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Nonlinear Analysis of Stochastic SI Vaccination Model

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Abstract The modeling of infectious disease has been a means of study of disease spread and predicting of an outbreak as well as evaluating strategies for the control of the epidemic. Epidemic models are normally classified based on the disease status. In this article, we study SI vaccination model. The threshold parameter R_0 is deduced which shows the disease would spread if its value exceeds one. The global stability of the disease-free and the endemic equilibrium is studied by using the theorem of a Lyapunov function. We adopt the stochastic version of the model and analyzed the stability of the stochastic positive equilibrium. Numerical simulation was done for the models which show the population dynamics of the SI models in the different compartments.

Keywords Boundedness, Basic Reproduction Ratio, Lyapunov function, Positive equilibrium

1. Introduction

The modeling of communicable diseases with stochastic differential equation (SDE) has gained grounds recently due to its wide range of applications and its ability to reflect reality in epidemiology [3]. Diseases outbreak in a population of susceptibles logically follows stochastic processes, but not the idea of the robustic deterministic as perceived [12]. Stochastic process arises naturally in many physical applications where randomness is to be included in the mathematical model [5, 8]. Stochastic models are adopted when a small number of reacting molecules is present in a modeling system. In such instants of small numbers reacting molecules, fluctuation becomes inevitable and deterministic models become inappropriate to use [2]. In recent years, major studies on stochastic differential equations (SDEs) that have been published by researchers have identified the growing importance of investigating the stability of stochastic positive equilibrium, as well as the global stability of the disease-free and the endemic equilibrium [1, 4, 9].

In this paper, we consider the *SI* vaccination model proposed by Gardon et al. [7] as follows

$$\begin{cases} \frac{dS}{dt} = (1-\rho)\pi - \mu s - (\beta I + \beta_v I_v)s \\ \frac{dS_v}{dt} = \rho\pi - \mu s_v - (1-r)(\beta I + \beta_v I_v)s_v \\ \frac{dI}{dt} = (\beta I + \beta_v I_v)s - (u+v)I \\ \frac{dI_v}{dt} = (1-r)(\beta I + \beta_v I_v)s_v - (u+v_v)I_v \end{cases} \tag{1}$$

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Where S, S_v, I and I_v represent unvaccinated susceptible, vaccinated susceptible, unvaccinated infectives and vaccinated infectives respectively. The model answers one important underlying research subjects; the determination of the existence of the threshold parameter R_0, which hints on the spreading or dying out of an invading epidemic into a population of susceptible, as studied by various authors [15, 16]. Our motivation lies in the works of Maroufy et al. [14], Adnani et al. [6], Kiouach and Omari [13] and Mukherjee et al [10], who extended their deterministic models to stochastic versions, and studied the stability of the stochastic models. In this research article, we first study the positivity and boundedness of the system (1). The basic reproduction ratio is determined. Applying the hypothetical theorem of the Lyapunov functional, we determine the global stability of the two equilibria for system (1). We extend our stability analysis to the stochastic system (5), which is obtained by random perturbation of the deterministic system (1) and find the stability of its positive equilibrium. Finally, numerical examples which shows the dynamics of systems (1) and (5) are given, which gives the explicit difference in the dynamics of the models.

2. Positivity, Boundedness and Basic Reproduction Ratio

2.1. Positivity and Boundedness

The theory of ordinary differential equations requires that, for every set of initial conditions $(S_0, S_{v_0}, I_0, I_{v_0})$, the state variables $(S(t), S_v(t), I(t), I_v(t))$ of the solution must remain non-negative.

Proposition 2.1. Let $(S(t), S_v(t), I(t), I_v(t))$ be the solution of the system (1).

(a) Given the initial condition $(S_0, S_{v_0}, I_0, I_{v_0}) \in \Omega$, then there exist a unique positive solution $(S(t), S_v(t), I(t), I_v(t))$ for every $t \geq 0$, such that the solution will remain in Ω with probability of one.

(b) The solution $(S,\ S_v, I, I_v)$ is defined in the interval $[0, \infty)$ and $\lim_{t\to\infty} Sup\, N(t) \leq \frac{\pi}{\mu}$, where $N(t) = S(t) +\ S_v(t) + I(t) +\ I_v(t)$.

