AB}$.

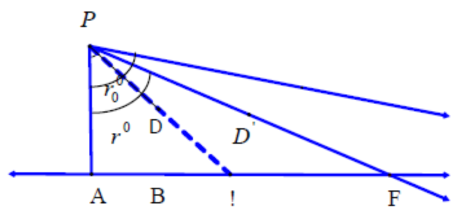


Figure 1.13.5

Proof: Since $r_0 = \sup K$ and $m\angle APD < r_0$, it follows that $m\angle APD$ not an upper bound of K. therefore, some r in K is greater than $m\angle APD$. Let D' be such that $m\angle APD' = r$. Then $\overrightarrow{PD'}$ intersects $\overrightarrow{AB}$ in a point of F. but $\overrightarrow{PD}$ is in the interior of $\angle APD'$. Therefore by the crossbar theorem $\overrightarrow{PD}$ intersects $\overrightarrow{AF}$. Therefore $\overrightarrow{PD}$ intersects $\overrightarrow{AB}$. Thus there is a certain "critical ray" $\overrightarrow{PD}$, with $m\angle APD = r_0$; $\overrightarrow{PD}$ does not intersects $\overrightarrow{AB}$. But if F is in the interior of $\angle APD$, then $\overrightarrow{PF}$ does not intersect $\overrightarrow{AB}$. (if F is in the interior of $\angle APD$, we shall say that $\overrightarrow{AF}$ is an interior ray of $\angle APD$)

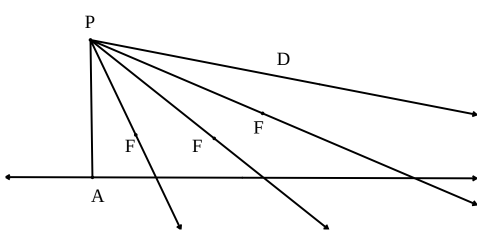
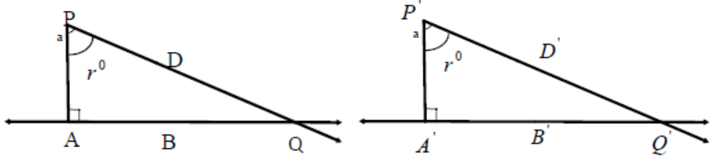


Figure 1.13.6

Note that r_0 was defined in terms of P, A and B. It turns out, however, that r_0 depends only on the distance AP.

Theorem 1.13.3: Let P, A, B and also P', A', B' be as in the definition of the critical number. If $AP = A'P'$, then the critical numbers r_0 and r'_0 are the same



Proof: Let $K = \{r : \overline{PD}..\text{intersects}...\overline{AB}\}$ and let $K = \{r : \overline{P'D'}..\text{intersects}...\overline{A'B'}\}$. If $r \in K$, let Q be the point where $\overrightarrow{PD}$ intersects $\overrightarrow{AB}$, and let Q' be the point of $\overrightarrow{A'B'}$ for which $A'Q' = AQ$. Then $m\angle A'P'D' = r$ (why?) Therefore $r \in K'$ Thus $K \subseteq K'$; and similarly $K' \subseteq K$. Therefore $K = K'$. And $\sup K = \sup K'$. We now have a function $AP \to r_0$. We shall denote this function by c, and call it the critical function. Thus, for every $a > 0, c(a)$ denotes the critical number corresponding to AP=a. thus $\overrightarrow{PD}$ intersects $\overrightarrow{AB}$ when $m\angle APD < c(a)$, but $\overrightarrow{PD}$ does not intersects $\overrightarrow{AB}$ when $m\angle APD > c(a)$

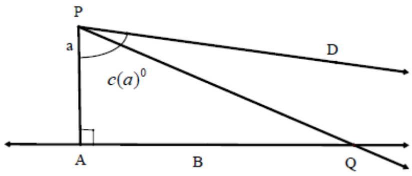


Figure 1.13.8

we shall now investigate the function c.

1.14. Open Triangle and Critically Parallel Rays

Given rays $\overrightarrow{AB}$, $\overrightarrow{PD}$, and the segment $\overline{AP}$, no two of these figure being collinear. Suppose that B and D are on the same side of $\overline{AP}$, and that $\overline{AB} // \overline{PD}$.

Then $\overline{PD} \cup \overline{PA} \cup \overline{AB}$ is called an open triangle, and is denoted by $\Delta DPAB$.

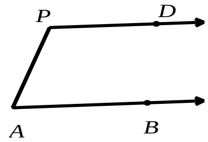


Figure 1.14.1

Here, when we write $\overline{AB} // \overline{PD}$, we mean that the lines are parallel in the usual sense of not intersecting one another.

Suppose now that $\Delta DPAB$ is an open triangle and every interior ray of $\angle APD$ intersects $\overline{AB}$:

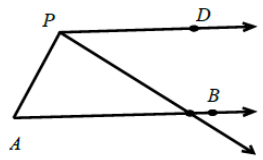


Figure 1.14.2

We then say that $\overline{PD}$ is critically parallel to $\overline{AB}$, and we write $\overline{PD} / \overline{AB}$. Here the single vertical stroke is supposed to suggest that $\overline{PD}$ is parallel to $\overline{AB}$, which no room to spare.

Note that $\overline{PD}$ and $\overline{AB}$ do not appear symmetrically in this definition. Thus if $\overline{PD} / \overline{AB}$, it does not immediately follow that $\overline{AB} / \overline{PD}$. Note also that the relation $\overline{PD} / \overline{AB}$ (as we have defined it) depends not only on the "directions" of the two rays, but also on the initial points.

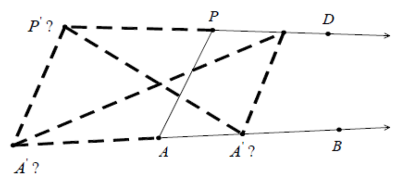


Figure 1.14.3

Thus if $\overline{PD}/\overline{AB}$ (as we have defined it) depends not only on the initial points. Thus if $\overline{PD}/\overline{AB}$, we cannot conclude immediate that $\overline{P'D}/\overline{A'B}$. We shall see, however, in the next few theorems, that the conclusion is true.

Theorem 1.14.1: If $\overrightarrow{PD}|\overrightarrow{AB}$, and C-P-D, then $\overrightarrow{CD}|\overrightarrow{AB}$

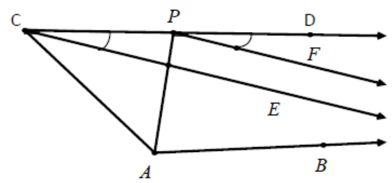


Figure 1.14.4

Proof: Let $\overrightarrow{CE}$ be an interior ray of $\angle ACD$, and suppose that $\overrightarrow{CE}$ does not intersect $\overrightarrow{AB}$. By the exterior angle theorem, we know that $\angle APD > \angle ACD$. Therefore, there is an interior ray $\overrightarrow{PF}$ of $\angle APD$ such that $\angle DPF \cong \angle DCE$. Therefore $\overrightarrow{PF}||\overrightarrow{CE}$. Therefore, $\overrightarrow{PF}$ does not intersect $\overrightarrow{AB}$, because these rays lie on opposite sides of $\overrightarrow{CE}$. This contradicts the hypothesis $\overrightarrow{PD}|\overrightarrow{AB}$.

Theorem 1.14.2: If $\overrightarrow{PD}|\overrightarrow{AB}$, and P-C-D, then $\overrightarrow{CD}|\overrightarrow{AB}$

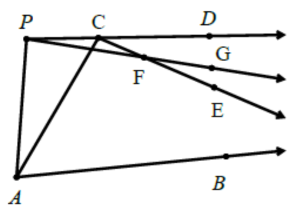


Figure 1.14.5

We give the proof briefly. Suppose that there is an interior ray $\overrightarrow{CE}$ of $\angle ACD$ such that $\overrightarrow{CE}$ does not intersect $\overrightarrow{AB}$. Let F be any point of $\overrightarrow{CE}$ -C, and take G so that P-F-G. Then

1. F is in the interior of $\angle APC$
2. $\overline{PF}$ does not intersect $\overrightarrow{AB}$
3. $\overline{PG}$ does not intersect $\overrightarrow{AB}$
4. $\overrightarrow{PF}$ does not intersect $\overrightarrow{AB}$

Statement (1) and (4) contradict the hypothesis $\overrightarrow{PD}|\overrightarrow{AB}$.

Two rays R and R' are called equivalent if one of them contains the other. We then write $R \sim R'$. Obviously the symbol $\sim$ represents an equivalence relation. Fitting together the preceding two theorems, we get:

Theorem 1.14.3: If $R|\overrightarrow{AB}$, and R and R' are equivalent, then $R'|\overrightarrow{AB}$. Somewhat easier proofs show that the relation $\overrightarrow{PD}|\overrightarrow{AB}$ depends only on the equivalence class of $\overrightarrow{AB}$. We leave these proofs to you.

Theorem 1.14.4: If $R_1|R_2$, $R_1' \sim R_1$ and $R_2' \sim R_2$, then $R'_1|R'_2$

Given $\overrightarrow{PD}|\overrightarrow{AB}$, let C be the foot of the perpendicular from P to $\overleftrightarrow{AB}$, and let PC=a

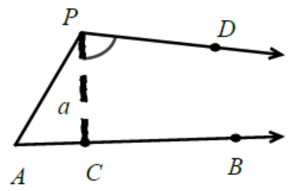


Figure 1.14.6

Then $\overrightarrow{PD}\big|\overrightarrow{CB}$ (providing, of course, that B is chosen so that A-C-B, as in the fig.). Therefore $m\angle CPD = C(a)$. Now on the side of $\overleftrightarrow{PC}$ that contains B there is only one ray $\overrightarrow{PD}$ for which $m\angle CPD = C(a)$. Thus we have:

Theorem 1.14.5: The critical parallel to the given ray, through a give external point, is unique. Two open triangles are called equivalent if the rays that from their sides are equivalent. An open triangle $\Delta DPAB$ is called isosceles if $\angle P \cong \angle A$

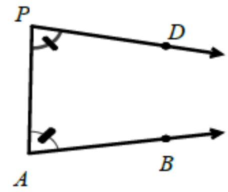


Figure 1.14.7

Theorem 1.14.6: If $\overrightarrow{PD}\big|\overrightarrow{AB}$, then $\Delta DPAB$ is equivalent to an isosceles open triangle which has P as a vertex.

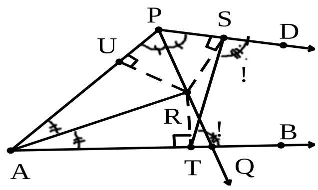


Figure 1.14.8

Proof: Since $\overrightarrow{PD}\big|\overrightarrow{AB}$, the bisecting ray of $\angle APD$ intersects $\overrightarrow{AB}$ in a point Q. By the crossbar theorem, the bisecting ray of $\angle PAB$ intersects $\overline{PQ}$ at a point R. let S, T, and U be the feet of the perpendicular from R to $\overleftrightarrow{PD}$, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AP}$. Then $RU = RT$ and $RU = RS$. Therefore $RS = RT$ and $\angle RST \cong \angle RTS$. Hence (by addition or subtraction) $\angle DST \cong \angle BTS$ and $\Delta DSTB$ is isosceles. To make P a vertex, we take V on the ray opposite to $\overrightarrow{TB}$, such that TV=SP.

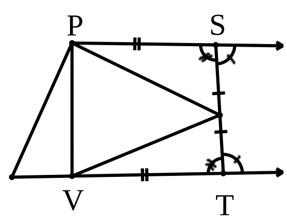


Figure 1.14.9

Theorem 1.14.7: Critical parallelism is a symmetric relation. That is, if $\overrightarrow{PD}\big|\overrightarrow{AB}$, then $\overrightarrow{AB}\big|\overrightarrow{PD}$

Proof: By theorem 1.14.4 and 1.14.6, we may suppose that $\Delta DPAB$ is an isosceles open triangle:

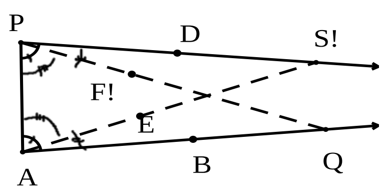


Figure 1.14.10

Let $\overrightarrow{AE}$ be any interior ray of $\angle PAB$. Let $\overrightarrow{PF}$ be an interior ray of $\angle APD$, such that $\angle DPF \cong \angle BAE$. Then $\overrightarrow{PF}$ intersects $\overrightarrow{AB}$ at a point Q. It follow that $\overrightarrow{AE}$ intersect $\overrightarrow{PD}$ at the point S where PS=AQ.

Theorem 1.14.8: If two nonequivalent rays are critically parallel to a third ray, then they are critically parallel to each other.

Restatement: If $\overrightarrow{AB}|\overrightarrow{CD}$, $\overrightarrow{CD}|\overrightarrow{EF}$, and $\overrightarrow{AB}$ and $\overrightarrow{EF}$ are not equivalent, then $\overrightarrow{AB}|\overrightarrow{EF}$.

1. Suppose that $\overrightarrow{AB}$ and $\overrightarrow{EF}$ lie on opposite sides of $\overrightarrow{CD}$. Then $\overline{AE}$ intersects $\overrightarrow{CD}$, and by theorem 1.14.4 we can assume that the point of intersection is C.

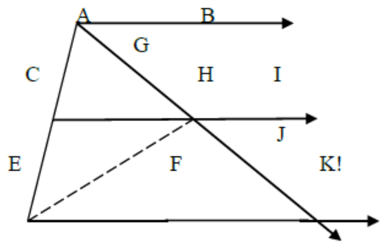


Figure 1.14.11

Let $\overrightarrow{AG}$ be any interior ray of $\angle EAB$. Then $\overrightarrow{AG}$ intersects $\overrightarrow{CD}$ at appoint H. take I so that C-H-I and take J so that A-H-J. Then $\overrightarrow{HI}|\overrightarrow{EF}$, by theorem 1.14.4; and $\overrightarrow{HJ}$ is an interior ray of $\angle EHI$. Therefore $\overrightarrow{HJ}$ intersects $\overrightarrow{EF}$ at appoint K. therefore $\overrightarrow{AG}$ intersects $\overrightarrow{EF}$, which was to be proved.

2. If $\overrightarrow{CD}$ and $\overrightarrow{EF}$ are on opposite sides of $\overleftrightarrow{AB}$, then the same conclusion follows. Here we may suppose that $\overrightarrow{AB} \cap \overline{EC} = A$, for the same reasons as in the first case. Through E there is exactly one ray $\overrightarrow{EF'}$ critically parallel to $\overleftrightarrow{AB}$, by the result in case (1), $\overrightarrow{EF'}|\overrightarrow{CD}$. Since critical parallels are unique $\overrightarrow{EF'} = \overrightarrow{EF}$ and $\overrightarrow{EF}|\overrightarrow{AB}$, which was to be proved.

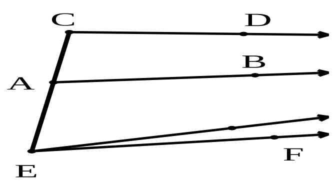


Figure 1.14.12

Given three nonintersecting lines, it can easily happen that every two of them are on the same side the third. Therefore the conditions $\overrightarrow{AB}\|\overrightarrow{CD}, \overrightarrow{CD}\|\overrightarrow{EF}$ are not enough for our purpose; to get a valid proof, we need to use the full force of the hypothesis $\overrightarrow{AB'}|\overrightarrow{CD}, \overrightarrow{CD}|\overrightarrow{EF}$. We shall show, under these conditions, that

3. Some lines intersects all three of the ray $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$. (surely this will be enough)

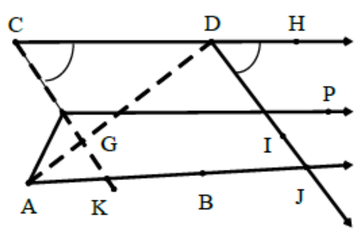


Figure 1.14.13

If A and E are opposite sides of $\overrightarrow{CD}$, then $\overline{AE}$ intersects $\overrightarrow{CD}$, and (3) follows. Suppose, then that

a. A and E are on the same sides of $\overrightarrow{CD}$. If A and D are on the same side of $\overrightarrow{EC}$, then $\overrightarrow{CA}$ is an interior ray of $\angle C$, so that $\overrightarrow{CA}$ intersects $\overrightarrow{EF}$, and (3) follows. If A lies on $\overrightarrow{CE}$ then (3) holds. We may therefore suppose that.

b. A and D are on opposite sides of $\overrightarrow{CE}$. Therefore $\overline{AD}$ intersects $\overrightarrow{CE}$ at a point G. take H so that C-D-H. Then $\overrightarrow{DH}\Big|\overrightarrow{AB}$. By the exterior angle theorem $\angle HDA > \angle C$. Therefore there is an interior ray $\overrightarrow{DI}$ of $\angle HDA$ such that $\angle HDI \cong \angle C$. Then $\overrightarrow{DI}\Big|\overrightarrow{CE}$ but $\overrightarrow{DI}$ intersect $\overrightarrow{AB}$ at a point J. Now $\overrightarrow{CE}$ intersects $\overrightarrow{AD}$ at G. therefore $\overrightarrow{CE}$ intersects another sides of ΔADJ. Since $\overrightarrow{CE}$ does not intersect $\overline{DJ}$, $\overrightarrow{CE}$ intersect $\overline{AJ}$ at a point K. now (3) follows; the line that we wanted is $\overrightarrow{CE}$.

Exercises

By the interior of an open triangle $\Delta DPAB$, we mean the intersection of the interiors of $\angle P$ and $\angle A$. If a line intersects the interior of an open triangle, does it follow that the line intersects one of the sides? Why or why not?

