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The Restrained Monophonic Domination Number of Graph

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Abstract A set *M* of vertices of a connected graph G is a *monophonic set* if every vertex of G lies on a *x* - *y* monophonic path for some elements *x* and *y* in *M*. The minimum cardinality of a monophonic set of G is the *monophonic number* of G, denoted by *m*(G). A set *M* of vertices of a connected graph G is a *monophonic dominating set* if *M* is both a monophonic set and a dominating set. The minimum cardinality of a monophonic dominating set of G is the *monophonic domination number* of *G*, denoted by γ_m(G). A set *M* of vertices of a connected graph G is a *restrained monophonic dominating set* if either V = *M* or *M* is a monophonic dominating set with the subgraph G[V – *M*] induced by V – *M* has no isolated vertices. The minimum cardinality of a restrained monophonic dominating set of G is the *restrained monophonic domination number* of G and is denoted by γm_r(G). The restrained monophonic domination number of some connected graph are realized. It is shown that, for any positive integers *a*, *b* and *c* with $3 \le a \le b < c$, there exists a connected graph G such that *m*(G) = *a*, γ_m(G) = *b* and γm_r(G) = *c*.

Keywords Monophonic set, Monophonic number, Monophonic dominating set, Monophonic domination number, Restrained monophonic dominating set, Restrained monophonic domination number

1. Introduction

By a graph G = (V, E), we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to [2, 4, 5]. The neighborhood of a vertex v is the set N(v) consisting of all vertices u which are adjacent with v. The closed neighborhood of a vertex v is the set N[v] = N(v) ∪{v}. A vertex v is an extreme vertex if the subgraph induced by its neighbors is complete. A vertex v is a semi-extreme vertex of G if the subgraph induced by its neighbors has a full degree vertex in N(v). In particular, every extreme vertex is a semi-extreme vertex and a semi-extreme vertex need not be an extreme vertex.

A *chord* of a path $u_1, u_2, \ldots, u_k$ in G is an edge u_iu_j with $j \ge i + 2$. A $u - v$ path *P* is called a *monophonic path* if it is a chordless path. A set *M* of vertices is a *monophonic set* if every vertex of *G* lies on a monophonic path joining some pair of vertices in *M*, and the minimum cardinality of a monophonic set is the *monophonic number* *m*(G). A monophonic set of cardinality *m*(G) is called an *m*-set of G. The monophonic number of a graph G was studied in [3]. A dominating set in a graph G is a subset of vertices of G such that every vertex outside the subset has a neighbor in it. The size of a minimum dominating set in a graph G is called the domination number of G and is denoted $\gamma(G)$ [6].

A set of vertices of G is said to be monophonic domination set if it is both monophonic set and a dominating set of G.The minimum cardinality among all the monophonic dominating sets of G is called a monophonic domination number and is denoted by $\gamma_m(G)$ [8]. The restrained edge monophonic number of a graph was studied in [9].

Theorem 1.1 [10] Each extreme vertex of a connected graph G belongs to every monophonic set of G.

Throughout this paper *G* denotes a connected graph with at least two vertices.

2. Restrained Monophonic Domination Number

Definition 2.1 A set *M* of vertices of a connected graph G is a restrained monophonic domination set if either V= *M* or *M* is a monophonic dominating set with the subgraph G[V – *M*] induced by V – *M* has no isolated vertices. The minimum cardinality of a restrained monophonic dominating set of G is the *restrained monophonic domination number* of G, and is denoted by γm_r (G).

Example 2.2 For the graph G given in Figure 1, $M_1 = \{v_1, v_4\}$ is a monophonic set of G and so *m*(G) = 2*;* $M_2 = \{v_1, v_4,$

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$v_{10}\}$ is a monophonic dominating set of G and so $\gamma_m(G) = 3$; $M_3 = \{v_1, v_4, v_{10}, v_{11}\}$ is a restrained monophonic dominating set of G and so $\gamma m_r(G) = 4$.