Proof:

In (a) we let $(S_0, S_{v_0}, I_0, I_{v_0}) \in \Omega$. Evidently, the coefficients of system (1) are locally Lipschitz continuous. Hence, for any given initial condition $(S_0, S_{v_0}, I_0, I_{v_0}) \in \Omega$, there exist a unique local solution $(S(t), S_v(t), I(t), I_v(t))$ for every $t \in [0, T)$, where T is the final time. Here, it can be deduced that $S(t) +\ S_v(t) +\ I(t) + I_v(t) \leq \frac{\pi}{\mu}$ for every $t \in [0, T)$. Summing the total population of system (1) gives $dN(t) \leq (\pi - \mu N)dt$. Suppose $x(t)$ is the solution of the differential equation $dx(t) = (\pi - dx(t))dt$, $x(0) = N(0)$, where $N(0) = S(0) +\ S_v(0) +\ I(0) + I_v(0)$. Hence, by comparison theorem; $N(t) \leq x(t) \leq \frac{\pi}{\mu}$ for $t \in [0, T)$ as required.

Again, we can verify in (b) that

$$\frac{dN}{dt} \leq \pi - \mu s - \mu s_v - (u+v)I - (u+v_v)I_v$$

$$\frac{dN}{dt} \leq \pi - \mu N \tag{2}$$

Integrating inequality (2) gives $N(t) \leq \frac{\pi}{\mu}(1 - e^{-\mu t})$ for every $t \in [0, T]$, which implies $N(t) \leq \frac{2\pi}{\mu}$. It can therefore be verified that the solution (S, S_v, I, I_v) is bounded within the interval $[0, T]$. This implies $N(t) \leq \frac{\pi}{\mu}(1 - e^{-\mu t})$ for every $t \in [0, \infty)$. Hence $\lim_{t\to\infty} Sup\, N(t) \leq \frac{\pi}{\mu}$. Hence, employing the same intuition used in proving proposition 2.1, we see that system (1) with non-negative initial conditions $S_0 \geq 0, S_{v_0} \geq 0,\ I_0 \geq 0, I_{v_0} \geq 0$ has a non-negative solution defined in R and the set $\Omega = \{(S, S_v, I, I_v)/S > 0, S_v > 0, I > 0, I_v > 0 \text{ and } S + S_v + I + I_v = \frac{\pi}{\mu}\}$ is invariant by system (1).

2.2. The Basic Reproduction Ratio

The basic reproduction ratio (R_0) is defined as an infections originating from an infected individual that invades a population originally of susceptible individuals. Our model calculation would be based on the approach of Diekmann and Heesterbeek ([17]). Here, the functions (F) and (V) denote the rate of new infection term and the rate of transfer into and out of the unvaccinated infectives and vaccinated infectives respectively.

The disease compartments are

$$\frac{dI}{dt} = (\beta I + \beta_v I_v)s - (u+v)I$$

$$\frac{dI_v}{dt} = (1-r)(\beta I + \beta_v I_v)s_v - (u+v_v)I_v$$

Hence $f = \begin{bmatrix} (\beta I + \beta_v I_v)s \\ (1-r)(\beta I + \beta_v I_v)s_v \end{bmatrix}$ and $\mathscr{v} = \begin{bmatrix} (u+v)I \\ (u+v_v)I_v \end{bmatrix}$

The deterministic system (1) has a unique disease-free equilibrium, given by $E_0 = (\rho N_0, (1-\rho)N_0, 0, 0)$, where $N_0 = \frac{\pi}{\mu}$.

The matrices F and V evaluated at $E_0 = (\rho N_0, (1-\rho)N_0, 0, 0)$ are given by

$F = \begin{pmatrix} \beta\rho N_0 & \beta_v\rho N_0 \\ (1-r)\beta(1-\rho)N_0 & (1-r)\beta_v S_v \end{pmatrix}$ and $V = \begin{pmatrix} u+v & 0 \\ 0 & u+v_v \end{pmatrix}$.