1. The same question, for the case where $\overrightarrow{PD}\Big|\overrightarrow{AB}$

2. In a Euclidean plane, if a line intersects the interior of an angle, does it follow that the line intersects the angle?

2. Euclidean Geometry

Though in schools most students learn plane geometry/Euclidean geometry, there are actual many different types. These different types were developed by other mathematicians who developed theories and research that may have contradicted the work of other. But, here our concern is Euclidean geometry which is based on rules called postulates as stated below. It is different from other geometries, such as absolute/neutral geometry, hyperbolic geometry, elliptic geometry and the like where no parallel lines exist, because of the parallel postulate. E of

Euclid's Axioms of geometry: The Euclidean geometry is based on the following postulates

Postulate 1: We can draw a unique line segment between any two points.

Postulate 2: Any line segment can be continued indefinitely.

Postulate 3: A circle of any radius and any center can be drawn.

Postulate 4: Any two right angles are congruent.

Postulate 5: Let l and m be two lines cut by a transversal in such a way that the sum of the measures of the two interior angles on one side of t less than 180. Then l and m intersect on that side t.

2.1. Euclidean Parallel Postulate and Some Consequences

The parallel postulate was the most controversial of Euclid's postulate for geometry. Many mathematicians felt that it should be possible to deduce the parallel postulate from Euclid's other postulates. It was later proved to be impossible to deduce the parallel postulate from the other postulates, efforts to do so led the invention of various non-Euclidean geometries in which the parallel postulate is violated. Here below it will be given the statement of Euclidean parallel postulate and some of its consequences.

Definition 2.1.1: Two distinct lines are parallel if they have no points in common. We also say that any line is parallel to itself.

The word parallel simply means that two lines have no points in common. It doesn't say anything about being in the same direction, or being equidistance from each other, or anything else.

Euclidean parallel postulate: For every line l and for every point P that does not lies on ℓ there is exactly one line $\overleftrightarrow{AB}$ such that P is on α and α. The parallel postulate in its equivalent form:

[P](Play fair's Axiom): For each point P and each line l, there exist at most one line through P parallel to l. Indeed, in Euclid's development of geometry; this is not an Axiom, but, a theorem that can be proved from the axioms. However, some mathematicians like to take the statement [P] as an axiom instead of using Euclid's parallel postulate. As a result, it is very important to explain in what sense we can say that Euclid's parallel postulate is equivalent to fair's play axiom. Since the parallel postulate plays such a special rule in Euclid's geometry, let us make a special point of being aware when we use this postulate, and which theorems are dependent on its use. Let us recall neutral geometry the collection of all postulates and common notations except parallel postulate together with all theorems that can be proved without using parallel postulate. If we take neutral geometry, and add back the parallel postulate, then we recover the ordinary Euclidean geometry and we can prove [P] as a theorem.

Euclid has proved, using the parallel postulate, that the angle sum in triangle is always two right angles. This property of triangles is equivalent to the parallel postulate, that is one can also prove that the converse implication, that if the angle sum is assumed to be two right angles, then the parallel postulate follows. Thus, proving the parallel postulate is equivalent to proving the angle theorem.

Theorem 2.1.1: Given two lines and a transversal. If the lines are parallel, then each pair of alternate interior angles are congruent.

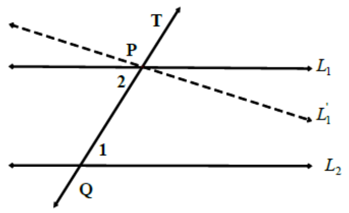


Figure 2.1.1

Proof: There is exactly one line L_1' through P, for which the alternate interior angles are congruent, and by theorem in chapter one, we have $L_1' \| L_2$. Since there is only one such parallel line, we have $L_1' = L_2$. Therefore, $\angle 1 \cong \angle 2$ this was to be proved.

The proof of the following theorem is entirely analogous.

Theorem 2.1.2: Given two lines and a transversal. If the lines are parallel, then each pair of corresponding angles is congruent. The inequality $m\angle A + m\angle B + m\angle C \le 180$ now becomes an equation.

Theorem 2.1.3: In any triangle ΔABC we have $m\angle A + m\angle B + m\angle C = 180$.

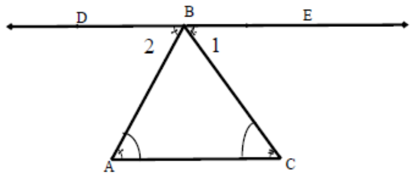


Figure 2.1.2

Proof: let L be the parallel to $\overleftrightarrow{AC}$ through B. Let D and E be points of L such that D-B-E and such that D and A are on the same side of $\overleftrightarrow{BC}$. Then $m\angle 2 + m\angle B = m\angle DBC$ and $m\angle DBC + m\angle 1 = 180^0$. Therefore, $m\angle 1 + m\angle B + m\angle 2 = 180$. By theorem 2.1.1 $m\angle 1 = m\angle C$ and; $m\angle 2 = m\angle A$; Therefore $m\angle A + m\angle B + m\angle C = 180$. This was to be proved.

Theorem 2.1.4: The acute angles of a right triangle are complementary.

Theorem 2.1.5: Every Saccheri quadrilateral is a rectangle.

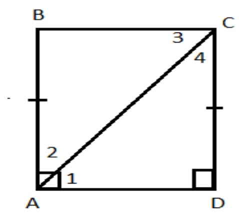


Figure 2.1.3

Proof: By theorem 2.1.1 $m\angle 1 = m\angle 2$. Since AB=DC and AC=AC, it follows that $\Delta BAC \cong \Delta DCA$. Therefore, $m\angle B \cong m\angle D$, is a right angle. The proof that $\angle C$ is a right angle is obtained merely by permuting the notations. Thus we have finally shown that rectangles exist. Note that in this proof we are using a figure to explain the notation. If the reader (or the writer) sees no other way to explain, say, the idea of alternate interior angles, then it is worthwhile to fight our way through the problem as we did in the previous chapter. But once we done this, we have earned the right to speak in the abbreviated language of pictures. A quadrilateral is a trapezoid if at least one pair of opposite sides are parallel (It is sometimes required that the other pair of sides be nonparallel, but this is artificial, just as it would be artificial to require that an isosceles triangle be nonequidrilateral). If both pairs of opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram. If two adjacent sides of a parallelogram are congruent, then the quadrilateral is rhombus. The proofs of the following theorems are omitted. (They are not much harder to write than to read.)

2.2. Equivalent Form of the Euclidean Parallel Postulate

In this section we consider some statements that are equivalent to Euclid’s parallel postulate. When says that two statements are equivalent in this sense we mean that if we add either statement to the axioms of neutral Geometry, we can prove the other statement. It does not mean that the two statements are precisely logically equivalent.

Euclidean parallel postulate: For every line ℓ and for every point P that does not lie on ℓ there is exactly one line m such that $P \in m$ and $m \| \ell$.

Equivalent Axiom (Euclid's Fifth Postulate)

Let ℓ and m be two lines cut by a transversal in such a way that the sum of the measures of the two interior angles on one side of t is less than 180. Then ℓ and m intersect on that side of t.

Euclid's Fifth postulates states that if $\alpha+\beta<180$, then ℓ intersects m at a point C that is on the same of t as α and β

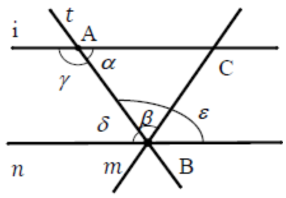


Figure 2.2.1

Proof: (Euclid's fifth postulate is equivalent to the Euclidean Parallel Postulate)

$(\Rightarrow)$ [The Euclidean Parallel Postulate $\Rightarrow$ Euclid's fifth postulate]

Let l, m, t, α, β be as indicated in figure 2.1, i.e, construct the lines l, m and n as shown; then $\alpha+\beta<180$. There is a line n through β such that $\gamma+\delta=180$ (by the protractor postulate). By the linear pair theorem, then $\varepsilon+\delta=180$ and $\gamma+\delta=180$.

Hence, $\alpha+\varepsilon=180-\delta+180-\gamma=360-(\delta+\gamma)=360-180=180$..........................(*)

Thus, both pairs of non-alternating interior angles formed by t sum to 180. By assumption $\alpha+\beta<180$ substituting equation (*) gives $180-\varepsilon+\beta<180$, $\beta<\varepsilon$. In particular, since $\beta\neq\varepsilon$, then $m\neq n$. Since $\delta=180-\gamma=\alpha$, $n||\ell$ (alternate interior angle theorem). Since $m\neq n$ this means m is not parallel to ℓ (this is because we are assuming the Euclidean parallel postulate, that there is only one line through B that is parallel to ℓ). Since m is not parallel to ℓ, they intersect at a point C, and there must be such a point C on the same side of t as the angles α and β. This is Euclid's fifth postulate.

$(\Leftarrow)$ [Euclid's fifth postulate $\Rightarrow$ Euclidean parallel postulate]

See proof that Euclid's fifth postulate implies the Euclidean parallel postulate.

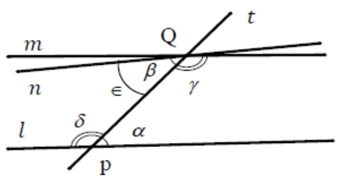


Figure 2.2.2

Assume Euclid's fifth postulate. Let ℓ be a line and P be a point such that $p\notin\ell$. Drop a perpendicular line from P to ℓ, and all the foot of the line Q. Construct m through P such that $m\perp\overleftrightarrow{PQ}$. By the alternate interior angles theorem, $\ell||m$. Assumer $n\neq m$ is a second line through P such that $\ell||n$. Then $\overleftrightarrow{PQ}$ is a transversal to n and ℓ. Since, $n\neq m$ the interior angles $\gamma\neq 90$ and $\delta\neq 90$. Since, they form a linear pair $\gamma+\delta=90$. Hence one of γ, δ is less than 90 and another is greater than 90. By Euclid's fifth postulate, lines n and ℓ meet on whichever side of $\overleftrightarrow{PQ}$ the smaller of angles γ and δ lies. Thus, n is not parallel to ℓ. Hence there is only one line through P that is parallel to ℓ. Hence, the Euclidean parallel postulate follows from Euclid's fifth postulate.

2.3. The Euclidean Parallel Projections

We know that the perpendicular from a point to a line always exists and is unique. Furthermore, the parallel projection theorem is one consequence of Euclidean Parallel Postulate. We will discuss the general notion of parallel projection in plane as follows. And we also show that parallel projection preserves betweens, congruence and ratios, let us first consider the special case indicated in the following figure, and treated in next the theorem.

Theorem 2.3.1: Every parallel projection is a one-to-one correspondence.

Proof: Given $f: L\rightarrow L^{'}$ the projection of L onto $L^{'}$ in the direction T (see figure 2.3.1). Let g be the projection L onto $L^{'}$ in the direction of T. Obviously g reverses the action of L. that is if $P=g(P^{'})$, then $P^{'}=f(p)$,: $L^{'}\rightarrow L$. Therefore f

is a one-to-one correspondence $L^{'} \leftrightarrow L$, which was to be proved (another way of putting it is to say that every point $P^{'}$ of $L^{'}$ is equal $f(P)$ for one and only one point P of L.)

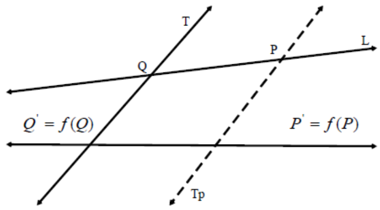


Figure 2.3.1

Theorem 2.3.2: Parallel projection preserve betweeness.

Restatement: Let $f: L \to L^{'}$ be a parallel projection. If P-Q-R on L, then $P^{'} - Q^{'} - R^{'}$ on $L^{'}$

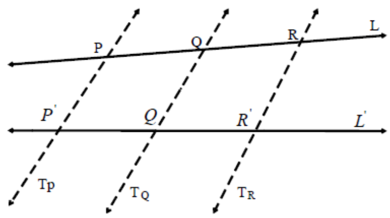


Figure 2.3.2

Here, of course $P^{'} = f(P)$, $Q^{'} = f(Q)$ and $R^{'} = f(R)$

Proof: Let T_P, T_Q, T_R be as in the definition of a parallel projection, so that $T_P \| T_Q \| T_R$. Then R and $R^{'}$ are on the same side of T_Q, because $\overrightarrow{RR^{'}}$ does not intersect T_Q. Similarly, P and $P^{'}$ are on the same side of T_Q. But P and R are on opposite sides of T_Q, because P-Q-R, and $P^{'}$ and $R^{'}$ are on opposite sides of T_Q. Therefore, $\overleftrightarrow{P^{'}R^{'}}$ intersects T_Q in a point X. since $T_Q \neq L^{'}$, there is only one such point of intersection. Therefore, $X = Q^{'}$. Therefore, $Q^{'}$ lies on $\overleftrightarrow{P^{'}R^{'}}$, and $P^{'} - Q^{'} - R^{'}$, which was to be proved.

2.4. Basic Similarity Theorem

Here we will revise the preliminary notions that could be used in this section like ration and proportion while we study similarity of triangles.

Definition 2.4.1: - A comparison of the magnitudes of two quantities of the same kind in the same unit is called a **ratio**. It is usually expressed as quotient of two numbers. For instance, if we are given lengths of two line segments as $\overline{AB} = 16cm$ and $\overline{DC} = 7cm$, then the ratio of their lengths is 16:7.

Definition 2.4.2: - Any equality of two ratios is called a **proportion**.

Remark:

- ✓ A proportion is usually expressed as $\frac{a}{b} = \frac{c}{d}$ or $a:b = c:d$
- ✓ The constant ratio $k = \frac{a}{b} = \frac{c}{d}$ is called the proportionality constant (Common values of each ratio)
- ✓ If three quantities a,b,c are such that, $\frac{a}{b} = \frac{b}{c}$ then b is called a mean proportional between a and c. Thus, if b is the mean proportional between a and c, then $b^2 = ac$.

Ratio of segment of a line: Let p be a point on the line segment $\overline{AB}$.

If $A - P - B$, then $\overline{AB}$ is said to be divided internally at p in the ratio $AP:PB$.

Figure 2.4.1

If $P-A-B$ or $A-B-P$, then $\overline{AB}$ is said to be divided externally at p in the ratio $AP:PB$ (See figure below ii and iii)

ii iii

Theorem 2.4.1: If a line parallel to one side of a triangle intersects the other two sides (at points that divides the sides internally), then it divides each of these sides in segments which are proportional.

Let us investigate some important proportions that can be deduced from this theorem.

In figure 2.4.2. Let $\overline{DE} \parallel \overline{BC}$. Then it follows that

$$\frac{AD}{DB} = \frac{AE}{EC} \text{..................................}(1)$$

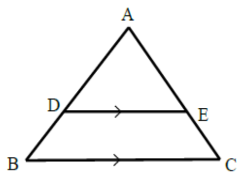


Figure 2.4.3

But $\frac{AD}{DB} = \frac{AE}{EC} \Leftrightarrow \frac{AD}{DB} + 1 = \frac{AE}{EC} + 1 \Leftrightarrow \frac{AD+DB}{DB} = \frac{AE+EC}{EC}$

$$\Leftrightarrow \frac{AB}{DB} = \frac{AC}{EC} \text{..}(2)$$

Also from (1), it follows that $\frac{DB}{AD} = \frac{EC}{AE} \Leftrightarrow 1 + \frac{DB}{AD} = 1 + \frac{EC}{AE} \Leftrightarrow \frac{AD+DB}{AD} = \frac{AE+EC}{AE}$

$$\Leftrightarrow \frac{AB}{AD} = \frac{AC}{AE} \text{..}(3)$$

Theorem 2.4.2: (Basic Similarity theorem)

If l_1, l_2 and l_3 are three parallel lines, with common transversal m and n, then $\frac{BC}{AB} = \frac{EF}{DE}$

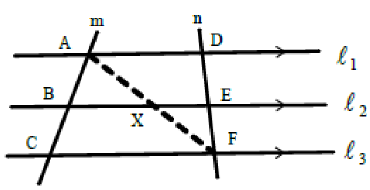


Figure 2.4.4

Here l_1, l_2 and l_3 are parallel lines, with common transversal m and n. We want to prove that $\frac{BC}{AB} = \frac{EF}{DE}$

Proof: Let m and n be transversals to l_1, l_2, l_3, where $l_1 \parallel l_2 \parallel l_3$.

To show that $\frac{AB}{BC} = \frac{DE}{EF}$ (see figure below)

Join A with F and apply the above theorem in ΔACF and ΔFDA, to get $\frac{AB}{BC} = \frac{AX}{XF}$ and $\frac{FX}{XA} = \frac{FE}{ED}$ (X is the point of intersection of $\overleftrightarrow{AF}$ and l_2)

But $\frac{XF}{AX} = \frac{FX}{XA} = \frac{FE}{ED} = \frac{EF}{DE} \Rightarrow \frac{AX}{XF} = \frac{DE}{EF}$

$\therefore \frac{AB}{BC} = \frac{DE}{EF}$

NOTE: The theorem stated above is one of the basic theorems in proving similarity of two triangles.

Theorem 2.4.3: if M and N are two points on sides $\overline{XY}$ and $\overline{XZ}$ of ΔXYZ, respectively such that $\overleftrightarrow{MN} \parallel \overleftrightarrow{YZ}$, then $\frac{XM}{XY} = \frac{MN}{YZ} = \frac{XN}{XZ}$

Proof: By theorem 3, we have $\frac{XM}{XY} = \frac{XN}{XZ}$

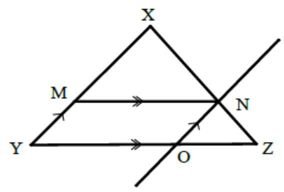


Figure 2.4.5

Now since N is not on the $\overleftrightarrow{XY}$, there exists a unique line l through N parallel to $\overleftrightarrow{XY}$. Let this line intersect $\overline{YZ}$ at O (see the above figure) then by theorem (1) $\frac{XZ}{XN} = \frac{YZ}{YO}$

That is $\frac{XN}{XZ} = \frac{OY}{YZ}$ But $\overline{OY} \equiv \overline{NM}$ (why?).