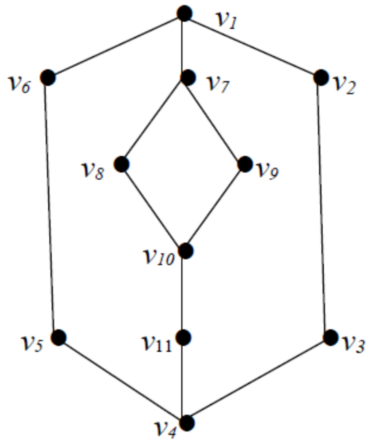

Figure 1. A graph G for restrained monophonic domination numbers

Theorem 2.3 Each semi-extreme vertex of a graph G belongs to every restrained monophonic dominating set of G. In particular, if the set M of all semi-extreme vertices of G is a restrained monophonic dominating set, then M is the unique minimum restrained monophonic dominating set of G.

Proof. Let M be the set of all semi-extreme vertices of G, and let N be any restrained monophonic dominating set of G. Suppose that there exists a vertex $u \in M$ such that $u \notin$ N. Since $\Delta(<N(u)>) = |N(u)| - 1$, there exists $v \in N(u)$ such that $deg_{<N(u)>}(v) = |N(u)| - 1$. Since N is a restrained monophonic dominating set of G, the edge $e = uv$ lies on a x - y monophonic path $P : x = x_0, x_1, \ldots, x_{i-1}, x_i = u, x_{i+1} = v, \ldots, x_n = y$ with $x, y \in$ N. Since $u \notin$ N, it is clear that u is an internal vertex of a path P. Since $deg_{<N(u)>}(v) = |N(u)| - 1$, we see that v is adjacent to x_{i-1}, which is a contradiction to the fact that P is a $x - y$ monophonic path. Hence M is contained in every restrained monophonic dominating set of G.

Theorem 2.4 For any connected graph G, $2 \le m(G) \le \gamma_m(G) \le \gamma m_r(G) \le p$.

Proof. A monophonic set needs at least two vertices and therefore $m(G) \ge 2$. Also every monophonic dominating set is a monophonic set of G, and then $m(G) \le \gamma_m(G)$. If $\gamma_m(G) = p$ or $p - 1$, then $\gamma m_r(G) = p$ the converse need not be true. Also since every restrained monophonic dominating set of G is an monophonic dominating set of G, and then $\gamma_m(G) \le \gamma m_r(G)$. The complement of each restrained monophonic dominating set has cardinality different from 1 we have $\gamma m_r(G) \ne p - 1$. Thus there is no graph G of order p with $\gamma m_r(G) = p - 1$. Hence $2 \le m(G) \le \gamma_m(G) \le \gamma m_r(G) \le p$.

Theorem 2.5 If a graph G of order p has exactly one vertex of degree $p - 1$, then $\gamma m_r(G) = p$.

Proof. Let G be a Graph of order p with exactly one vertex of degree p – 1, and let it be u. Since the vertex u is adjacent to all other vertices in G, then any vertex v where $v \in V(G) - \{u\}$, is not an internal vertex of any monophonic path joining two vertices of G other than u and v. Hence $\gamma m_r(G) = p$.

Corollary 2.6 For the complete graph K_p ($p \ge 2$), $\gamma m_r(K_p) = p$.

Proof. Since every vertex of the complete graph K_p ($p \ge 2$) is a extreme vertex, by Theorem 2.3, the vertex set of K_p is the minimum restrained monophonic dominating set of K_p. Thus $\gamma m_r(K_p) = p$.

Theorem 2.7 For the complete bipartite graph $G = K_{m,n}$, $\gamma m_r(G) = 4$ if $3 \le m \le n$.

Proof. Let X and Y be the partite sets with $|X| = m$, $|Y| = n$. If $3 \le m \le n$, then any minimum restrained monophonic dominating set of G is got by choosing two elements from X and Y so that $\gamma m_r(G) = 4$.