The matrix F is a rank one matrix, and its next generation matrix also has rank one ([16]). The spectral radius of a rank one matrix is its trace.

Hence

$$R_0 = \rho(FV^{-1}) = \begin{pmatrix} \frac{\beta\rho N_0}{(u+v)} & \frac{\beta_v\rho N_0}{(u+v_v)} \\ \frac{(1-r)\beta(1-\rho)N_0}{(u+v)} & \frac{(1-r)\beta_v(1-\rho)N_0}{(u+v_v)} \end{pmatrix}$$

Hence $R_0 = \frac{\beta\rho\pi}{\mu(u+v)} + \frac{(1-r)\beta_v(1-\rho)\pi}{\mu(u+v_v)}$ (3)

2.3. Global Stability of the Disease-free Equilibrium

Theorem 2.3: The disease-free equilibrium $E_0 = (\rho N_0, (1-\rho)N_0, 0, 0)$ is globally asymptotically stable in ${R_+}^4$ whenever $R_0 < 1$.

Proof: We consider the Lyapunov function

$L(w) = w_1 S + w_2 S_V + w_3 I + w_4 I_v$, where w_i, $i = 1, 2, 3, 4$ are constants that would be chosen in the course of the proof.

Hence, calculating the rate of change of L along the solution of (1) gives,

$$\frac{dL}{dt} = \frac{\partial L}{\partial s}.\frac{ds}{dt} + \frac{\partial L}{\partial S_v}.\frac{dS_v}{dt} + \frac{\partial L}{\partial I}.\frac{dI}{dt} + \frac{\partial L}{\partial I_v}.\frac{dI_v}{dt}$$

$$\frac{dL}{dt} = w_1\big((1-\rho)\pi - \mu S - (\beta I + \beta_v I_v)S\big) + w_2\big(\rho\pi - \mu S_v - (1-r)(\beta I + \beta_v I_v)\big)S_v + w_3\big((\beta I + \beta_v I_v)S - (u+v)I\big) + w_4\big((1-r)(\beta I + \beta_v I_v)S_v - (u+v_v)I_v\big)$$

$$\frac{dL}{dt} = (w_2 - w_1)\rho\pi - \mu w_1 S - \mu w_2 S_v + (\beta I + \beta_v I_v)(w_3 - w_1)S - w_3(\mu+v)I - (\mu+v_v)w_4 I_v + (1-r)(\beta I + \beta_v I_v)(w_4 - w_2)S_v$$

Choosing $w_1 = w_2 = w_3 = w_4$ gives the following

$$-\mu w_1 S - \mu w_2 S_v - w_3(\mu+v)I - (u+v_v)w_4 I_v\ .$$

It follows that L is positive definite and $\frac{dL}{dt}$ is negative definite. It can therefore be ascertained that the function L is a

Lyapunov function for system (1). Hence by Lyapunov asymptotic stability theorem [9], the equilibrium E_0 is globally asymptotically stable.

2.4. The Global Stability of the Endemic Equilibrium

Theorem 2.4: The unique endemic equilibrium E^* is globally asymptotically stable in ${R_+}^4$ whenever $R_0 > 1$.

Proof: We consider the Lyapunovfunction

$L(m) = \frac{1}{2}m_1(s+\beta)^2 + m_2 S_v + \frac{1}{2}(I+\beta_v)^2 + m_3 I_v,\ m_i, i = 1,2,3$ are constants to be chosen in the course of the proof. The derivative of w along the solution of (1) gives

$$\frac{dL}{dt} = m_1(S+\beta).\frac{ds}{dt} + m_2.\frac{dS_v}{dt} + (I+\beta_v).\frac{dI}{dt} + m_3.\frac{dI_v}{dt}$$