Hence, $\frac{XN}{XZ} = \frac{MN}{YZ}$

Therefore, $\frac{XM}{XY} = \frac{MN}{YZ} = \frac{XN}{XZ}$

Theorem 2.4.4: If points D and E are respectively on sides $\overline{AB}$ and $\overline{AC}$ of ΔABC such that $\frac{AD}{AB} = \frac{AE}{AC}$, then $\overline{DE} \parallel \overline{BC}$.

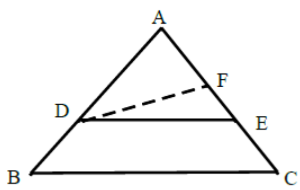


Figure 2.4.6

Proof: Suppose $\overline{DE}$ is not parallel to $\overline{BC}$. Then by parallel axiom there exists a point F on $\overline{AC}$ different from E such that $\overline{DF} \parallel \overline{BC}$. Hence $\frac{AD}{AB} = \frac{AF}{AC}$ by theorem (1)

But from the hypothesis of the theorem we have $\frac{AD}{AB} = \frac{AE}{AC}$. Thus $\frac{AF}{AC} = \frac{AE}{AC}$ and hence AF=AE. This in turn implies $\overline{AF} \equiv \overline{AE}$, contrary to axiom of segment construction as $F \neq E$. Therefore the supposition $\overline{DE}$ is not parallel to $\overline{BC}$ is wrong.

Consequently $\overline{DE} \parallel \overline{BC}$

2.5. Similarities between Triangles

We will discuss the mathematical notion of similarity which describes the idea of change of scale that is found in such forms as map making, perspective drawing, photographical enlargement and indirect measurement of distance. In this section we will mainly discuss some important theorem that could be used to prove similarity of triangles. The proofs of similarity theorem are based on the use of basic similarity theorem. Recall from high school that geometric figures are similar when they have the same shape, but not necessary same size.

Definition 2.5.1: Two triangles ΔABC and ΔDEF are said to be similar, written as

$\Delta ABC \sim \Delta DEF$ if and only if

i. All three parts of corresponding angles are congruent

ii. Lengths of all three pairs of corresponding sides are proportional.

Note: To establish similarity of triangles, however, it is not necessary to establish congruence of all pairs of angles and proportionality of all pairs of sides. It also important to note that triangle similarity do require Euclid's parallel postulate.

Note:

1) If ΔABC is similar to ΔDEF, we denote this by $\Delta ABC \sim \Delta DEF$.
2) Similar triangles should always be named in such way that so that the order of the letters indicates the correspondence between the two triangles.
3) $\Delta ABC \sim \Delta DEF$ if and only if
   i. $\angle A \equiv \angle D, \quad \angle B \equiv \angle E, \quad \angle C \equiv \angle F$, and
   ii. $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$
4) The common value of each ratio in (ii) is called proportionality constant.
5) Intuitively speaking, two triangles are similar if they have the same shape, although not necessary the same size. It looks as if the shape ought to be determined by the angles alone, and this is true.

Theorem 2.5.1: The AAA similarity theorem.

Given a correspondence between two triangles. If correspondence angles are congruent, then the correspondence is a similarity.

Restatement: Given ΔABC, ΔDEF and correspondence $ABC \leftrightarrow DEF$

IF $\angle A \cong \angle D, \quad \angle B \cong \angle E, \quad \angle C \cong \angle F$, then $\Delta ABC \sim \Delta DEF$

Proof: Let E' and F' be points of $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ as shown in figure

By SAS, we have $\Delta ABC \cong \Delta DEF$,

Therefore $\angle AE'F \cong \angle E$ Since $\angle B \cong \angle E$, we have $\angle AE'F' \cong \angle B$; thus $E'F' \parallel BC$ and A, F', C

Correspond to A, E', and B under parallel projection.

Since parallel projection preserves ratios, we have $\dfrac{AE'}{AB} = \dfrac{AF'}{AC}$

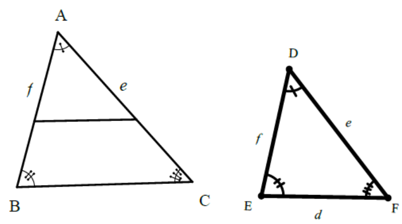


Figure 2.5.1

In exactly the same, merely changing the notations, we can show that $\dfrac{AF'}{AC} = \dfrac{EF}{BC}$

Therefore, $\dfrac{AF'}{AC} = \dfrac{EF}{BC} = \dfrac{AE'}{AB}$

Hence, corresponding angles are congruent and corresponding sides are proportional. By definition, $\Delta ABC \sim \Delta DEF$

Theorem 2.5.2: AA similarity theorem

If two angles of one triangle are congruent to the corresponding two angles of another triangle, the triangles are similar.

Proof: Let ΔABC and ΔXYZ be two triangles such that $\angle A \equiv \angle X$ and $\angle B \equiv \angle Y$. we need to show that $\Delta ABC \sim \Delta XYZ$. Since $\angle A = \angle X$ and $\angle B = \angle Y$, then $\angle C \equiv \angle Z$ (why?). So it remains to show that the corresponding sides are proportional.

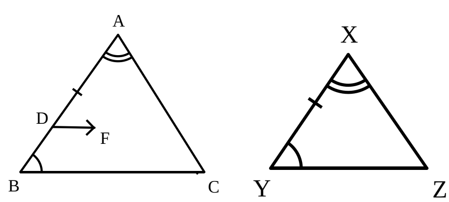


Figure 2.5.2

If $\overline{AB} \equiv \overline{XY}$, then $\Delta ABC \sim \Delta XYZ$ (why?)

If $\angle C \equiv \angle Z$, $\overline{AB} \not\equiv \overline{XY}$, then either $AB < XY$ or $XY < AB$. Without loss of generality assume that $XY < AB$. Then there exists a point D on $\overline{AB}$ such that A-D-B and $\overline{AD} = \overline{XY}$. By axiom of angle construction there exists a point F on the half plane determined by $\overline{AB}$ containing C such that $\angle ADF \equiv \angle XTZ$

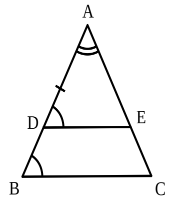


Figure 2.5.3

Since $\angle ABC \equiv \angle XTZ$, $\angle ADF \equiv \angle ABC$ by transitivity. Hence $\overline{DF} \parallel \overline{BC}$ (why?)

Since ray $\overrightarrow{DF}$ does not pass through the vertices of ΔABC and does not intersect $\overline{BC}$, it must intersect $\overline{AC}$ at some point E. Thus, $\Delta ADE \equiv \Delta XYZ$ by ASA. Hence $\overline{AD} \equiv \overline{XY}$, $\overline{DE} \equiv \overline{YZ}$, and $\overline{EA} \equiv \overline{ZX}$. But $\overline{DF} \parallel \overline{BC}$ as $E \in \overline{DF}$ and $\overline{DF} \parallel \overline{BC}$.

It then follows from theorem (3) that this in turn implies $\frac{XY}{AB} = \frac{YZ}{BC} = \frac{ZX}{CA}$.

Therefore $\Delta ABC \sim \Delta XYZ$

Theorem 2.5.3: SAS similarity theorem

Given a correspondence between two triangles. If two pairs of corresponding sides are proportional, and the included angles are congruent, then the correspondence is a similarity.

Proof: Given two triangles ΔABC and ΔPQR such that $\angle A \equiv \angle P$ and $\frac{AB}{PQ} = \frac{AC}{PR}$.

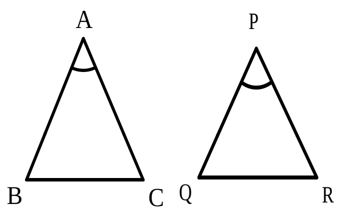


Figure 2.5.4

To show that $\Delta ABE \sim \Delta PQR$. Since it is given that $\angle A \equiv \angle P$, it is sufficient to show that $\angle B \equiv \angle Q$. Let D and E be two points on $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively such that $\overline{AD} = \overline{PQ}$ and $\overline{AE} = \overline{PR}$ (this is possible by axiom of segment construction).

Then $\Delta ADE \equiv \Delta PQR$ by SAS

Hence $\angle ADE \equiv \angle PQR$

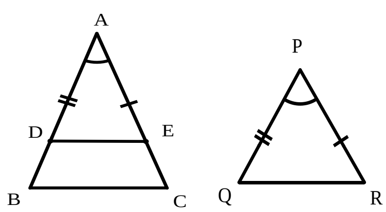


Figure 2.5.5

From $\frac{AB}{PQ} = \frac{AC}{PR}$, $\overline{AD} = \overline{PQ}$ and $\overline{AE} = \overline{PR}$ it follow that $\frac{AB}{AD} = \frac{AC}{AE}$,

Thus, $\overline{DE} \parallel \overline{BC}$ by theorem (4) and hence $\angle ADE \equiv \angle ABC$

SO $\angle ADE \equiv \angle PQR$ by transitivity therefore $\Delta ABC \sim \Delta PQR$ by AA similarity as $\angle A \equiv \angle P$ and $\angle B \equiv \angle Q$

Theorem 2.5.4: The SSS similarity theorem

If two triangles are such that the corresponding sides are proportional, then the two triangles are similar.

Proof: Left for reader.

Theorem 2.5.5: The bisectors of an angle of a triangle divide the opposite side into segments which are proportional to the adjacent sides.

Restatement: If in ΔABC, $\overrightarrow{AD}$ is the bisector of $\angle BAC$ where D is point on $\overline{BC}$, then $\frac{AB}{AC} = \frac{BD}{CD}$

Proof: Left as exercise.

Theorem 2.5.6: If an external bisector of an angle of a triangle intersects the line containing the opposite side, then the point of intersection divides the opposite side externally into segment whish are proportional to the adjacent sides.

Proof:

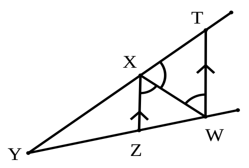


Figure 2.5.6

Let the external bisector of angle $\angle X$ of ΔXYZ intersect the opposite side YZ at W (externally).

Draw a line l through W parallel to $\overline{XZ}$. Then l and $\overleftrightarrow{XY}$ are not parallel (why?). So they intersect at some point, say T. $\angle ZXW \equiv \angle TWX$ Since they are alternate interior angles.

Also, $\angle ZXW \equiv \angle TXW$ so we have $\angle TXW \equiv \angle TWX$ by transitivity. Thus, XT=TW.

Since $\overline{XZ} \parallel \overline{TW}$, we have $\frac{YW}{WZ} = \frac{YT}{TX} = \frac{YT}{TW}$ ……………………………. (1)

From $\Delta YTW \backsim \Delta YXZ$ it follow that $\frac{YW}{YZ} = \frac{YT}{TX} = \frac{TW}{XZ}$, which implies that $\frac{YT}{TW} = \frac{YX}{XZ}$ …….. (2)

From (1) and (2) we conclude that $\frac{YW}{WZ} = \frac{YX}{XZ}$. this completes the proof.

Example: Let $\overleftrightarrow{BE}$ and $\overleftrightarrow{DC}$ are angle bisectors of $\angle CBF$ and $\angle ACB$ respectively. If AD=21cm, AC=30cm, and BC=20cm, then find DB and EC.

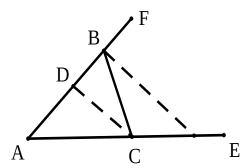


Figure 2.5.7

Solution: Since $\overrightarrow{CD}$ is the bisector of $A\hat{C}B$, $\frac{AC}{BC} = \frac{AD}{DB}$ (why?)

$\therefore DB = \frac{AD \times BC}{AC} = \frac{21 \times 20}{30} = 14$

Since $\overrightarrow{BE}$ is the external bisector of angle $\angle B$ of ΔABC, $\frac{AB}{BC} = \frac{AE}{EC}$ (why?).

Thus $\frac{35}{20} = \frac{30 + EC}{EC}$,

$\therefore EC = 40$

Theorem 2.5.7: In a right triangle, if an altitude is drawn to the hypotenuse, then

I. The triangle is divided into two similar right triangles and which are also similar to each other.

II. The altitude is the mean proportional between the segments of the hypotenuse.

III. Either leg is the mean proportional between the hypotenuse and the segment of the hypotenuse adjacent to the leg.

Proof: Let ΔABC be a right triangle with right angle at C and $\overrightarrow{CD}$ be altitude to the hypotenuse $\overline{AB}$.

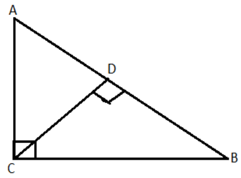


Figure 2.5.8

i) Then $\angle ACD$ and $\angle CBD$ are congruent as they are complements of the same angle $\angle CAB$. similarly $\angle CAD \equiv \angle BCD$. Thus $\Delta ABC \backsim \Delta ACD$ and $\Delta ABC \backsim \Delta CBD$ by AA. Then $\Delta ACD \backsim \Delta CBD$

ii) From $\Delta ACD \sim \Delta CBD$, it follow that, $\frac{CD}{BD} = \frac{AD}{CD}$. That is $CD^2 = AD.BD$

iii) From $\Delta ABC \sim \angle ACD$ it follows that, $\frac{AB}{AC} = \frac{AC}{AD}$

That is $AC^2 = AB.AD$. From $\Delta ABC \sim \angle CBD$, we have $\frac{AB}{CB} = \frac{BC}{BD}$.

That is $BD^2 = AB.BD$

2.6. Pythagorean Theorem

In mathematics, the Pythagorean Theorem is a relation in Euclidean geometry among the three sides of a right triangle (right-angle triangle)

In terms of areas, it states that in any right triangle, the area of the square whose side is the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares whose sides are the two legs (the two sides that meet at a right angle).

Euclid's version of Pythagorean Theorem: The sum of the areas of the squares on the legs (a, b) equals the area of the square on the hypotenuse(c).

The theorem can be written as an equation relating the length of the sides a, b, and c often called the Pythagorean equation $a^2 + b^2 = c^2$ where c represents the length of the hypotenuse, and a, b represent the lengths of the other two sides.

These two formulations show two fundamental aspects of this theorem: it is both a statement about areas and about lengths. The Pythagorean Theorem has been modified to apply outside its original domain. A number of these generations are found in more advanced mathematics courses including extension to many-dimensional Euclidean spaces, to spaces that are not Euclidean, to objects that are not right triangles, and indeed, to objects that are not triangles at all, but n-dimension solids.

Theorem 2.6.1: (Pythagorean Theorem)

Let ΔABC be a right triangle with right angle at vertex C. The square of the hypotenuse of a right triangle is equal to the sum of the square of the other two sides.

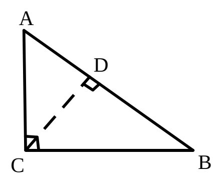


Figure 2.6.1

Proof: Let ΔABC be a right triangle with right angle at C. To show that $BC^2 + AC^2 = AB^2$. Draw altitude $\overline{CD}$ to $\overline{AB}$. Then from theorem (2.4.1) (iii) we have

$$AC^2 + BC^2 = (AB).(AD) + (AB).(BD)$$
$$= AB(AD + BD)$$
$$= AB.AB \quad \text{As A-D-B}$$
$$\therefore AB^2 = BC^2 + AC^2$$

Theorem 2.6.2: (Converse of Pythagorean Theorem)

If $a^2 + b^2 = c^2$, then $\angle c$ is a right angle.

Proof:

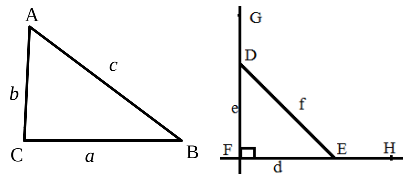


Figure 2.6.2

We are given with $a^2 + b^2 = c^2$. Construct a right angle at point F on rays $\overrightarrow{FG}$ and $\overrightarrow{FH}$.

Define point $E \in \overrightarrow{FH}$ such that FE=a, and define point $D \in \overrightarrow{FG}$ such that FD =b. Then $\triangle DEF$ is a right triangle

By the Pythagorean Theorem $f^2 = d^2 + e^2 = a^2 + b^2 = c^2$. This mean that $f = c$ and hence by SSS. $\Delta ABC \cong \triangle DEF$. Hence $\angle C = \angle F = 90$.

Definition 2.6.1 (Trigonometry)

Let ΔABC be a right triangle with right angle at vertex C, and let $\theta = \angle CAB$. Then if θ is a acute, we define $\sin\theta = \dfrac{BC}{AB}$ and $\cos\theta = \dfrac{AC}{AB}$. If θ is obtuse, then let $\theta' = 180 - \theta$ and define $\sin\theta = \sin\theta'$ and $\cos\theta = \cos\theta'$

Also, define $\sin 0 = 0$ and $\cos 0 = 1$; $\sin 90 = 1$ and $\cos 90 = 0$

Theorem 2.6.3: (Pythagorean identity)

$\sin^2\theta + \cos^2\theta = 1$

Proof: Exercise

Theorem 2.6.4: (law of Sines)

Let ΔABC be any triangle with sides a, b, c opposite vertices A, B, C. Then $\dfrac{a}{\sin\angle A} = \dfrac{b}{\sin\angle B} = \dfrac{c}{\sin\angle C}$

Proof: Exercise

Theorem 2.6.5: (Law of cosines)

Let ΔABC be any triangle with sides a, b, c opposite vertices A, B, C. Then

$c^2 = a^2 + b^2 - 2ab\cos\angle C$

Proof: Exercise

Euclid, of Course, did not state the Pythagorean Theorem in terms of the sum of the squares of the edges; to do so would have required algebra, which was not invented for another thousand years after Euclid. Instead, the theorem was expressed in terms of area.