3. Realization Result

Theorem 3.1 For any integers a, b, c with $3 \le a \le b < c$, then there exists a connected graph G such that $m(G) = a$, $\gamma_m(G) = b$ and $\gamma m_r(G) = c$.

Proof. Case 1 $3 \le a = b < c$

Let $P : x, y, z$ be a path of order 3. We first add $a - 1$ new vertices $u_1, u_2, \ldots, u_{a-1}$ to P and join these to x. We then add $c - a$ new vertices $w_1, w_2, \ldots, w_{c-a}$ and join these to both x and z, there by producing the graph G given in Figure 2.

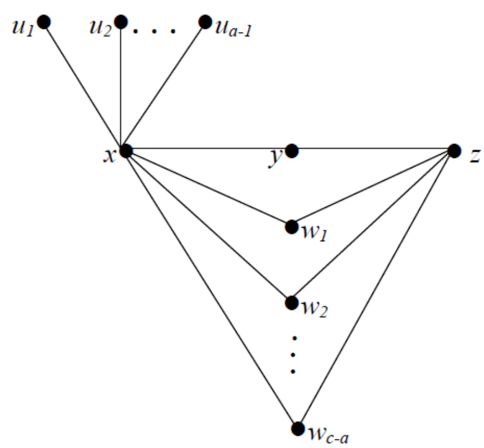

Figure 2. The graph G in Case 1 of Theorem 3.1

First we show that $m(G) = a$. Since each u_i ($1 \le i \le a$-1) is a extreme vertex of G, by Theorem1.1, each u_i ($1 \le i \le a$-1) belongs to every monophonic set of G. Let $M = \{u_1, u_2, \ldots, u_{a-1}\}$. Then M is not a monophonic set of G and so $m(G) \ge a$. However, $M_1 = M \cup \{z\}$ is a monophonic set of G, and so $m(G) = a$. Clearly $\gamma_m(G) = a$. Next, we show that $\gamma m_r(G) = c$. $M_2 = M \cup \{w_1, w_2, \ldots, w_{c-a}\}$ is a minimum restrained monophonic dominating set of G, so that $\gamma m_r(G) = c$.

Case 2 $a + 1 = b < c$

Let $C_6 : v_1, v_2, v_3, v_4, v_5, v_6$ be a cycle of order 6, and let P_3: x, y, z be a path of order 3. Let H be a graph obtained from C_6 and P_3 by identifying the vertex v_1 in C_6 and the vertex z in P_3. We first add $a - 1$ new vertices $u_1, u_2, \ldots, u_{a-1}$ to H, and join these to x. We then add $c - b$ new vertices $w_1, w_2, \ldots, w_{c-b}$ and join these to both v_2 and v_4, there by producing the graph G given in Figure 3.