$$\begin{aligned}\frac{dL}{dt} &= m_1(S+\beta)\big((1-\rho)\pi - \mu S - (\beta I + \beta_v I_v)S\big) + m_2\big(\rho\pi - \mu S_v - (1-r)(\beta I + \beta_v I_v)\big)S_v \\ &\quad +(I+\beta_v)\big((\beta I + \beta_v I_v)S - (u+v)I\big) + m_3\big((1-r)(\beta I + \beta_v I_v)S_v - (u+v_v)I_v\big) \\ &= \big(Sm_1\pi(1-\rho) + m_1\pi\beta(1-\rho)\big) - \mu m_1(S+\beta)S - m_1(S+\beta)(\beta I + \beta_v I_v)S - \mu m_2 S_v \\ &\quad -m_2(1-r)(\beta I + \beta_v I_v)S_v - (I+\beta_v)(u+v)I + m_3(1-r)(\beta I + \beta_v I_v)S_v - m_3(u+v_v)I_v \\ &\quad +(I+\beta_v)(\beta I + \beta_v I_v)S\end{aligned} \tag{4}$$

Choosing m_1, m_2 and m_3 such that $(I+\beta_v) - m_1(S+\beta) = 0$, $m_3(1-r) - m_2(1-r) = 0$, and $\rho = 1$. Then, the relation (4) can be expressed as

$$-\mu m_1(S+\beta)S - \mu m_2 S_v - ((I+\beta_v)(u+v))I - m_3(u+v_v)I_v\ .$$

L is a positive definite and $\frac{dL}{dt}$ is negative definite. Therefore the function L is a Lyapunov function for system (1) and consequently, by Lyapunov asymptotic stability theorem [13], the equilibrium state E^* is globally asymptotically stable. Hence this completes the proof.

3. The Stochastic Model

Here, we introduce stochastic perturbations in the main parameters of the deterministic model (1). Thus we permit stochastic perturbations of the variable $S,\ S_v,\ I,\ I_v$ around their values at positive equilibrium E^*.

Hence, we assume that the white noise of the stochastic perturbations of the variable around values of E^* are proportional to the distances of $S,\ S_v,\ I,\ I_v$ from $S^*,\ {S_v}^*,\ I^*,\ {I_v}^*$. Hence the stochastic version of model (1) is

$$\begin{cases} dS = [(1-\rho)\pi - \mu s - (\beta I + \beta_v I_v)s]dt + \rho_1(s - s^*)dB_1 \\ dS_v = [\rho\pi - \mu s_v - (1-r)(\beta I + \beta_v I_v)s_v]dt + \rho_2(s_v - {s_v}^*)dB_2 \\ dI = [(\beta I + \beta_v I_v)s - (u+v)I]dt + \rho_3(I - I^*)dB_3 \\ dI_v = [(1-r)(\beta I + \beta_v I_v)s_v - (u+v)I_v]dt + \rho_4(I_v - {I_v}^*)dB_4 \end{cases} \tag{5}$$

With ρ_i, where $i = 1,\ 2,\ 3,\ 4$ are real constants, and $B_i = 1,\ 2,\ 3,\ 4$ are independent wiener processes. We investigate the asymptotic stability behavior of the equilibrium E^* of the stochastic equation (5) and compare results with the deterministic model (1).

3.1. Stochastic Stability of the Positive Equilibrium

It can be shown clearly that, the deterministic model (1) has one disease-free equilibrium $E^0\big(\rho N_{0,}\ (1-\rho)N_0, 0, 0\big)$ which is globally asymptotically stable when $R_0 \le 1$. However, when $R_0 > 1$, the disease-free equilibrium E^0 is unstable . Obviously, there is also a unique positive endemic equilibrium $E^*(S^*, {S_v}^*, I^*, {I_v}^*) =$

$$\left(\frac{(1-\rho)\pi}{((\beta I + \beta_v I_v) + \mu)}, \frac{\rho\pi}{((1-\rho)(\beta I + \beta_v I_v) + \mu)}, \frac{\beta_v I_v S}{((u+v) - \beta S)}, \frac{\beta I S_v(1-\rho)}{(r\beta_v S_v + (u+v) - \beta_v S_v)}\right)$$

This equilibrium is globally asymptotically stable. The stochastic system (5) has the same equilibria as the deterministic system (1). Assuming that $R_0 \le 1$, we investigate the stability of the endemic equilibrium E^* of (5). The stochastic differential equation (5) can be centered at its positive equilibrium E^* by the change of variables

$$x_1 = S - S^*,\ x_2 = S_v - {S_v}^*,\ x_3 = I - I^*,\ x_4 = I_v - {I_v}^* \tag{6}$$