Theorem 2.6.6: (Euclid's version of the Pythagorean Theorem)

The area of the square on the hypotenuse of a right triangle is equal to the sum of the squares on the legs.

Theorem 2.6.7: In any triangle, the product of a base and the corresponding altitude is independent of the choice of the base.

Restatement: Given ΔABC. Let $\overline{AD}$ be the altitude from A to $\overline{BC}$ and let $\overline{BE}$ be the altitude from B to $\overline{AC}$. Then $AD.BC = BE.AC$

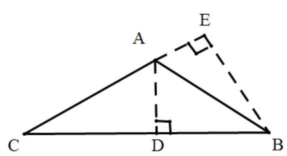


Figure 2.6.3

Proof: Suppose that $E \neq C$ and $D \neq C$, as shown in the figure. Then $\angle D = \angle E$ and $\Delta BEC \cong \Delta ADC,$ because both are right angles Therefore $\Delta BEC \sim \Delta ADC$

Hence $BE, BC \sim AD, AC.$ Thus $\dfrac{AD}{BE} = \dfrac{AC}{BC}$ and $AD.BC = BE.AC$ which was to be proved.

If E=C, then ΔABC is a right triangle with its right angle C and we also have D=C.

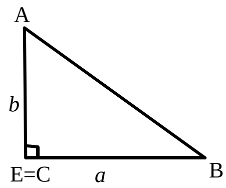


Figure 2.6.4

Theorem 2.6.8: For similar triangles, the ratio of any two corresponding altitudes is equal to the ratio of any two corresponding sides.

Restatement: Suppose that $\Delta ABC \sim \Delta A^{'}B^{'}C^{'}$. Let h be the altitude from A to $\overline{BC}$, and let $h^{'}$ be the altitude from $A^{'}$ to $\overline{B^{'}C^{'}}$. Then $\frac{h}{h^{'}} = \frac{AB}{A^{'}B^{'}}$

Proof:

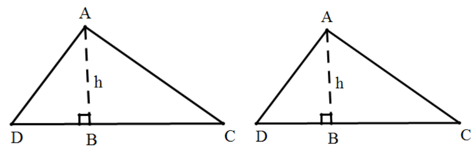


Figure 2.6.5

Let $\overline{AD}$ and $\overline{A^{'}D^{'}}$ be the altitudes whose lengths are h and $h^{'}$. If $D = B$, then $D^{'} = B^{'}$, and there is nothing to prove. If not, $\Delta ABD \sim \Delta A^{'}B^{'}D^{'}$ and the theorem follows.

Theorem 2.6.9: The area of a right triangle is half the product of the length of its legs.

Proof:

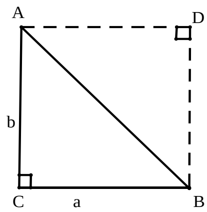


Figure 2.6.6

Given ΔABD, with the right angle at C. Let D be the point such that *ADBC* is a rectangle.

By the additivity postulate, area ($\square\ ADBC$) $= area(\Delta ABC) + area(\Delta ABD)$

By the rectangle formula, area ($\square\ ADBC) = ab$

Therefore $2area(\Delta ABC) = ab$ and area $area(\Delta ABC) = \frac{1}{2}ab$

Theorem 2.6.10: The area of the triangle is half the product of any base and the corresponding altitude.

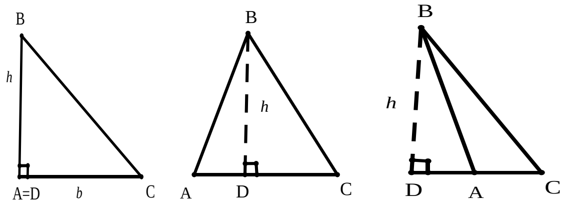


Figure 2.6.7

Proof: Given ΔABC. Let D be the foot of the perpendicular from B to $\overline{AC}$; let AC=b and let BD=h (as in each of the figures).

There are essentially, three cases to consider.

1. If A=D, then ΔABC is the right triangle and $ABC = \frac{1}{2}bh$, by theorem 1.

2. A-D-C. Let $AD = b_1$ and $DC = b_2$. by theorem 1, $BDA = \frac{1}{2}b_1h$, $BDC = \frac{1}{2}b_2h.$

By the additivity postulate $\Delta ABC = \Delta BDA + \Delta BDC$

Therefore $ABC = \frac{1}{2}b_1h + \frac{1}{2}b_2h = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}bh,$ this was to be proved

3. D-A-C. let $b^' = AD$ by theorem () $Area(\Delta BDC) = \frac{1}{2}(b^' + b)h.$

Also, by theorem $Area(\Delta BDA) = \frac{1}{2}(b^' + b)h.$

By the additivity postulate $\Delta ABC = \Delta BDA + \Delta BDC$

Therefore, $ABC = BDC - BDA = \frac{1}{2}(b^' + b)h - \frac{1}{2}b^'h = \frac{1}{2}bh$ this was to be proved

Theorem 2.6.11: If two triangles have the same altitude, then the ratio of their areas is equal to the ratio of their bases.

This theorem follows immediately from the area formula. If the triangles ΔABC and ΔDEF have bases b_1, b_2 and the corresponding altitude for each of them is h, then $\frac{ABC}{DEF} = \frac{\frac{1}{2}b_1h}{\frac{1}{2}b_2h} = \frac{b_1}{b_2}$ this was to be proved. In the same way, we get the following theorem.

Theorem 2.6.12: If two triangles have the same base, then the ratio of their areas is the ratio of their corresponding altitudes. The next theorem is a corollary of each preceding theorems.

Theorem 2.6.13: If two triangles have the same base and the same altitude, then they have the same area.

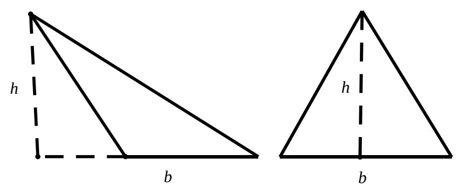


Figure 2.6.8

Theorem 2.6.14: If two triangles are similar, then the ratio of their areas is the square of the ratio of any two corresponding sides. That is if $\Delta ABC \sim \Delta DEF$, then $\frac{ABC}{DEF} = \left(\frac{a}{b}\right)^2$

Proof: if the altitude to $\overline{AC} = \overline{DF}$ are h and $h^'$, as in the above figure , then we know from theorem()that $\frac{h}{h^'} = \frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

Now, $\frac{ABC}{DEF} = \frac{\frac{1}{2}bh}{\frac{1}{2}eh^'} = \left(\frac{b}{e}\right)\left(\frac{h}{h^'}\right) = \left(\frac{b}{e}\right)^2 = \left(\frac{a}{d}\right)^2 = \left(\frac{c}{f}\right)^2$ which was to be proved

3. Hyperbolic Geometry

Introduction

Until the 19th century the Euclidean Geometry was the only known system of geometry which concerned with measurement, concepts of congruence, parallelism and perpendicularity. Then early in that century, a new system dealing with the same concepts was discovered. This new system was called Non-Euclidean System which contained theorems that disagreed with the Euclidean Theorems. For instance, hyperbolic geometry, and elliptic geometry are some examples of Non-Euclidean Geometry.

Non-Euclidean Geometry is not Euclidean Geometry. The term is usually applied only to the special geometries that are obtained by negating the parallel postulate, but, keeping the same the other axioms of Euclidean Geometry.

Since the first 28 postulates of Euclid's Elements do not use parallel postulate, then these results will also be valid in our first example of non-Euclidean Geometry called hyperbolic geometry.

Remember that one of Euclid's unstated assumptions was that lines are infinite. This will not be the case in our other version of Euclidean Geometry called elliptic geometry and so not all 28 propositions will hold there (for example, in elliptic geometry the sum of the angles of a triangle is always more than two right angles and two of the angles together can be greater than two right angles, contradicting proposition 17).

Hyperbolic geometry is the geometry you get by assuming all the postulates of Euclid, except the fifth one, which replaced by its negation.

3.1. The Poincare Model

In this section we shall assume that there is a mathematical system satisfying the postulates of Euclidean Plane Geometry, and we shall use Euclidean Geometry to describe a mathematical system in which the Euclidean parallel postulate fail, but in which the other postulates of Euclidean Geometry hold.

Consider a fixed circle C in an Euclidean Plane. We assume that, merely for the sake of convenience, that C is a unit circle. Let E be the interior of C.

Consider the following figure:

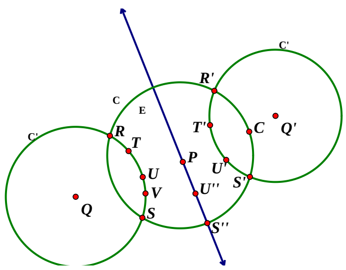


Figure 3.1.1

By hyperbolic circle we mean a circle C' which is orthogonal to C. When we say that two circles are orthogonal to each other, we mean that their tangents at each intersection point are perpendicular. If this happens at one intersection point R, then it happens at the other intersection point S. But, we shall not stop to prove this; this chapter is purely descriptive and proofs will come later.

The points of our hyperbolic plane will be the points of the interior E of C. By hyperbolic line we mean:

1. The intersection of E and a hyperbolic circle, or
2. The intersection of E and a diameter of C.

It is a fact that

Every two points of E lie on exactly one hyperbolic line. *

We are going to define a kind of "Plane geometry" in which the "plane" is the set E and the lines are the hyperbolic lines. In our new geometry we already know what is meant by point and line. We need next to define distance and angular measure. For each pair of points X, Y, either on C or in the interior C, let XY be the usual Euclidean distance.

Notice that if R, S, T, and U are as in the above figure, then R and S are not points of our hyperbolic plane, but they are points of the Euclidean plane that we started with. Therefore, all of the distances TS, TR, US, UR are defined, and * tells us that R and S are determined when T and U are determined.

There is one and only one hyperbolic line through T and U, and this line cuts the circle C in the points R and S. We shall use these four distances TS, TR, US, UR to define a new distance d (T, U)in our" plane" E, by the following formula:

$$d(T, U) = \left| log_e \frac{\frac{TR}{TS}}{\frac{UR}{US}} \right|$$

Evidently we have the following:

d is a function which can be defined as

$$d: ExE \longrightarrow R$$

Let us now look at the ruler postulate in chapter 1. On any hyperbolic line L, take a point U and regard this point as fixed. For every point T of L, let

$$f\,(T) = log_e \frac{\frac{TR}{TS}}{\frac{UR}{US}}$$

That is, f (T) is what we get by omitting the absolute value signs in the formula for d (T, U). We now have a function,

$$f: L \longrightarrow \Re$$

Where L is a hyperbolic line

We shall now show that f is a coordinate system for L.

If V is any other point of L, then

$$f(V)=log_e \frac{\frac{VR}{VS}}{\frac{UR}{US}}$$

Let x=f(T) and y=f(V). Then

$|x-y| = \left|log_e \frac{\frac{TR}{TS}}{\frac{UR}{US}} - log_e \frac{\frac{VR}{VS}}{\frac{UR}{US}}\right| = \left|log_e \frac{\frac{TR}{TS}}{\frac{VR}{VS}}\right|$. Since the absolute value of the difference of the logarithms is the absolute value of the logarithm of the quotient of the fractions.

Therefore, $|x-y| = d(T,V)$, which means that our new distance function satisfies the ruler postulate.

Since the ruler postulate in chapter one holds, the other distance postulates automatically hold.

We define betweenness, segment, rays, and so on, exactly as in chapter one. All of the Theorems of chapter one also hold in our new geometry. Because the new geometry satisfies the postulates on which the proofs of the theorems were based.

It is rather easy to convince yourself that the plane-separation postulate holds E.

To discuss congruence of angles, we need to define an angular-measure function. Given "hyperbolic angle" in our new geometry, we form an angle in the old geometry by using the two tangent rays:

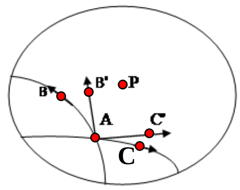


Figure 3.1.2

We then define the measure m<BAC to be the measure (in the old sense) of the Euclidean angle $<'BAC'$.

It is a fact that the resulting structure [E, L, d, m] satisfies all the postulates of chapter one, including the SAS postulate. The proof of this takes time, however, and it requires the use of more Euclidean Geometry than we know so far.

Granted that the postulates hold, it follows that the theorems also hold. Therefore, the whole theory of congruence, and of geometric inequalities, applies to the Poincare model of hyperbolic geometry.

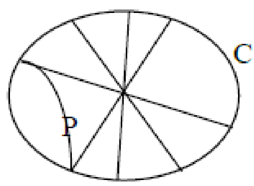


Figure 3.1.3

On the other hand, the Euclidean parallel postulate obviously does not hold for the Poincare model. Consider, for example a hyperbolic line L which does not pass through the center p of C (figure 3.3). Through p there are infinitely many hyperbolic lines which are parallel to L.

3.2. The Hyperbolic Parallel Postulate

Hyperbolic geometry (also called Lobachevskian geometry) is the kind represented by the Poincare model. In such geometry, when the familiar parallel postulate fails, it pulls down a great many familiar theorems with it. A few samples of the theorems in hyperbolic geometry which are quite different from the analogous theorems of Euclidean Geometry follow.

1. No quadrilateral is a rectangle. In fact, if a quadrilateral has three right angles, the fourth angle is always acute.
2. For any triangle, the sum of the measures of the angles is always strictly less than 180°.
3. No two triangles are ever similar, except in the case where they are also congruent.

The third of these theorems means that two figures cannot have exactly the same shape, unless they also have exactly the same size. Thus, in hyperbolic geometry, exact scale models are impossible.

In fact, each of the above three theorems characterizes hyperbolic geometry.

If the angle-sum inequality, m<A+m<B+m<C< $180°$ holds, even for one triangle, then the geometry is hyperbolic; if the angle sum inequality holds, even for one triangle, then the geometry is Euclidean and similarly for (1) and (3).

This has a curious consequence in connection with our knowledge of physical space. If physical space is hyperbolic, which it may be, it is theoretically possible for the fact to be demonstrated by measurement. For example, suppose that you measure the angles of a triangle with an error less than" 0.0001" for each angle. Suppose that the sum of the measures turns out to be $179°59'59.999''$. The difference between this and $180°$ is 0.001". This discrepancy could not be due to errors in measurement, because the greatest possible cumulative error is only "0.0003". Our experiment therefore, proves that the space that we live in is hyperbolic.

On the other hand, no measurement however exact can prove that the space is Euclidean. The point is that every physical measurement involves some possible error. Therefore, we can never show by measurement that an equation, r+s+t=180°, holds exactly; and this is what we would have to do to prove that the space we live in is Euclidean.

Thus, there are two possibilities:

1. The Euclidean parallel postulate does not hold in physical space, or
2. The truth about physical space will never be known.

The Hyperbolic parallel postulate: Given a line L and a point P not on L, there are at least two lines L' and L'' which contain P and are parallel to L.

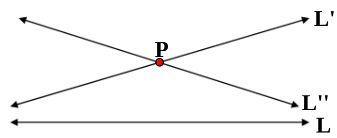


Figure 3.2.1

3.3. Closed Triangles and Angle Sum

In this section we deal specifically with the hyperbolic case. To avoid confusion, throughout this chapter, we shall mention the hyperbolic parallel postulate in every theorem whose proofs requires it.

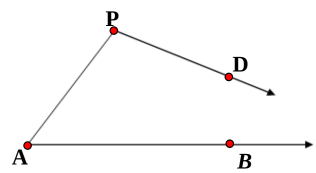


Figure 3.3.1

If $\overrightarrow{PD}$ *is parallel to* $\overrightarrow{AB}$, then ΔPABD is called a closed triangle.

Note that every closed triangle is an open triangle, but under hyperbolic parallel postulate the converse is false, because through P there is more than one line parallel to $\overrightarrow{AB}$.

Closed triangles have important properties in common with genuine triangles.

Theorem 3.3.1: The Exterior Angle Theorem

Under hyperbolic parallel postulate, in every closed triangle, each exterior angle is greater than its remote interior angle. That is, if $\overrightarrow{PD}$ *is parallel to* $\overrightarrow{AB}$ and Q-A-B, then $<$QAP$>$ $\angle$P.

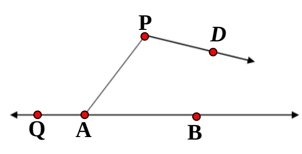


Figure 3.3.2

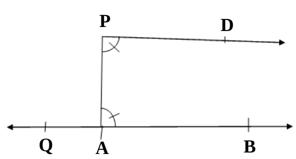


Figure 3.3.3

Proof:

If ΔDPAB is an isosceles triangle, this is obvious. Here, if hyperbolic parallel postulate holds, then $\angle$P and $<$PAB are acute angles (because c (a) $\angle 90°$ for every a), and therefore, $\angle$QAP is obtuse angle. ΔDPAB is equivalent to an isosceles open triangle ΔDPCB, and this open triangle is also closed:

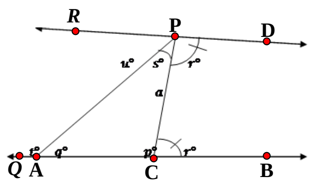


Figure 3.3.4

If C=A, there is nothing to prove. For the case A-C-B, let the degree measures of the various angles be as in the figure. Then $P > r$, because c $(a) < 90°$, and $P + q + s \leq 180°$, by theorems in chapter one.

Therefore, t=180°-q≥ p+s>r+s, and t>r+s, which proves half of our theorem.