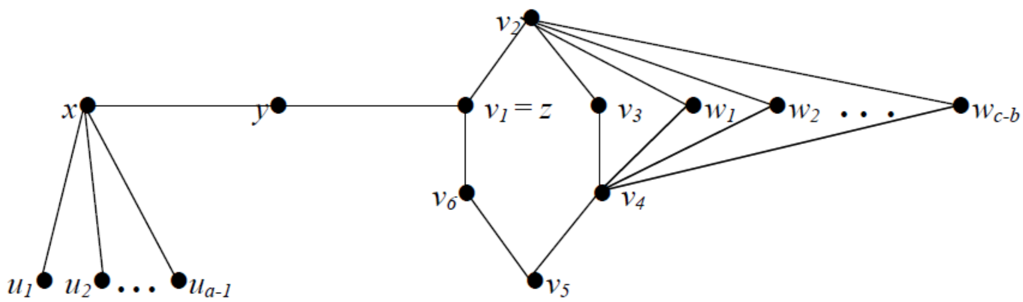

Figure 3. The graph G in Case 2 of Theorem 3.1

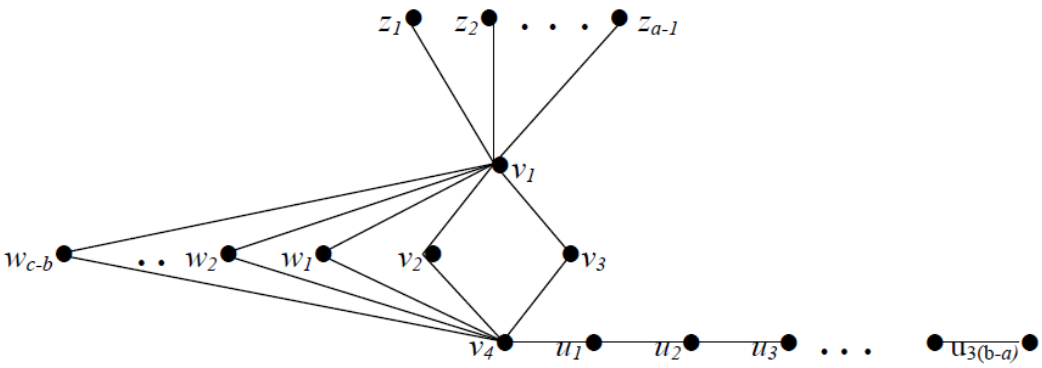

Figure 4. The graph G in Case 3 of Theorem 3.1

First we show that $m(G) = a$. Since each u_i $(1 \le i \le a-1)$ is a extreme vertices of G, by Theorem 1.1, each u_i $(1 \le i \le a-1)$ belongs to every monophonic set of G. Let $M = \{u_1, u_2,...,u_{a-1}\}$. M is not a monophonic set of G, and so $m(G) \ge a$. However, $M_1 = M \cup \{v_4\}$ is a monophonic set of G, and so $m(G) = a$. Next, we show that $\gamma_m(G) = b$. Let $M_2 = M_1 \cup \{v_1\}$ is a monophonic dominating set of G, and so $\gamma_m(G) = b$. It is clear that $M_3 = M \cup \{w_1, w_2,....,w_{c-b}\}$ is a minimum restrained monophonic dominating set of G, so that $\gamma m_r(G) = c$.

Case 3 $a + 2 \le b < c$

Let $C_4 : v_1, v_2, v_3, v_4$ be a cycle of order 4, and let $P : u_1, u_2,....,u_{3(b-a)}$ be a path of order $3(b-a)$. Let H be a graph obtained from C_4 and P by identifying the vertex v_4 in C_4 and u_1 in P, and then joining v_4 and u_1. We first add $a - 1$ new vertices $z_1, z_2,...z_{a-1}$ to H and join these to v_1. We then add $c - b$ new vertices $w_1, w_2,....,w_{c-b}$ and join these to both v_1 and v_4, there by producing the graph G given in Figure 4.

First we show that $m(G) = a$. Since each u_i $(1 \le i \le a-1)$ is a extreme vertices of G, by Theorem 1.1, each u_i $(1 \le i \le a-1)$ belongs to every monophonic set of G. Let $M = \{z_1, z_2, ..., z_{a-1}\}$. M is not a monophonic set of G, and so $m(G) \ge a$. However, $M_1 = M \cup \{u_{3(b-a)}\}$ is a monophonic set of G, and so $m(G) = a$. Next, we show that $\gamma_m(G) = b$. It is clear that M_1 is not a monophonic dominating set of G. Clearly $M_2 = M_1 \cup \{v_4, u_2, u_5, ..., u_{3(b-a)}\}$ is a monophonic dominating set of G, and so $\gamma_m(G) = b$. Next, we show that $\gamma m_r(G) = c$. It is clear that $M_3 = M_2 \cup \{w_1, w_2,....,w_{c-b}\}$ is a minimum restrained monophonic dominating set of G, and so that $\gamma m_r(G) = c$.

4. Conclusions

In this paper the authors introduced the restrained monophonic domination number and the restrained monophonic domination number of some connected graph are realized. The authors obtained the restrained edge monophonic domination number, connected restrained monophonic domination number, and forcing restrained monophonic domination number in the subsequent papers. It has so many application in security of buildings and communication networks.

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