The linearized system of the stochastic model (5) around E^* takes the form

$$dx(t) = f\big(x(t)\big)dt + g\big(x(t)\big)dB(t) \tag{7}$$

Where $x(t) = col(x_1(t), x_2(t), x_3(t), x_4(t))$

and equals

$$f(x(t)) = \begin{pmatrix} -(\beta I^* + \beta_v I_v{}^*) - \mu & 0 & -\beta S^* & -\beta_v S^* \\ 0 & -(1-r)(\beta I^* + \beta_v I_v{}^*) - \mu & -(1-r)\beta S_v{}^* & -(1-r)\beta_v S_v{}^* \\ (\beta I^* + \beta_v I_v{}^*) & 0 & -((\mu + v) - \beta S^* & \beta S^* \\ 0 & -(1-r)(\beta I^* + \beta_v I_v{}^*) & (1-r)\beta S_v{}^* & -((\mu + v_v) - (1-r))\beta_v S^* \end{pmatrix}$$

$$g(x) = \begin{pmatrix} \rho_1 x_1 & 0 & 0 & 0 \\ 0 & \rho_2 x_2 & 0 & 0 \\ 0 & 0 & \rho_3 x_3 & 0 \\ 0 & 0 & 0 & \rho_4 x_4 \end{pmatrix}$$

Clearly, the endemic equilibrium E^* corresponds to the trivial solution $x(t) = 0$ in (7). We denote L to be the differential operator associated with (7), defined for the family of nonnegative functions $\mu(t,x) \in C^{1,2}(R \times R^n)$, such that it is continuously differentiable with respect to t and twice with respect to x.

According to Afanas' ev et al [11], the differential operator L for a function $u(t,x) \in C^{1,2}(R \times R^n)$ is given by

$$Lu(t,x) = \frac{\partial u\,(t,x)}{\partial t} + f^T(x)\frac{\partial u\,(t,x)}{\partial x} + \frac{1}{2}Tr\left[g^T(x)\frac{\partial^2 u(t,x)}{\partial x^2}g(x)\right] \tag{8}$$

Where $\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right)^T$ and $\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{ij}$,

Where $i, j = 1, 2, 3, 4$, ''T'' and ''T_r'' are the transposition and trace respectively. With reference to Afanas' ev et al [11], the following results hold.

Theorem 3.1: Suppose a function $u(t,x) \in C^{1,2}(R \times R^n)$ exist, satisfying the following inequalities

$$k_1|x|^p \le u(t,x) \le k_2|x|^p$$
$$Lu(t,x) \le -k_3|x|^p \tag{9}$$

Where $k_i > 0$, $i = 1, 2, 3, 4$ and $p > 0$. Then the trivial solution of (7) is p^{th} moment exponentially stable. Again, given that $p = 2$, the trivial solution is said to be exponentially stable in mean square and the equilibrium $x = 0$ is globally asymptotically stable.

From theorem 3.1, the conditions for stochastic asymptotic stability of trivial solution of (7) are given theorem 3.2.

Theorem 3.2: Suppose $\rho_1{}^2 < 2\left((\beta I^* + \beta_v I_v{}^*) + \mu\right)$, $\rho_2{}^2 < 2\left((1-r)(\beta I^* + \beta_v I_v{}^*) + \mu\right)$, $\rho_3{}^2 < 2\left((u+v) + \beta S^*\right)$ and $\rho_4{}^2 < 2\left((u + v_v) + (1-r)\beta_v S_v{}^*\right)$ hold, then the zero solution of (7) is asymptotically mean square stable.

Proof: We consider the Lyapunov $u(t,x) = \frac{1}{2}(w_1 x_1{}^2 + w_2 x_2{}^2 + w_3 x_3{}^2 + w_4 x_4{}^2)$

With w_i, for $i = 1, 2, 3, 4$ non-negative constants that will be chosen in the course of the proof. It can be easily ascertained that inequality (9) hold true when $p = 2$.