To prove that the other half, we need to show that $\mu > q$. This follows from

$t = 180° - q > 180° - \mu = r + s.$

We found, in chapter one, that the critical function c was non- increasing. That is, if $a' > a$, then c (a') ≤c (a). Using the exterior angle theorem, we can shorten this result.

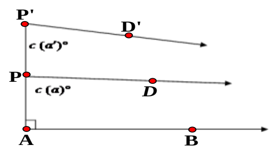


Figure 3.3.5

Theorem 3.3.2: Under hyperbolic parallel postulate, the critical function is strictly decreasing. That is, if a' >a then c(a') ≤c (a).

Proof:

In the above figure 3.9, AP=a and AP'=a', $\overrightarrow{PD}\overrightarrow{AB}$ and $\overrightarrow{P'D'}\overrightarrow{AB}$, so that $\overrightarrow{PD}\overrightarrow{P'D'}$.

Therefore, $\Delta D'P'$PD is a closed triangle. Therefore, c (a) >c (a'), this was to be proved.

Theorem 3.3.3: Under hyperbolic parallel postulate, the upper base angles of a saccheri quadrilateral are always acute.

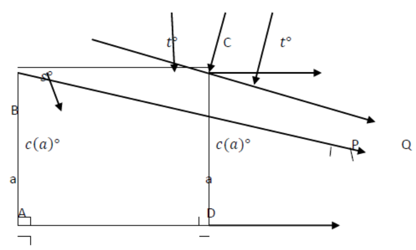


Figure 3.3.6

We already know, from chapter one, that they are congruent, and cannot be obtuse.

In the figure, $\overrightarrow{BQ}$ and $\overrightarrow{CP}$ are the critical parallels to $\overrightarrow{AD}$ through B and C.

Therefore, m∠ABQ=c (a) =m∠DCP, as indicated. Applying the exterior angle theorem to the closed triangle PCBQ, we see that t>s.

Therefore, t +c (a) >s + c (a).

Therefore, s +c (a) <90°, which proves our theorem.

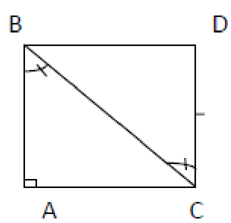


Figure 3.3.7

Theorem 3.3.4: Under hyperbolic parallel postulate, in every right triangle ABC, we have

m∠A + m∠B + m∠C ∠ 180°.

Proof: Suppose not. Then, if ∠A is the right angle, ∠B and ∠C must be complementary angles. Take point D on the opposite side of $\overline{BC}$ from A, so that ∠ BCD ≅ ∠ ABC and CD=AB. Then ΔABC ≅ ΔDCB by SAS; and quadrilateral

ABDC is a Saccheri quadrilateral. This is impossible, because ∠D is a right angle.

Theorem 3.3.5: Under hyperbolic parallel postulate, for every triangle ABC, we have m∠A + m∠B + m∠C ∠ 180°.

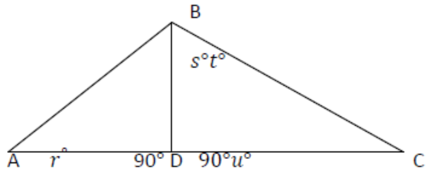


Figure 3.3.8

Proof:

Let $\overline{AC}$ be a longest side of ΔABC, and let $\overline{BD}$ be the altitude from B to $\overline{AC}$. Then r + s + 90°∠ 180°, and t + μ + 90°∠ 180°.

Therefore, r + (s + t) + μ∠ 180°, which proves the theorem.

Soon we shall see that under hyperbolic parallel postulate this theorem has a true converse:

For every number x∠ 180° there is a triangle for which the angle sum is x. Thus, 180° is not an upper bound for the angle sums of triangles, but is precisely their supremum.

3.4. The Defect of a Triangle and the Collapse of Similarity Theory

The defect of ΔABC is defined to be 180°-m∠A - m∠B - m∠C. The defect of ΔABC is denoted by δΔABC. Under hyperbolic parallel postulate we know that the defect of any triangle is positive, and obviously it is less than 180°. (Later we shall see that the converse holds: every number between 0° and 180° is the defect of some triangle.)

The following theorem is easy to check, regardless of under hyperbolic parallel postulate.

Theorem 3.4.1: GivenΔABC, with B – D – C. Then δΔABC = δΔABD + δΔADC

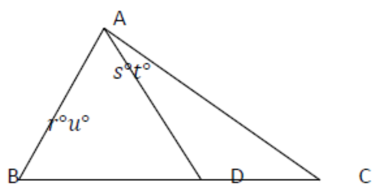


Figure 3.4.1

It has, however, an important consequence.

Theorem 3.4.2: Under hyperbolic parallel postulate, every similarity is congruence. That is, If ΔABC ~ΔDEF, then ΔABC ≅ ΔDEF.

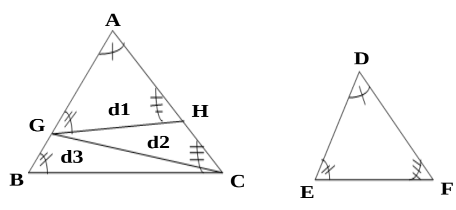


Figure 3.4.2

First we take G on $\overline{AB}$ so that AG =DE; and we take H and $\overline{AC}$ so that AH = DF. We then have ΔAGH ≅ ΔEDF, by SAS; therefore, ΔAGH ~ΔABC

If G = B, then H = C, and the theorem follows. We shall show that the contrary assumption G≠B and H≠ C (as shown in the figure) leads to a contradiction.

Let the defects of ΔAGH ~ΔGHC, and ΔGBC be d_1, d_2 and d_3 respectively, as indicated in the figure; let d be the defect of ΔABC. By two applications of the preceding theorem, we have: d=$d_1 + d_2 + d_3$. This is impossible, because the angle congruence's given by the similarity ΔABC ~ΔAGH tell us that d =d_1.

The additivity of the defect, described in theorem 3.4.1, gives us more information about the critical function. What we know so far is that

1. $0 < c(a) < 90$ for every $a > 0$, and
2. c decreases as a increases.

There remains the question of how small the numbers c (a) eventually become when a is very large. We might have either of the following situations:

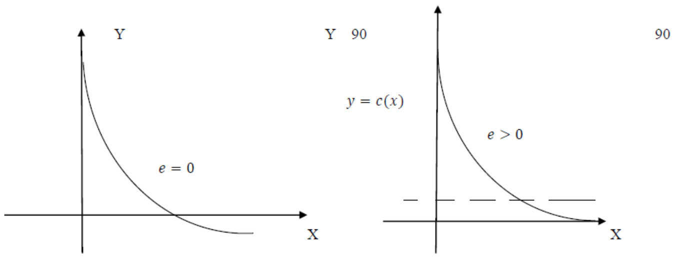


Figure 3.4.3

In each case, e= inf{c (a)}, that is, the greatest lower bound of the numbers c (a). In each case, it follows from (2) that $\lim_{a\to\infty} c(a)$=e. To prove the following theorem, therefore, we need merely show that e>0 is impossible.

Theorem 3.4.3: $\lim_{a\to\infty} c(a) = 0.$

Proof:

Suppose that c (a)>e>0 for every a.

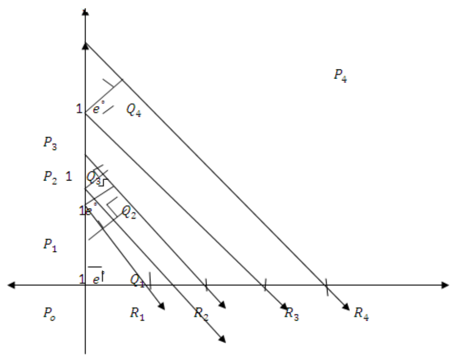


Figure 3.4.4

The markings in the figure should be self-explanatory. For each n, $\overrightarrow{P_nQ_n}$ intersects $\overrightarrow{P_oR_1}$, because e<c (n). The right triangles $\Delta P_nP_{n+1}Q_{n+1}$ all are congruent, and therefore have the same defect d_o. Consider now what happens to the defect d_n of $\Delta P_oP_nR_n$ where, n is increased by 1. In the figure below, the letters in the interiors of the triangles denote their defects. We have:

$\delta\Delta P_oP_nR_{n+1} = d_n + y,$

$\delta\Delta P_oP_nR_{n+1} = d_n + x,$

d_{n+1} =$(d_n + y) + (d_o + x),$

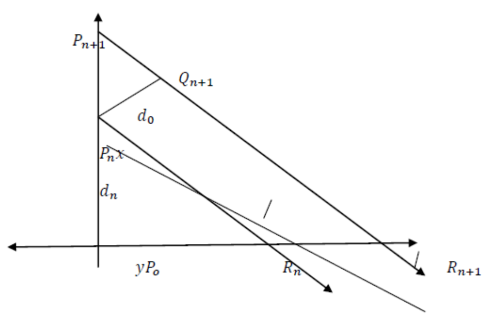


Figure 3.4.5

By theorem 3.4.1 in each case, Therefore, $d_{n+1} > d_n + d_o$

Thus, d_2>$d_1 + d_o$, $d_3 > d_2$+d_o>$d_1 + 2d_o$.

And by induction, we have $d_n > d_1$+ (n-1)d_o.

When n is sufficiently large, we have $d_n > 180°$, by the Archimedean postulate.this is possible, because the defect of a triangle is 180° minus the angle sum. Therefore, $c\ (a) > e > 0$ is impossible, which was to be proved.

Consider now what happens to the measure $r\ (a)$ of the base angles of an isosceles right triangle, as the length a of the legs becomes large.

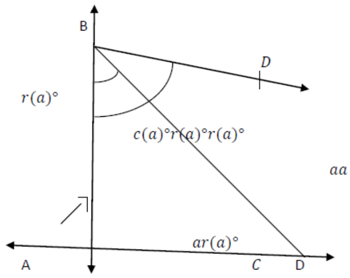


Figure 3.4.6

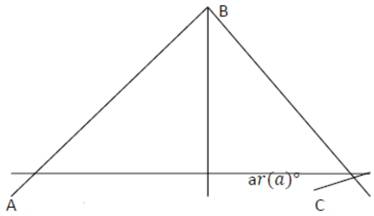


Figure 3.4.7

Here, $\overrightarrow{BD}/\overrightarrow{AC}$. Therefore, we always have r (a) <c (a). Therefore, $\lim_{a\to\infty} c(a) = 0$.

Let us now make the figure symmetrical by copying ΔABC on the other side of $\overleftrightarrow{AB}$. ForΔDBC, the angle sum is 4r(a). Therefore, the defect $180° - 4r(a)$ can be made as close to $180°$ as we please; we merely need to take a sufficiently large. Thus, $180°$ is not merely an upper bound of the numbers which are the defects of triangles; $180°$ is precisely their supremum.

Theorem 3.4.4: For every number x<$180°$ there is a triangle whose defect is greater than x.

4. The Consistency of the Hyperbolic Geometry

Under this chapter, we shall show that the Poincare Model satisfies all the postulates of hyperbolic geometry. In the analysis of the model we will depend, on Euclidean geometry, and so our consistency proof will be conditional. At the end of the chapter we shall know not that the hyperbolic postulates are consistent, but merely that they are as consistent as the Euclidean postulates.

4.1. Inversions of a Punctured Plane

Given a point A of a Euclidean plane E and a circle C with center at A and radius a. The set E-A is called a punctured plane. The inversion of E-A about C is a function,

f: E-A ↔ E-A,

defined in the following way. For each point P of E-A, let P'=*f*(P) be the point of $\overrightarrow{AP}$ for which

AP'= $\frac{a^2}{AP}$.

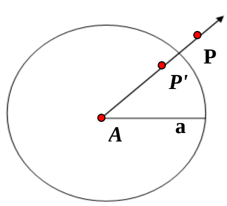


Figure 4.1.1

(Thus, for a=1, we have AP'=$\frac{1}{AP}$.). Since $\frac{a^2}{a} = a$, we have the following theorems.

Theorem 4.1.1: If P∈ C, then$f(P) = P$.

Theorem 4.1.2: If P is in the interior of C, then $f(P)$ is in the interior of C, and conversely.

Theorem 4.1.3: For every $P, f(f(P)) = P$.

That is, when we apply an inversion twice, this gets us back to wherever we started.

Proof:

$f(P)$ Is the point of $\overrightarrow{AP}$ for which $Af(P) = \frac{a^2}{AP}$, and $f(f(P))$ is the point of the same ray for which $Af(fP) = \frac{a^2}{Af(P)} = \frac{a^2}{a^2/_{AP}} = AP$

Therefore, $f\big(f(P)\big) = P$.

Theorem 4.1.4: If L is a line through A, then $f(L - A) = L - A$.

Here by $f(L - A)$ we mean the set of all image point $f(P)$, where $P \in A$. In general,

If $K \subset E - A$, then $f(K) = \{P' = f(P)/P \in K\}$.

It is also easy to see that "if P is close to A, then P' is far from A, and conversely; the reason is that "$\frac{a^2}{AP}$ is large when AP is small." In studying less obvious properties of inversion, it will be convenient to use both rectangular and polar coordinates, taking the origin of each coordinate system at A.

The advantage of polar coordinates is that they allow us to describe the inversion in the simple form. $f\colon E - A \leftrightarrow E - A$ $(r, \theta) \leftrightarrow (s, \theta)$

Where

$s = \frac{a^2}{r}$ and $r = \frac{a^2}{s}$

In rectangular coordinates we have, $P = (x, y) = (rcos\theta, rsin\theta), f(P) = (u\ v) = (scos\theta, ssin\theta)$,

Where r and s are related by the same equation as before evidently.

Just as $u^2 + v^2 = s^2$

$x^2 + y^2 = r^2$

These equations will enable us to tell what happens to lines and circles under inversions. We allow the cases in which the lines and circles contain the origin A, so that they appear in E-A "as punctured lines" and "punctured circles." Thus, we shall be dealing with four types of figures, namely, lines and circles, punctured and unpunctured. For short, we shall refer to such figures as k-sets. The rest of this section will be devoted to the proof that if K is a k-set, then so also is f (K). Let us look first, however, at a special case.

Let K be the line x=a.

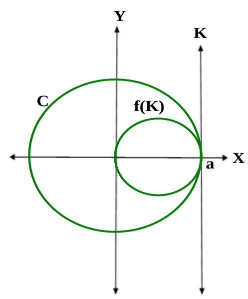


Figure 4.1.2

Then K is the graph of the polar equation $rcos\theta = a$

Since $r = \frac{a^2}{s}$, where $f(r, \theta) = (s, \theta)$, it follows that $f(K)$ is the graph of the condition

$\frac{a^2}{s} cos\theta = a, \qquad s \neq 0$

Or $s = acos\theta, \qquad s \neq 0$

$s^2 = ascos\theta, \qquad s \neq 0$

In rectangular form, this is

$u^2 + v^2 = au, u^2 + v^2 \neq 0.$

Replacing u and v by x and y (to Mach the labels on the axes), we see that $f(K)$ is the graph of

$x^2 - ax + y^2 = 0,\ x^2 + y^2 \neq 0$

And is hence the punctured circle with center at $(\frac{a}{2}, 0)$ and radius $\frac{a}{2}$. Thus, f has pulled the upper half of the line K on to the upper semicircle, and the lower half on to the lower semicircle. It is to see that points far from the x-axis either above or below go on to points near the origin.

More generally, we have the following theorem.

Theorem 4.1.5: If K is a line in E-A, then f(K) is a punctured circle.

Proof: Since we can choose the x-axis any way we want, we are free to assume that K is the graph of a rectangular equation

$x = b > 0$

And hence of a polar equation

$rcos\theta = b > 0.$

As before, setting $r = \frac{a^2}{s}$, we conclude that $f(K)$ is the graph of

$\frac{a^2}{x} cos\theta = b, \qquad s \neq 0$

$s^2 = \frac{a^2}{b} scos\theta$

Or

$$u^2 - \frac{a^2}{b} u + v^2 = 0,\ u^2 + v^2 \neq 0$$

Or $u^2 - \frac{a^2}{b} u + v^2 = 0,\ x^2 + y^2 \neq 0$

Therefore, $f(K)$ is a punctured circle, with center at $(\frac{a^2}{2b}, 0)$ and radius $\frac{a^2}{2b}$.

It is easy to see that (1) every punctured circle is described by the above formula for some choice of b and some choice of the axes. Therefore, (2) every punctured circle L is = $f(K)$ for some line K. Theorem 3 tells us that $f\big(f(P)\big) = P$ for P. Therefore,

$f(L) = f\big(f(K)\big) = K.$

Thus, we have the following theorem.

Theorem 4.1.6: If L is a punctured circle, then $f(L)$ is a line in $E - A$.

We now know, from theorem 4, that under f, punctured lines go on to punctured lines; and we know, by theorem 5 and 6, that lines go on to punctured circles and vice versa. Now we must see what happens to circles.

Theorem 4.1.7: If M is a circle in $E - A$, then $f(M)$ is a line in $E - A$.

Proof: M is the graph of a rectangular equation

$$x^2 + y^2 + Ax + By + C = 0,$$

Where, C$\neq 0$ since the circle is not punctured.

In polar form, this is

$$x^2 + Axcos\theta + Brsin\theta + C = 0,$$

Since $r = \frac{a^2}{s}$, this tells us that $f(M)$ is the graph the equation

$$\frac{a^4}{s^2} + A\frac{a^2}{s}cos\theta + B\frac{a^2}{s}sin\theta + C = 0.$$

Or $a^4 + Aa^2scos\theta + Ba^2ssin\theta + Cs^2 = 0,$

Or $a^4 + Aa^2u + Ba^2v + C(u^2 + v^2) = 0,$

Replacing u and v by x and$y$, to match the labels on the axes, we get an equation for $f(M)$ in the form

$$x^2 + y^2 + \frac{Aa^2}{C}x + \frac{Ba^2}{C}y + \frac{a^2}{C} = 0.$$

The graph of $f(M)$ is a circle; this circle is punctured, because $\frac{a^2}{C} \neq 0$

Theorem 4.1.8: If K is a k-set, then so is $f(K)$.

4.2. Cross Ratio and Inversions

We recall, from chapter 1, the definition of distance in the Poincare model.

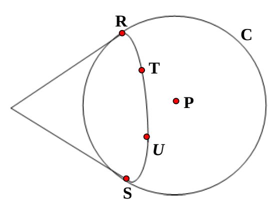


Figure 4.2.1

If T and U are points of the hyperbolic line with end points R, S on the boundary circle C, then the non-Euclidean distance is defined by the formula.

$$d(R, U) = \left|\log\frac{TR/_{TS}}{UR/_{US}}\right|$$

The fraction whose logarithm gets taken in this formula is called the cross ration the quadruplet R, S, T, U, and is commonly denoted by (R, S, T, U).

Thus, $(R, A, T, U) = \log\frac{TR/_{TS}}{UR/_{US}}$

And changing the notation slightly, we have $(P_1, P_2, P_3, P_4) = \frac{P_1P_3.P_2P_4}{P_1P_4.P_2P_3}$

We shall show that inversions preserve the cross ratio. In the following theorem, f is an inversion of a punctured plane $E - A$ about a circle with center at A and radius a, as in the preceding section.

Theorem 4.2.1: If $P \coloneqq P'_i = f(P_1)(i = 1, 2, 3, 4)$, then

$$(P_1, P_2, P_3, P_4) = (P_1', P_2', P_3', P_4')$$

Proof: For each i from 1 to 4, let the polar coordinates of P_i be (r_i, θ_i). By the usual polar distance formula, we have

$$P_iP_j^{\ 2} = r_i^{\ 2} + r_j^{\ 2} - 2r_ir_j\cos(\theta_i - \theta_j).$$

Now $P_i^{'} = (s_i, \theta_i) = (\frac{a^2}{r_i}, \theta_i)$

Therefore,

$$(P_1, P_2, P_3, P_4)^2 = \frac{[r_1^{\ 2} + r_3^{\ 2} - 2r_1r_3\cos(\theta_1 - \theta_3)][r_2^{\ 2} + r_4^{\ 2} - 2r_2r_4(\theta_2 - \theta_4)]}{[r_1^{\ 2} + r_4^{\ 2} - 2r_1r_4\cos(\theta_1 - \theta_4)][r_2^{\ 2} + r_3^{\ 2} - 2r_2r_3(\theta_2 - \theta_3)]}$$

And

$$(P_1{}', P_2{}', P_3{}', P_4{}')^2 = \frac{\left[\frac{a^4}{r_1^{\ 2}} + \frac{a^4}{r_3^{\ 2}} - 2\frac{a^4}{r_1r_3}cos(\theta_1 - \theta_3)\right][\frac{a^4}{r_2^{\ 2}} + \frac{a^4}{r_4^{\ 2}} - 2\frac{a^4}{r_2r_4}cos(\theta_2 - \theta_4)]}{\left[\frac{a^4}{r_1^{\ 2}} + \frac{a^4}{r_4^{\ 2}} - 2\frac{a^4}{r_1r_4}cos(\theta_1 - \theta_4)\right]\left[\frac{a^4}{r_2^{\ 2}} + \frac{a^4}{r_3^{\ 2}} - 2\frac{a^4}{r_2r_3}cos(\theta_2 - \theta_3)\right]}$$

To reduce the second of these fractions to the first, we multiply in both the numerator and denominator by $\frac{r^2{}_1r^2{}_2r^2{}_3r^2{}_4}{a^8}$.

This theorem will tell us, in due course, that inversions applied to the Poincare model are isometries, relative to the non-Euclidean distance.

4.3. Angular Measure and Inversions

A re-examination of Section 4.1 will indicate that the image of an angle under inversion is never an angle. The point is that every angle in E-A has at least one side lying on a non-punctured line and the image of a non-punctured line is always a punctured circle. Therefore, the following theorem doesn't mean it might seem to mean.

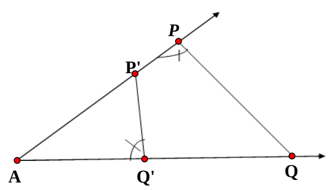


Figure 4.3.1

Theorem 4.3.1: If A and Q are non-collinear, then

$$P^{'} = f(P) and\ Q^{'} = f(Q), m\angle APQ = m\angle AQ'P'$$

Proof: Consider $\Delta PAQ\ and\ \Delta Q'AP'$. They have the angle $\angle$A in common. Since

$$AP^{'} = \frac{a^2}{AP'}\ and\ AQ^{'} = \frac{a^2}{AQ}$$

We have $AP.AP^{'} = AQ.AQ^{'} = a^2$,

So that

$$\frac{AP}{AQ'} = \frac{AQ}{AP'}$$

By the SAS similarity, $\Delta PAQ\ similar\ to\ \Delta Q'AP'$

Note the reversal of order of vertices here. Since $\angle APQ\ and\ \angle AQ'P'$ are corresponding angles, they have the same measure.

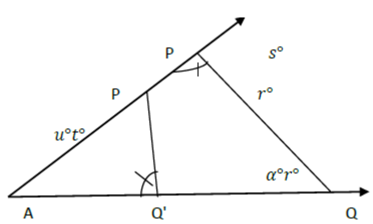


Figure 4.3.2

In the figure above, $P' = f(P)\ and\ Q' = f(Q)$ as before. Here we have

$u = 180 - \alpha - r$

$= (180 - r) - \alpha$

$= s - \alpha.$

Therefore, $s = u - \alpha.$

The order of s and u depends on the order in which P and P' appear on the ray. If P and P' are interchanged, then we should interchange s and u, getting $u - s = \alpha.$

Thus, in general we have $|s - u| = \alpha.$

Consider next the situation illustrated in the figure below:

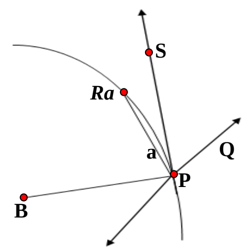


Figure 4.3.3

Here B is the center of a circular arc; $\underset{PQ}{\leftrightarrow}$ is a line intersects in the arc at P; $\underset{PS}{\leftrightarrow}$ is a tangent ray at P:$R_\alpha P = a$. We assert that

$\lim_{a\to 0} m\angle R_a PQ = m\angle SPQ.$

Proof:

The first step is to show that $\lim_{a\to 0} m\angle R_a = PS = 0.$

Consider now a circular arc $\overline{QS}$ with end point at a point Q.

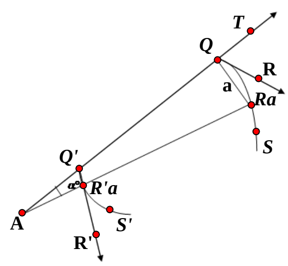


Figure 4.3.4

For small positive numbers a, let R_α be the point of the arc for which Q$R_\alpha = a$.

Let $\overline{Q'S'}$ be the image of $\overline{QS}$; that is $\overline{Q'S'} = f(\overline{QS})$;

Let $\overline{Q'R'}$ be the tangent ray at Q'.

We assert that $\angle AQ'R' \cong \angle TQR.$

To see this, we observe that m $\angle TQR_a\ and\ m\angle AQ'R_a{}'$ is the s and u that we discussed just after Theorem 4.3.1. Therefore,

$$\left| m\angle TQR_a - m\angle AQ'R_a{}' \right| = \alpha$$

Now

$$\lim_{a\to 0} m\angle TQR_a = m\angle TQR,$$

and

$$\lim_{a\to 0} m\angle AQ'R_a{}' = m\angle AQ'R',$$

Therefore, $\lim_{a\to 0}[m\angle TQR_a - m\angle TQR - m\angle AQ'R']$.

But, the absolute value of the quantity indicated in square brackets is equal to α and $a \to 0.$

Therefore, $m\angle TQR = m\angle AQ'R'.$

Given two intersecting circles or lines, the tangent rays give us “tangent angles” like this:

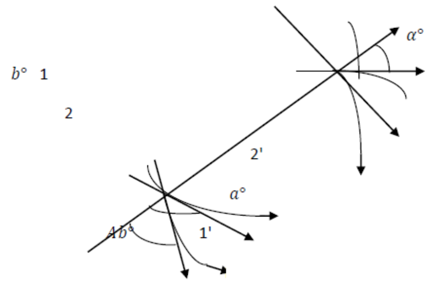


Figure 4.3.5

By the preceding result, we have the following theorem.

Theorem 4.3.2: Under inversions, corresponding tangent angles are congruent. That is, if $\overline{AB}$ and $\overline{AC}$ are arcs with a tangent angle of measure r, then their images $f(\overline{AB})$ and, $f(\overline{AC})$ have a tangent angle of measure r. Similarity for an arc and a segment or a segment and a segment.

Proof: Left for reader.

4.4. Reflections across L-Lines in the Poincare Model

We recall that the points in the Poincare model are the points of the interior E of a circle C with center at P; the L-lines are:

1. The intersection of E with lines through P and
2. The intersection E with circles C' orthogonal to C.

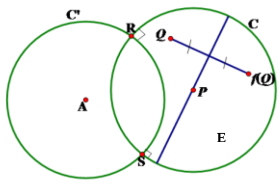


Figure 4.4.1

If L is hyperbolic line of the first type, then the reflection of E across L is defined in the familiar fashion as a one correspondence: $f: E \leftrightarrow E$ Such that for each point Q of E, Q and *f*(Q) are symmetric across L.

If L is hyperbolic line of the second type, then the reflection of E across L is the inversion of E about C'.

To justify this definition of course we have to show that if *f* is an inversion about a circle C'

Orthogonal to C, then *f*(E) =E. But, this is not difficult to show. In the next few theorems, it should be understood that f is an inversion about C'; C' has center at A and intersects C orthogonally at R and S: and $L = E \cap C'$.

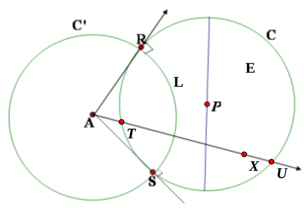


Figure 4.4.2

Theorem 4.4.1: $\boldsymbol{f(C) = C}$.

Proof: $f(C)$ is a circle. This circle contains R and S because $f(R) = R$ and $f(S) = S$. By Theorem 2 of the preceding section, $f(C)$ *and* C' are orthogonal. But there is only one circle C which crosses C' orthogonally at R and S. It is clear that P must be the center of any circle. Therefore, $f(C) = C$, this was the required.

Theorem 4.4.2: $f(E) = E$.

Proof: Let X be any point of E. Then $\overrightarrow{AX}$ intersects C at points T and U. Since $f(C) = C$ we have $U = f(T)$ *and* $T = f(U)$, but, inversions preserve betweenness on rays starting at A. Therefore, $f(\overline{TU}) = \overline{TU}$ and $f(X) \in E$. Thus, $f(E) \subset E$.

We need to show conversely that $E \subset f(E)$. This is trivial: given that $f(E) \subset E$ we have $f\big(f(E)\big) \subset f(E)$. Since $f\big(f(E)\big) = E$, this gives $E \subset f(E)$.

Theorem 4.4.3: If M is hyperbolic line, then so also is $f(M)$.

Proof:

M is the intersection $E \cap D$ where D is either a circle orthogonal to E or a line orthogonal to C. Now $f(D)$ is orthogonal to C and is a line or circle (punctured or unpunctured). Let D' be the corresponding complete line or circle. Thus, $D' = f(D), or\ D' = f(D) \cup A.$

Then

$f(M) = f(D) \cap E = D' \cap E,$ Which is hyperbolic line.

We recall that hyperbolic angle is the angle formed by two "ray" in the Poincare model.

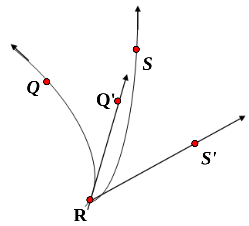


Figure 4.4.3

The measure of hyperbolic angle is the measure of the angle formed by the tangent rays. We now sum up nearly all of the preceding discussion in the following theorem.

Theorem 4.4.4: Let *f* be a reflection across E and hyperbolic line. Then

1. f is a one-to-one correspondence $E \leftrightarrow E$
2. f preserves the non-Euclidean distances between points
3. f preserves hyperbolic lines
4. f preserves measures of hyperbolic angles

For hyperbolic lines of the first type passing through P all this is trivial because in this case f is an isometry in the Euclidean sense. It therefore preserves distances of both types lines, circles, orthogonally, and angular measure. For hyperbolic lines of the second type conditions 1 to 4 follows from the theorems of this section and the preceding two sections.

4.5. Uniqueness of the hyperbolic lines Through Two Points

Given the center P of C, and some other point Q of E. We know that P and Q lie on only one straight line in the Euclidean Plane. Therefore, P and Q lie on only one hyperbolic line of the first kind. But P doesn't on any hyperbolic line of the second kind. The reason is that on the right triangle ARP in the figure, the hypotenuse $\overline{AP}$ is the longest side. It follows that the hyperbolic line through two points of E is unique, in the case where one of the points is P.

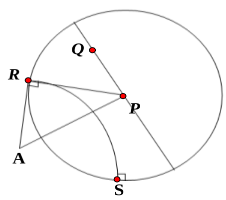


Figure 4.5.1

To prove that uniqueness always holds we need the following theorem.

Theorem 4.5.1: For each point Q of E there is a reflection *f* such that *f*(Q)=P.

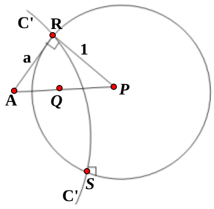


Figure 4.5.2

Proof:

We start by the method of wishful thinking. If the inversion f about C' gives $f(Q) = P$, then

$AP = \frac{a^2}{AQ}.$

We recall that the radius PR=1. Let K=QP, and let x be the unknown distance AP (figure 4.13)

Suppose that L passes through P, and let its end points on C be R and S. For every point Q of L, Let $f(Q) = \log_e \frac{QR/_{QS}}{PR/_{PS}}$ $= \log_e \frac{QR}{QS}$ (Because PR=PS).

Let QS=x. Then QR=2-QS=2-x, and we have $f(Q) = \log_e \frac{2-x}{x}$.

Obviously f is a function $L \to \mathbb{R}$ in to the real numbers. We need to verify that f is a one-to-one correspondence $L \to \mathbb{R}$. Then we need to show every real number K is equal to $f(Q)$ for exactly one point Q. Thus, we want $k = \log_e \frac{2-x}{x}$

Or

$$e^k = \frac{2-x}{x}$$

Or $(e^k + 1)x = 2$ Or $x = \frac{2}{e^k+1}$

For Every K there is exactly one such x, and 0<x<2 as it should be. Therefore every k is equal to $f(Q)$ for exactly one point Q of L.

We have already checked that when the coordinate system f defined in this way, the distance formula $d(T, U) = |f(T) - f(U)|$ is always satisfied.

Before proceeding to generalize the following theorem we observe that the formulas give us some more information.

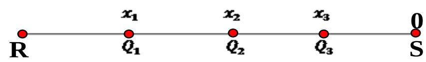


Figure 4.5.3

The figure $x_i = QS \; for \; i = 1, 2, 3$. It is easy to check that $\frac{2-x}{x}$ is a function. Its derivative is $-\frac{2}{x^2} < 0$ and the logarithm is an increasing function.

Therefore, if $x_1 < x_2 < x_3$ as in the figure, it follows that $f(Q_1) < f(Q_2) < f(Q_3)$, and conversely. We recall that betweenness is defined in terms of distance and one point of a line is between two others if and only if its coordinate is between their coordinates.

Theorem 4.5.2: Let Q_1, Q_2, Q_3 be points of hyperbolic line through P. Then $Q_2 - Q_3$ under the non-Euclidean distance if and only if $Q_1, Q_2, and \; Q_3$ are in the Euclidean plane.

Theorem 4.5.3: Every hyperbolic line has a coordinate system.

Proof:

Given hyperbolic line L. If L contains P, we use theorem 4.6.1. If not, let Q be point of L; let g be a reflection such that g(Q) =P; let L'=g(L) and let $f: L' \leftrightarrow \mathcal{R}$ be a coordinate system for L'. For each point T of L, let $f'(T) = f(g(T))$.

That is, the coordinate of T is the coordinate of the corresponding point $g(T)$ of L'. Since f and g are one-to-one correspondences, so also is their composition f(g). Given points such as T, and U of L. We know that $d(T, U) = d(g(T), g(U)$ because inversions preserve the non-Euclidean distance. This in turn is equal to

$\left|f(g(T)) - f(g(U))\right|$. Because f is a coordinate system for L'.

Therefore, $d(T, U) = |f'(T) - f'(U)|$ this was to be proved.

Theorem 4.5.4: Every hyperbolic line through P separates E in to two sets H_1 and H_2 such that

1. H_1 and H_2 are convex sets
2. If Q∈H_1 andR ∈H_2, then $\overline{QR}$ intersects L.

Here $\overline{QR}$ means of course the non-Euclidean segment.

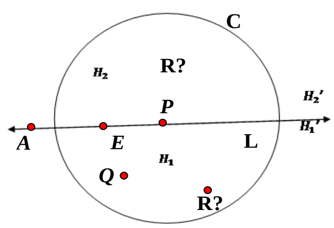


Figure 4.5.4

Proof:

We know that the Euclidean line containing L separates the Euclidean Plane in to two half-planes $H_1' \; and \; H_2'$. Let $H_1 \; and \; H_2$ be the intersections $H_1' \cap E \; and \; H_2' \cap E$ as indicated in the figure.