Applying the operator L on $u(t,x)$ gives

$$\begin{aligned} Lu(t,x) = &-\left((\beta I^* + \beta_v I_v{}^*) + \mu\right)w_1 x_1{}^2 - \beta S^* w_1 x_1 x_3 - \beta_v S^* w_1 x_1 x_4 - \left((1-r)((\beta I^* + \beta_v I_v{}^*) + \mu\right)w_2 x^2 \\ &-(1-r)\beta S_v{}^* w_2 x_2 x_3 - (1-r)\beta_v S_v{}^* w_2 x_2 x_4 + (\beta I^* + \beta_v I_v{}^*)w_3 x_3 x_1 - \\ &\left((u+v) - \beta S^*\right)w_3 x_3{}^2 + \beta S^* w_3 x_3 x_4 - (1-r)(\beta I^* + \beta_v I_v{}^*)w_4 x_4 x_2 \\ &+(1-r)\beta S_v{}^* w_4 x_4 x_3 - \left((u + v_v) - (1-r)\beta_v S_v{}^*\right)w_4 x_4{}^2 + \frac{1}{2}Tr\left[g^T(x)\frac{\partial^2 u}{\partial x^2}g(x)\right] \end{aligned} \tag{10}$$

Further

$$\begin{aligned} &w_1(-(\beta I^* + \beta_v I_v{}^*) + \mu)x_1 - \beta S^* x_3 - \beta_v S^* x_4)x_1 \\ &+w_2(-\left((1-r)((\beta I^* + \beta_v I_v{}^*) + \mu\right)x_2 - (1-r)\beta S_v{}^* x_3 - (1-r)\beta_v S_v{}^* x_4)x_2 \\ &+w_3\left((\beta I^* + \beta_v I_v{}^*)x_1 - \left((u+v) - \beta S^*\right)x_3 + \beta S^* x_4\right)x_3 \\ &+w_4(-(1-r)((\beta I^* + \beta_v I_v{}^*)x_2 + (1-r)\beta S_v{}^* x_3 - \left((u + v_v) - (1-r)\beta_v S_v{}^* x_4\right)x_4 \\ &+\frac{1}{2}Tr\left[g^T(x)\frac{\partial^2 u}{\partial x^2}g(x)\right] \end{aligned} \tag{11}$$

Now remark that

$$\frac{\partial^2 u}{\partial x^2} = \begin{pmatrix} w_1 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 \\ 0 & 0 & w_3 & 0 \\ 0 & 0 & 0 & w_4 \end{pmatrix}$$

and

$$g^T(x)\frac{\partial^2 u}{\partial x^2}g(x)=\begin{pmatrix} w_1{\rho_1}^2{x_1}^2 & 0 & 0 & 0 \\ 0 & w_2{\rho_2}^2{x_2}^2 & 0 & 0 \\ 0 & 0 & w_3{\rho_3}^2{x_3}^2 & 0 \\ 0 & 0 & 0 & w_4{\rho_4}^2{x_4}^2 \end{pmatrix}$$

with $\frac{1}{2}Tr\left[g^T(x)\frac{\partial^2 u}{\partial x^2}g(x)\right]=\frac{1}{2}(w_1{\rho_1}^2{x_1}^2+w_2{\rho_2}^2{x_2}^2+w_3{\rho_3}^2{x_3}^2+w_4{\rho_4}^2{x_4}^2)$ (12)

Now, from equation (10), if we choose
$w_2(1-r)\beta {S_v}^*=w_3(\beta I^*+\beta_v {I_v}^*)$, $w_2(1-r)\beta {S_v}^*=w_4(1-r)\beta {S_v}^*$ and $w_1(\beta S^*)=w_3(\beta S^*)$, then from equation (10), it is easy to verify that,

$$\begin{aligned} Lu(t,x) &= w_1(-(\beta I^*+\beta_v {I_v}^*)+\mu)x_1-\beta_v S^* x_4)x_1\ +w_2(-(1-r)(\beta I^*+\beta_v {I_v}^*)+\mu)x_2)x_2 \\ &+w_3(-(u+v)-\beta S^*)x_3)x_3 \\ &+w_4\big(-(1-r)(\beta I^*+\beta_v {I_v}^*)\big)x_2-\big((u+V_v)-(1-r)\beta_v S^* x_4\big)x_4. \end{aligned}$$

Hence, according to theorem 3.1, the proof is completed.