Suppose that $Q, R \in H_1$ and suppose that $\overline{QR}$ intersects L in a point S. Let f be an inversion $E \leftrightarrow E$ about a circle with center A on the line containing L such that *f*(S) =P. Then $f\,(\overline{QR})$ is hyperbolic line through P and f (Q) and f (R) belong to H_1. Since Q-S-R, in the non-Euclidean sense, because *f* preserves the non-Euclidean distance.

Therefore, $f(Q) - P - f(R)$ in the Euclidean sense, which is impossible because $f(Q)\ and\ f(R)$ are in the same Euclidean half-plane.

It follows that in the same way that H_1 is a convex set. Thus, we have verified half of the proof.

Suppose now that $Q \in H_1$ and $R \in H_2$. Let C' be the Euclidean circle that contains the hyperbolic line $\overline{QR}$.

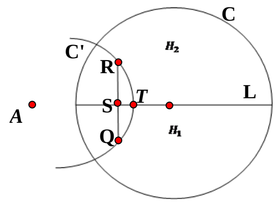


Figure 4.5.5

Then L contains a point S of the Euclidean segment from Q to R and S is in the interior of C'. It follows that the Euclidean line containing L intersects C' in points, one of which is a point T of L. Now we must verify that Q-T-R in non-Euclidean sense. [Hint: Use an inversion $f: E \leftrightarrow E, H_1 \longleftrightarrow H_1, H_2 \longleftrightarrow H_2, T \leftrightarrow P,$ and then apply theorem 4.6.2.]

To extend this result to hyperbolic lines in general, we observe that:

Theorem 4.5.5: Reflections preserve betweenness. Because, they preserve lines and distance.

Proof: Left for reader.

Theorem 4.5.6: Reflections preserve segments. Because, they preserve betweenness.

Proof: Left for reader.

Theorem 4.5.7: Reflections preserve convexity. Because, they preserve segments.

Proof: Left for reader.

Theorem 4.5.8: The plane separation postulate holds in the Poincare model.

Proof:

Let L be any hyperbolic line and let Q be any point of L. Let *f* be a reflection such that $f(Q) = P; let\ L' = f(L);\ and\ let\ H_1'\ and\ H_2{}'$ be the half-planes in E determined by L'. Let $H_1 = f^{-1}(H_1')\ and\ H_2 = f^{-1}(H_2{}')$.

f^{-1} is also a reflection and reflections preserve convexity. It follows that H_1 and H_2 are convex sets. This proves half of the plane separation postulate for L. It remains to show that if $R \in H_1\ and\ S \in H_2',\ then\ \overline{RS}$ intersects L.

If $R' = f(R)\ and\ S' = f(S),\ then\ R' \in H_1{}'$ and $S' \in H_2',$ so that $\overline{R'S'}$ intersects L' at a point T'. Therefore, $\overline{RS}$ intersects L at $T = f^{-1}(T')$.

Theorem 4.5.9: Reflections preserve segments Reflections preserve half planes.

That is, if $H_1\ and\ H_2$ are the half planes determined by L, then $f(H_1)\ and\ f(H_2)$ are the half planes determined by $f(L)$.

Proof: Left for reader.

Theorem 4.5.10: Reflections preserve segments Reflections preserve interior of angles.

Proof:

The interior of $\angle ABC$ is the intersection of

1. The side of $\overleftrightarrow{AB}$ that contains C
2. The side of $\overleftrightarrow{AB}$ that contains A

Since reflections preserve half planes, they preserve intersections of half planes.

We have defined the measure of non-Euclidean angle as the measure of the Euclidean angle formed by the two tangent rays. We need to check whether this measure function satisfies the postulate of section 1.5. For angles with vertex at P this is obvious. To verify it for angles with vertex at some other point Q, we throw Q on to P by a reflection f.

Now f preserves angles, angular measure, lines, and interior of angles. It is therefore trivial to check that if Postulates M-1 through M-5 holds at P, then they hold at Q.

5. The Consistency of Euclidean Geometry

Our proof of the consistency of hyperbolic geometry, in the preceding chapter, was conditional. We should that if there is a mathematical system satisfying the postulates for Euclidean geometry, and then there is a system satisfying the postulates for hyperbolic geometry. We shall now investigate that if, by describing model for the Euclidean postulates. Here again our consistency proof will be conditional. To set up our model, we shall need to assume that the real number system is given.

5.1. The Coordinate Plane and Isometries

Definition 5.1.1: $E = RxR$, where R the real number system is called a Cartesian model or coordinate system.

Then a point in a Cartesian model E is defined to be an ordered pair of real numbers.

Definition 5.1.2: A line in the Cartesian model E is a sub set of E which has the form

$L = \{(x, y)/ Ax + By + C = 0, A^2 + B^2 > 0\}$

That is a line is defined to be the graph of a linear equation in x and y.

Definition 5.1.3: If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, are two points in a Cartesian model E, then the distance between these two points from analytic geometry is given by

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We define betweenness in terms of distance. As usual, we abbreviate $d(P, Q)$ as PQ.

Segments and rays are defined in terms of betweenness; and angles are defined when rays are known. It turns out that setting up an angular measure function is a formidable technical chore. We hope, therefore, that the reader will settle for a congruence relation $\cong$ for angles, satisfying the congruence postulates for angles. This relation is defined in the following way.

Recall that: A one-to-one correspondence f from a set A to a set B is a function $f: A \to B$ such that, for each $b \in B$, there is a unique $a \in A$ for which $f(a) = b$. This is equivalent to the mapping $f: A \to B$ being both one-to one and on to. In other words, we have a "pairing" between elements of A and elements of B.

Definition 5.1.4: An isometry is a one-to-one correspondence $f: E \to E\ p$ reserving distance.

Definition 5.1.5: Two angles $\angle ABC\ and\ \angle DEF$ are congruent if there is an isometry $f: E \to E$ such that $f(\angle ABC) = \angle DEF$.

We have now given definitions, in the Cartesian model, for the terms used in the Euclidean postulates. Each of these postulates thus becomes a statement about a question of fact; and our task is to show that all of these statements are true.

5.2. The Ruler Postulate

Recall the following:

1. The ruler postulate: Every line has a coordinate system.
2. A vertical line is a line which is the graph of an equation $x = a$.
3. Every non-vertical line is the graph of an equation $y = mx + b$.
4. If $x = a$ and $x = b$ are equations of the same line, then $a = b$.
5. If $y = m_1x + b_1$ and $y = m_2x + b_2$ are equations of the same line, then $m_1 = m_2$ and $b_1 = b_2$.

Definition 5.2.1: A coordinate system f on a line L is a one-to-one correspondence

$$f: L \to R$$

Definition 5.2.2: Distance function

For each line L in the plane, fix a coordinate system $f_L: L \to R$. Then the distance function on the plane E is the function $d: ExE \to R$ which assigns to any two points P,Q a real number $d(P, Q) = PQ$

Defined by

$$d(P, Q) = PQ = \begin{cases} |f(P) - f(Q)|\ if\ P \neq Q \\ 0\ if\ P = Q \end{cases}$$

Theorem 5.2.1: Every vertical line L has a coordinate system.

Proof: Let $x = a$ be the vertical line and for each point $P = (a, y)$ of L, let $f(P) = y$

Then f is a one-to-one correspondence $L \leftrightarrow R$. If $P = (a, y_1)$ and $Q = (a, y_2)$, then

$$PQ = d(P, Q) = \sqrt{(a - a)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(y_2 - y_1)^2}$$
$$= |y_2 - y_1|$$
$$= |f(Q) - f(P)|, \text{ As desired.}$$

Theorem 5.2.2: Every non-vertical line has a coordinate system.

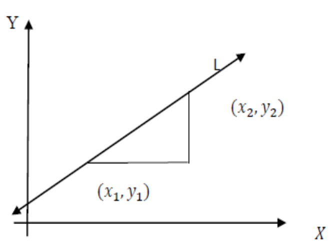


Figure 5.2.1

Proof: Let L be the graph of $y = mx + b$. If (x_1, y_1) and $(x_2, y_2) \in L$, then it is easy to check that $\frac{y_2 - y_1}{x_2 - x_1} = m, y_2 - y_1 = m(x_2 - x_1)$.

And

$PQ = d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= |x_2 - x_1|\sqrt{(1 + m^2)}$

From this we see how to define a coordinate system for L. For each point $P = (x, y) \in L$

Let $f(P) = f(x, y) = x\sqrt{(1 + m^2)}$

Then for $P = (x_1, y_1), Q = (x_2, y_2)$

We have

$PQ = |x_2 - x_1|\sqrt{(1 + m^2)}$

$= \left|x_{2\sqrt{(1+m^2)}} - x_{1\sqrt{(1+m^2)}}\right|$

$= |f(Q) - f(P)|$

as it should be.

These two theorems give us:

Theorem 5.2.3: In the Cartesian model, the ruler postulate holds.

Proof: Exercise

5.3. Incidence and Parallelism

Theorem 5.3.1: Every two points of the Cartesian model lie on a line.

Proof: Given $P = (x_1, y_1), Q = (x_2, y_2)$. If $x_1 = x_2$, then $P \; and \; Q$ lie on the vertical line

$x = a = x_1$.

If not, then $P \; and \; Q$ lie on graph of the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This is easily seen to be a line.

Theorem 5.3.2: Two lines intersect in at most one point.

Proof: Given L_1 and L_2 with $L_1 \neq L_2$. If both are vertical, then they do not intersect at all. If one is vertical and the other is not, then the graphs of

$x = a, y = mx + b$

Intersect at the unique point $(a, ma + b)$. Suppose finally, that L_1 and L_2 are the graphs of

$y = m_1x + b_1,\ y = m_2x + b_2$.

If $m_1 \neq m_2$, very elementary algebra gives us exactly one common solution and hence exactly one intersection point. If $m_1 = m_2$, then $b_1 \neq b_2$, and the graphs do not intersect at all.

We have already observed that if L is the graph of $y = mx + b$, then for every two points $(x_1, y_1), (x_2, y_2)$ of L, we have

$$\frac{y_2 - y_1}{x_2 - x_1} = m.$$

Thus, m is determined by the non-vertical line L. As usual, we call m the slope of L.

Theorem 5.3.3: Every vertical line intersects every non-vertical line.

Proof: Let L_1 be a vertical line $x = a$ and L_2 be non-vertical line $= mx + b$, then by theorem 2, L_1 and L_2 intersect at the point $(a, ma + b_2)$.

Theorem 5.3.4: Two lines are parallel if and only if (1) both are vertical, or (2) neither is vertical, and they have the same slope.

Proof: Given $L_1 \neq L_2$. If both are vertical, then $L_1 \parallel L_2$. If neither is vertical, and they have the same slope, then the equations

$y = mx + b_1,\ y = mx + b_2.\ (b_1 \neq b_2)$ have no common solution, and $L_1 \parallel L_2$.

Suppose, conversely, that $L_1 \parallel L_2$. If both are vertical, then (1) holds. It remains only to show that if neither line is vertical, they have the same slope.

Suppose not. Then $L_1 : y = m_1x + b_1, L_2 : y = m_2x + b_2\ (m_{1\neq}m_2)$

We can now solve for x and y:

$0 = (m_1 - m_2)x + (b_1 - b_2)$,

Solve for x we obtain

$$x = -\frac{b_1 - b_2}{m_1 - m_2},$$
$$y = -m_1\left(\frac{b_1 - b_2}{m_1 - m_2}\right) + b_1$$

We got this value of y by substituting in the equation of L_1. But, our x and y also satisfy the equation of L_2. This contradicts the hypothesis $L_1 \parallel L_2$.

Theorem 5.3.5: Given a point $P = (x_1, y_1)$ and a number m, there is exactly one line which passes through P and has slope $= m$.

Proof: The lines L with slope m are the graphs of equations $y = mx + b$.

If L contains (x_1, y_1), then $b = y_1 - mx_1$, and conversely. Therefore, our line exists and is unique.

Theorem 5.3.6: In the Cartesian model, the Euclidean parallel postulate holds.

Proof: Given a line L and a point $P = (x_1, y_1)$ not on L.

1. If L is the graph of $x = a$, then the line L': $x = x_1$ is the only vertical line through P, and by theorem 5.3.3, no non-vertical line is parallel to L. Thus, the parallel line L through P is unique.
2. If L is the graph of $y = mx + b$, then the only line parallel to L through P is the line through P with slope $= m$. This is unique.

5.4. Translations and Rotations

By a translation of the Cartesian model, we mean a one-to-one correspondence

$f: E \leftrightarrow E: (x, y) \leftrightarrow (x + a, y + b)$.

Merely by substitution in the distance formula, and observing that a and b cancel out, we have:

Theorem 5.4.1: Translations are isometries.

If L is the graph of the equation

$Ax + By + C = 0$, then the points$(x', y') = (x + a, y + b)$ of $f(L)$ satisfy the equation

$A(x' - a) + B(y' - b) + C = 0$,

Or $Ax' + By' + (-aA - bB + C) = 0$

This is linear. Thus, we have proven the theorem.

Theorem 5.4.2: Translation preserves lines.

Since translations preserve lines and distance, they preserve everything defined in terms of lines and distance.

Theorem 5.4.3: Translations preserve betweenness, segments, rays, angles, triangles, and angle congruences.

Rotations are harder to describe, because at this stage we have no trigonometry to work with. Let us first try using trigonometry, wistfully, to find out what we ought to be doing, and then find a way to do something equivalent, using only the primitive apparatus that we now have at our disposal in our study of the Cartesian model.

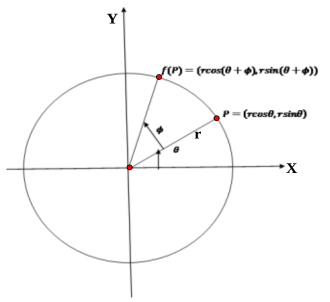


Figure 5.4.1

We want to rotate the Cartesian model through and angle of measure ∅. (Fig. 5.4.1)

Trigonometrically, this can be done by a one-to-one correspondence,

$f: E \leftrightarrow E$,

defined as the labels in the figure suggest.

Now $\cos(\theta+\phi) = cos\theta cos\phi - sin\theta sin\phi$

Let $a = cos\phi,\ b = sin\phi$

Now $r = \sqrt{x^2 + y^2}$, $cos\theta = \frac{x}{\sqrt{x^2+y^2}}$, $sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

We can therefore rewrite our formulas in the form

$f: (x, y) \leftrightarrow (x', y')$

Where $x' = rcos(\theta+\phi)$

$= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}}a - \frac{y}{\sqrt{x^2+y^2}}b\right)$
$= ax - by$
And $y' = \sqrt{x^2+y^2}\left(\frac{y}{\sqrt{x^2+y^2}}a + \frac{x}{\sqrt{x^2+y^2}}b\right)$
$= ay + bx.$
Any correspondence of this form, with $a^2+b^2=1,$ is called a rotation of the Cartesian model.
Theorem 5.4.4: Rotations preserve distance.
Proof: We have
$P = (x_1, y_1)$
$Q = (x_2, y_2)$
$P' = f(P) = (ax_1 - by_1, ay_1 + bx_1)$
$Q' = f(Q) = (ax_2 - by_2, ay_2 + bx_2)$
It is merely an exercise in patience to substitute in the distance formula, calculate
$P'Q'$, simplify with the aid of the equation $a^2+b^2=1$, and observe that $P'Q' = PQ$. Solving for $x \ and \ y$, we get
$x = ax' + by',\ y = ay' - bx'$
Comparing the formulas
$x' = ax - by,\ y' = bx + ay$
For f and the corresponding formulas for f^{-1}, we see that these have the same form:
$x = a'x' - b'y', y = a'y' + b'x'$, where $a' = a$ and $b' = -b$.
Therefore, we have the following theorem.
Theorem 5.4.5: The inverse of a rotation is a rotation.
Proof: Exercise
Theorem 5.4.6: Rotations preserve lines.
Proof: Exercise
Proof: L is the graph of an equation
1. $x = k,$
2. $y = k,$ Or
3. $y = mx + k)(m \neq 0)$
In case (1), If f(L) is the graph of $ax' + by' = k$, where a and b are not both equal to zero, because $a^2+b^2 =$ 1.Therefore, L is a line.
In case (2), $f(L)$ is the graph of $ay' - bx' = k$, which is a line.
In case (3), $f(L)$ is the graph of $ay' - bx' = max' + mby' + k,$
Or $(ma+b)x'$ +$(mb-a)y' + k = 0.$
If we had both $(ma+b) = 0, mb - a = 0,$
Then $ma^2 + ab = 0, mb^2 - ab = 0$
So that $m(a^2+b^2) = 0$, and $m = 0$, contradicting our hypothesis.
As for translations, once we know that rotations preserve lines and distance, it follows they preserve everything that is defined in terms of lines and distance.
Therefore, we have:
Theorem 5.4.7: Rotations preserve betweenness, segments rays, angles, triangles, and angle congruences.
We are going to use rotations in the Cartesian model in much the same way that we used reflections in the Poincare model, to show that postulates for angle congruence hold. To do this, we shall need to know that every ray starting at the origin $(0,0)$ can be rotated on to the positive end of the x-axis, and vice versa. By theorem 5.4.5, it will be sufficient to prove the following theorem.
Theorem 5.4.8: Let $= (x_0\ , 0), x_0 > 0$, let $Q = (x_1, y_1)$, and suppose that
$x_0 = \sqrt{{x_1}^2 + {y_1}^2}$
Then there is a function f such that $f(P) = Q.$

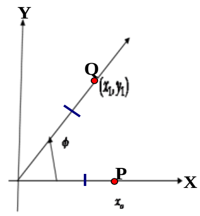


Figure 5.4.2

The equation in the hypothesis says, of course, such P and Q are equidistant from the origin. As a guide in setting up such a rotation, we note unofficially that we want to rotate E through an angle of measure ϕ, where

$$a = cos\phi = \frac{x_1}{\sqrt{{y_1}^2 + {x_1}^2}}$$

$$b = sin\phi = \frac{y_1}{\sqrt{{y_1}^2 + {x_1}^2}}$$

Thus, the rotation ought to be $f: E \leftrightarrow E$

$: (x, y) \leftrightarrow (x', y')$

where $x' = ax - by = \frac{x_1}{\sqrt{{y_1}^2+{x_1}^2}} x - \frac{y_1}{\sqrt{{y_1}^2+{x_1}^2}} y$

$y' = bx + ay = \frac{y_1}{\sqrt{{y_1}^2+{x_1}^2}} x + \frac{x_1}{\sqrt{{y_1}^2+{x_1}^2}} y$

Obviously, $a^2 + b^2 = 1$ in these equations, and so f is a rotation. And

$f(x_{0,}0) = \left(\frac{x_1}{\sqrt{{y_1}^2+{x_1}^2}} x_{0,} \frac{y_1}{\sqrt{{y_1}^2+{x_1}^2}} x_{0,}\right) = (x_1, y_1)$This is the result that we wanted.