4. Numerical Examples of the Models

Here, we illustrate with figures the dynamics of the systems (1) and (5), and gives an explicit difference in the models by carrying out numerical simulation for the hypothetical set of parameter values. To demonstrate the differences, we simulate the systems (1) and (5) by using the following set of parameter values; $\rho=0.09$, $\pi=3.14$, $\mu=0.6$, $\beta_v=0.5$, $v=0.04$, $r=0.9$, $\beta=0.01$. However, some of the parameters were allowed to change in the course of the simulations in order to bring out the dynamics of the models. The differences in the dynamics of the models are therefore given by the diagrams in the following:

Example 1:

Here, the dynamic behaviors of the four classes of individuals (S: blue, S_v: green, I: black, I_v: red) of the deterministic and its stochastic version are plotted against time. Here we assumed the following set of hypothetical parameter values; $\rho=0.09$, $\pi=3.14$, $\mu=0.6$, $\beta_v=0.5$, $v=0.04$, $r=0.9$, $\beta=0.01$. Calculating R_0 based on theses parameter values gives $R_0=0.2$. To confirm the deterministic plot of figure1a, we choose white noises ρ_1, ρ_2, ρ_3 and ρ_4 of equal strength 0.1, and shows the fluctuations in the trajectories of the plot of the stochastic system (5). We can see that the trajectories of the stochastic plot displayed on our graphs are the same as the trajectories of its deterministic model (1) during a finite time frame.

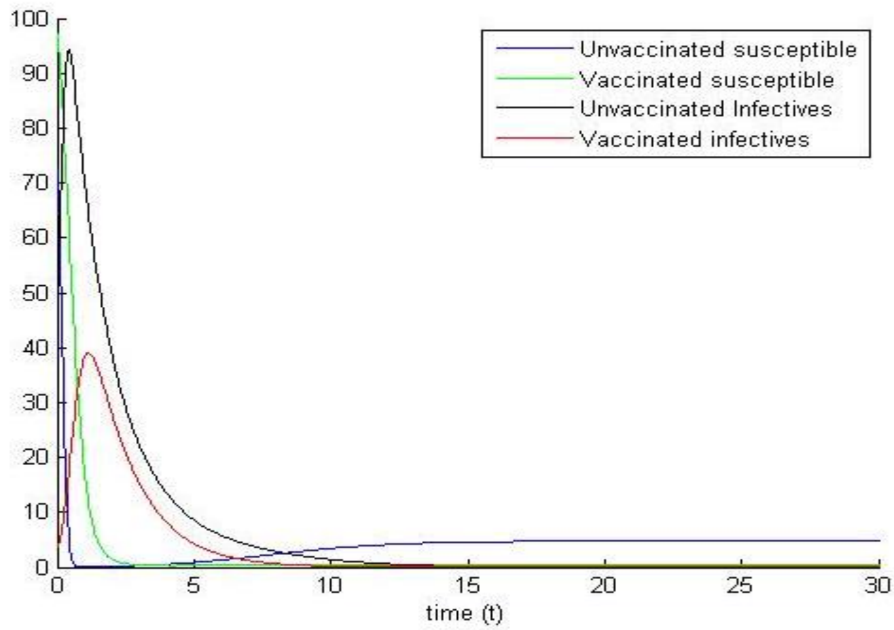

Figure 1a.