5.5. Plane Separation

We shall show first that the plane separation postulate holds for the case in which the given line is the x-axis. It will then be easy to get the general case.

Definition 5.5.1: A subset E^+ of the plane E is convex if, whenever $P\ and\ Q$ are two points of E^+, then the line segment PQ joining $P\ to\ Q$ is also contained in E^+.

Definition 5.5.2: The two non-empty convex sets E^+ and E^- formed by removing the line L from the plane are called half planes, and the line L is the edge of each half plane.

Let E^+ be the "upper half plane." That is,

$E^+ = \{(x, y): y > 0\}$.

Theorem 5.5.1: E^+ is convex.

Proof: Remember that, if A, B, and C are points of a line, with coordinates x, y and z, such $< y < z$, then A-B-C. (This was proved merely on the basis of the ruler postulate, and we can therefore apply it now). Since only one of the points A, B, C is between the other two, the lemma has a true converse: if A-B-C, then $x < y < z$ or $z < y < x$.

Consider now two points, $A = (x_1, y_1)$, $C = (x_2, y_2)$ of E^+.

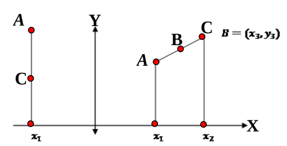


Figure 5.5.1

We need to show that $\overline{AC}$ lies in E^+. That is, if A-B-C, with $B = (x_3, y_3)$, then $y_3 > 0$.

Obviously, for the case $x_1 \neq x_2$ we may assume that $x_1 < x_2$, as in the figure; and for the case $x_1 = x_2$, We may assume that $y_1 > y_2$.

In the first case, the line $\overleftrightarrow{AC}$ is the graph of an equation $y = mx + b$.

And has a coordinate system of the form $f(x, y) = x\sqrt{1 + m^2}$.

In the second case, the line is the graph of the equation $x = x_1$ and has a coordinate system of the form $f(x, y) = y$.

It is easy to check that in the first case $f(A) < f(B) < f(C)$.

So that $f(x, y) = \sqrt{1 + m^2}$

$x_1 < x_3 < x_2$

For $m > 0$.

$mx_1 + b < mx_3 + b < mx_2 + b;$

For $m < 0$, the inequalities run the other way; but in either case y_2 lies between two positive numbers. In the second case $x_1 = x_2$, the same result follows even more easily.

Let E^- be the "lower half plane." That is, $E^- = \{(x, y): y < 0\}$.

Since the function, $f: (x, y) \leftrightarrow (x, -y)$ is obviously an isometry, it preserves segments. Therefore, it preserves convexity. Since $f(E^+) = E^-$, we have the following.

Theorem 5.5.2: E^- is convex.

It is an easy exercise in algebra to show that if $A = (x_1, y_1) \in E^+$ and $B = (x_2, y_2) \in E^-$, then $\overline{AB}$ contains a point (x, 0) of the x-axis.

Theorem 5.5.3: E and the line y=0 satisfy that conditions for E and L in the plane separation postulate.

Now let L be any line in E, and let $A = (x_1, y_1)$ be any point of L. By a translation *f*, we can move A to the origin. By a rotation *g*, we can move the resulting line on to the x-axis. Let

$H_1 = g^{-1}f^{-1}(E^+)$, $H_2 = g^{-1}f^{-1}(E^-)$.

Since all of the conditions of the plane separation postulate are preserved under isometries, we have the following theorems.

Theorem 5.5.4: E satisfies the conditions of the plane separation postulate.

Theorem 5.5.5: Isometries preserve half planes.

Proof:

Let H_1 be a half plane with edge L, and let H_2 be the other side of L. If f is an isometry, then $f(L)$ is a lineL'. Let $H_1' = f(H_1)$, $H_2' = f(H_2)$

Then H_1' and H_2' are convex, and every segment between two points $f(A)$ of H_1' and $f(B)$ of H_2' must intersect $f(L)$. Therefore, H_1' is a half plane with L' as edge.

From theorem 5.5.5 it follows that:

Theorem 5.5.6: Isometries preserve interior of angles**.**

That is, if I is the interior of $\angle ABC$, then $f(I)$ is the interior of $f(\angle ABC)$.

5.6. Angle Congruence

We want to verify that angle congruence, defined by means of isometries of E onto itself, satisfies the postulate of angle congruence, and also satisfies SAS. Only one of this verification is trivial.

Statement 1: For angles, congruence is an equivalence relation.

Proof:

1) $\angle A \cong \angle A$ always, because the identity functions $E \leftrightarrow E$ is an isometry.

2) If $\angle A \cong \angle B$, then $\angle B \cong \angle A$, because the inverse of an isometry is an isometry.

3) If $\angle A \cong \angle B$, and $\angle B \cong \angle C$, then $\angle A \cong \angle C$, because the composition of the isometries for which $\angle A \leftrightarrow \angle B$ and $\angle B \leftrightarrow \angle C$ is always an isometry for which $\angle A \leftrightarrow \angle B$ *and* $\angle B \leftrightarrow \angle C$ is always an isometry for which $\angle A \leftrightarrow \angle C$.

The other verification is very difficult. We begin with a lemma.

Lemma 5.6.1: Let *f* be an isometry of E on to itself. If $f(E^+) = E^+$, and $f(P) = P$ for every point P of the x-axis, then *f* is the identity.

Proof:

Let A be the origin (0, 0), and let $B = (1, 0)$. Let $Q = (a, b)$ be any point, and let $f(Q) = (c, d)$. Then $AQ = f(A)f(Q)$, $BQ = f(B)f(Q)$.

Taking the square of each of these distances, we get

$a^2 + b^2 = c^2 + d^2$,

$(a-1)^2 + b^2 = (c-1)^2 + d^2$,

$a^2 + b^2 - 2a + 1 = c^2 + d^2 - 2c + 1$,

So that $a = c$. Therefore, $b^2 = d^2$. Since $f(E^+) = E^+$, b *and* d are both positive, both zero, or both negative. Therefore, $b = d$. Thus, $f(Q) = Q$ for every Q, which was to be proved.

Lemma 5.6.2 Let A be the origin; Let $B = (a, 0), (a > 0)$ be a point of the x-axis; and let $C = (b, c)$, *and* $D = (d, e)$, be points of E^+ and E^- such that $AC = AD, BC = BD$.

Then there is an isometry $f: E \leftrightarrow E$ such that $f(A) = A, f(B) = B, f(C) = D$ *and* $f(D) = C$.

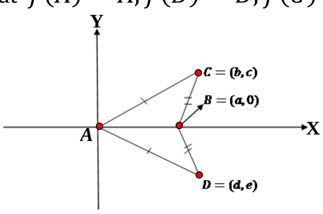


Figure 5.6.1

Proof: We shall show that d=b and e=-c. The desired isometry f will then be the function $(x, y) \leftrightarrow (x, -y)$.

Given $b^2 + c^2 = d^2 + e^2$,

$(b-a)^2 + c^2 = (d-a)^2 + e^2$,

We have $-2ab = -2ad$. Since $a > 0$, this gives $b = d$. Therefore, $c^2 = e^2$. Since $c > 0$ *and* $e < 0$ *we have* $e = -c$.

Lemma 5.6.3: Given $\angle ABC$, there is an isometry f of E on to itself such that

$f(\overrightarrow{BA}) = \overrightarrow{BC}$ and $f(\overrightarrow{BC}) = \overrightarrow{BA}$. That is, the sides of the angle can be changed by an isometry.

In the proof we may suppose that $BA = BC$, since A *and* C can always be chosen so as to satisfy the condition.

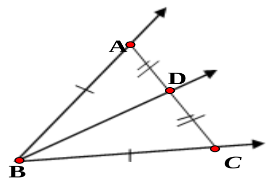


Figure 5.6.2

Let D be the midpoint of $\overline{AC}$. Using a translation followed by a rotation, we get an isometry $g: E \leftrightarrow E$ such that $g(\overrightarrow{BD})$ is the positive end of the x-axis (Fig. 6.7). First we translate B to the origin, and then we rotate.

By the preceding lemma there is an isometry $h: E \leftrightarrow E$, interchanging $A^{'} and\, C'$, and leaving $B^{'} and\, D'$ fixed. Let $f = g^{-1}hg$.

That is, f is the composition of $g, h\ and\ g^{-1}$. Then f is an isometry; $f(B) = B$, $f(A) = C, and\ f(C) = A$.

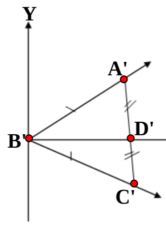


Figure 5.6.3

It is now easy to verify that the rest of our congruence postulates. Oddly enough, the easiest is SAS. We put this in the style of a restatement.

SAS. Given $\Delta \mathrm{ABC}, \Delta \mathrm{A^{'}B^{'}C^{'}}$, and a correspondence,

$\mathrm{ABC} \leftrightarrow \mathrm{A'B'C'}$.

If

1. $AB = A^{'}B^{'}, 2. \angle B \cong \angle B^{'}, 3. BC = B^{'}C^{'}, 4. \angle A = \angle A^{'}, 5. \angle C \cong \angle C^{'}, 6. AC = A'C'$

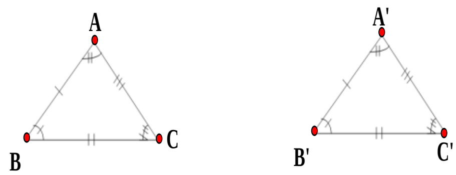


Figure 5.6.4

Proof:

By hypothesis 2, there is an isometry $E \leftrightarrow E, \angle B \leftrightarrow \angle B'$. By Lemma 3, it follows that there is an isometry

$f: E \leftrightarrow E,$

$: B \leftrightarrow B^{'}$

$: \overleftrightarrow{BA} \leftrightarrow \overleftrightarrow{B^{'}A^{'}}$

$: \overrightarrow{BC} \leftrightarrow \overrightarrow{B'C'}$

If the given isometry moves $\angle B$ on to $\angle B'$ in the wrong way, then we follow it by an isometry which interchanges the sides of $\angle B'$. From 1 it follows that

$$A^{'} = f(A) and\ C^{'} = f(C).$$

Therefore,$\angle A^{'} = f(\angle A)$, and $\angle A^{'} \cong \angle A$; $\angle C^{'} = f(\angle C), \angle C^{'} \cong \angle C, and\ also$
$AC = A^{'}C^{'}, f$ because is an isometry.

This proof bears a certain resemblance to Euclid's proof of SAS by supposition.

Statement 2: **The angle Construction Postulate**: Let $\angle ABC$ be an angle, let $\overrightarrow{B^{'}C^{'}}$ be a ray and let H be a half plane whose edge contains $\overrightarrow{B^{'}C^{'}}$. Then there is exactly one ray $\overrightarrow{B^{'}A^{'}}$ with $A^{'}$ in H such that $\angle ABC = \angle A^{'}B^{'}C^{'}$.

We give the proof merely in outline. It should be understood that all of the functions mentioned are isometries of E on to E and that the ray R is the positive x-axis.

1. Take f_1 so that $f_1\left(\overrightarrow{B^{'}C^{'}}\right) = R$
2. Take f_2 so that $f_2(R) = R$ and $f_2 f_1(H) = E^{+}$ (of course, if $f_1(H)$ is already $=E^{+}$, we let f_2 to be the identity.)
3. Take g_1 so that $g_1\left(\overrightarrow{BC}\right) = R$.

4. Take g_2 so that $g_2(R) = R$ and $g_2 g_1(A)$ is in E^+.
5. Let $\angle x = f_1^{-1} f_2^{-1} g_2 g_1(\angle ABC)$. Then $\angle x$ is the $\angle A'B'C'$ that we are interested.
6. Suppose that there are two rays $\overrightarrow{B'A'}$ and $\overrightarrow{B'A"}$ satisfying these conditions.

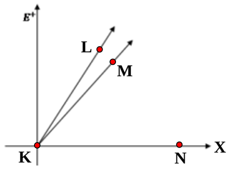


Figure 5.6.5

Then

$$f_2 f_1(\overrightarrow{B'A'}) = f_2 f_1(\overrightarrow{B'A"}) = \overrightarrow{KM}$$

Where K and M are in E^+ and $\overrightarrow{KL}$ and $\overrightarrow{KM}$ are different rays. Since $\angle LKN \cong \angle ABC \cong \angle MKN$,
We have $\angle LKN \cong \angle MKN$
Thus, there is an isometry f of E on to itself such that $f(\angle LKN) = \angle MKN$.
By Lemma 3, f can be chosen so that $f(\overrightarrow{KN}) = \overrightarrow{KN}$ and $f(\overrightarrow{KL}) = \overrightarrow{KM}$. It follows that for each point P of the x-axis, $f(P) = P$. Since isometries preserve half-Planes and $f(L)$ is in E^+, we have $f(E^+) = E^+$.
By Lemma 1, it follows that f is the identity. This contradicts the hypothesis $f(\overrightarrow{KL}) = \overrightarrow{KM} \neq \overrightarrow{KL}$.

Statement 3: The angle addition postulate:

If
1. D is the interior of $\angle BAC$
2. D' is in the interior of $\angle B'A'C'$
3. $\angle BAD \cong \angle B'A'D'$
4. $\angle DAC \cong \angle D'A'C'$
5. $\angle BAC \cong \angle B'A'C'$

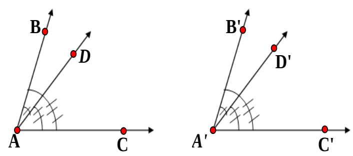


Figure 5.6.6

Proof:

1. By an isometry f we move $\overrightarrow{AD}$ on to R and B in to E^+. For this we need a translation followed by a rotation and perhaps a reflection $(x, y) \leftrightarrow (x, -y)$.
2. By an isometry g we move $\overrightarrow{A'D'}$ on to R and B' in to E^+.
3. By the uniqueness condition in the preceding postulate we know that $f(\overrightarrow{AB}) = g(\overrightarrow{A'B'})$ and $f(\overrightarrow{AC}) = g(\overrightarrow{A'C'})$.
4. Therefore, $f(\angle BAC) = g(\angle B'A'C')$. Hence, $\angle BAC \cong \angle B'A'C'$; the required isometry is $g^{-1}f$.

Statement 4: The angle Subtraction Postulate:

If
1. D is the interior of $\angle BAC$
2. D' is the interior of $\angle B'A'C'$
3. $\angle BAD \cong \angle B'A'D'$
4. $\angle BAC \cong \angle B'A'C'$, *then*
5. $\angle DAC \cong \angle D'A'C'$

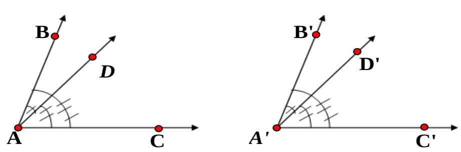


Figure 5.6.7

Proof:

Let f be the isometry given by (4) so that $\angle BAC \cong \angle B'A'C'$. By Lemma 3, we may suppose that $f(\overrightarrow{AB}) = \overrightarrow{A'B'}$ and

$f(\overrightarrow{AC}) = \overrightarrow{A'C'}$. Then surely $f(\angle BAD) \cong \angle BAD$.

The uniqueness condition in C-7 therefore tells us that $f(\overrightarrow{AD}) = \overrightarrow{A'D'}$. Therefore, $f(\angle DAC) = \angle D'A'C'$, and $\angle DAC = \angle D'A'C'$ which was to be proved.

Exercise:

1. Let L be a line and let A, B, and C be three distinct points of L with coordinates x, y, and z, respectively. If the point B is between the points A and C, then the number y is between the numbers x and z.
2. Show that there are at least three points in a plane E which are not contained in any single line.
3. Given two distinct points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then show that there is exactly one line L in a plane E containing both points.
4. Suppose L is a line in the real Cartesian plane defined by the equation, $Ax + By + C = 0$ and $f: L \to R^2$ is the function given by

$$f((x,y)) = \begin{cases} y \; if \; B = 0 \\ x\sqrt{1 + (\frac{A}{B})^2} \; if \; B \neq 0 \end{cases}$$

   a. Prove that f is a coordinate system on L.
   b. If every line L in R^2 is given the coordinate L as defined in (a), prove that the distance function defined on R^2 is the standard distance studied in analytic geometry: $d((x_1, y_1), (x_2, y_2)) = \sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2}$.
5. Let L be a line and let A, B, C be three distinct points of L with coordinates x, y, z, respectively. Then the point B is between the points A and C if and only if the number y is between the numbers x and z.

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