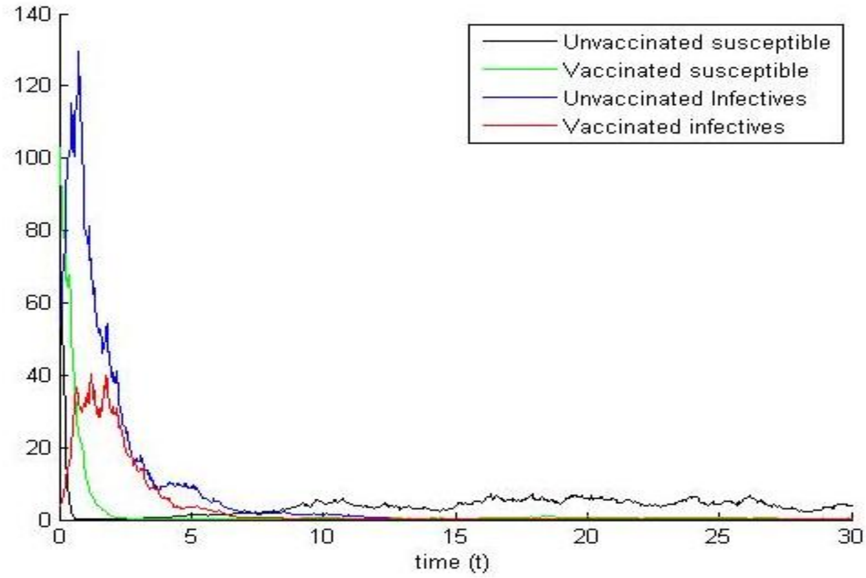

Figure 1b.

Example 2:

Here, we choose the same choice of parameter values: $\rho = 0.09,\ \pi = 3.14,\ \mu = 0.6,\ \beta_v = 0.5,\ v = 0.04,\ r = 0.9$, except for $\beta = 0.04$, and the same strength of white noise $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0.1$. Again, calculating R_0 on these parameter values gives $R_0 = 0.3$. We observe that the path of the trajectories of the systems (1) and (5) are eventually absorbed in the stable point (see figure 1a, 1b, 2a, 2b).

Examples 3

Choosing $\rho = 0.09,\ \pi = 0.14,\ \mu = 0.6,\ \beta_v = 0.05,\ v = 0.003,\ r = 0.9$, and $\beta = 0.01$, we observe that system (1) and (5) are stable (see figures 3a and 3b). Again, the path of the stochastic processes leaves the trajectories and is absorbed in the equilibrium (see figure 3b).

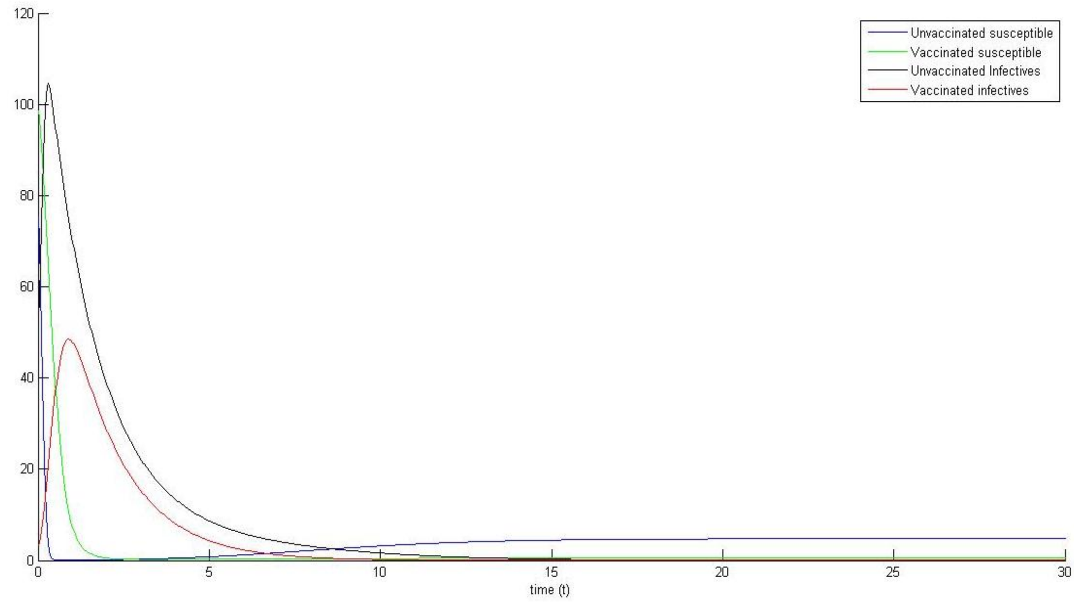

Figure 2a.

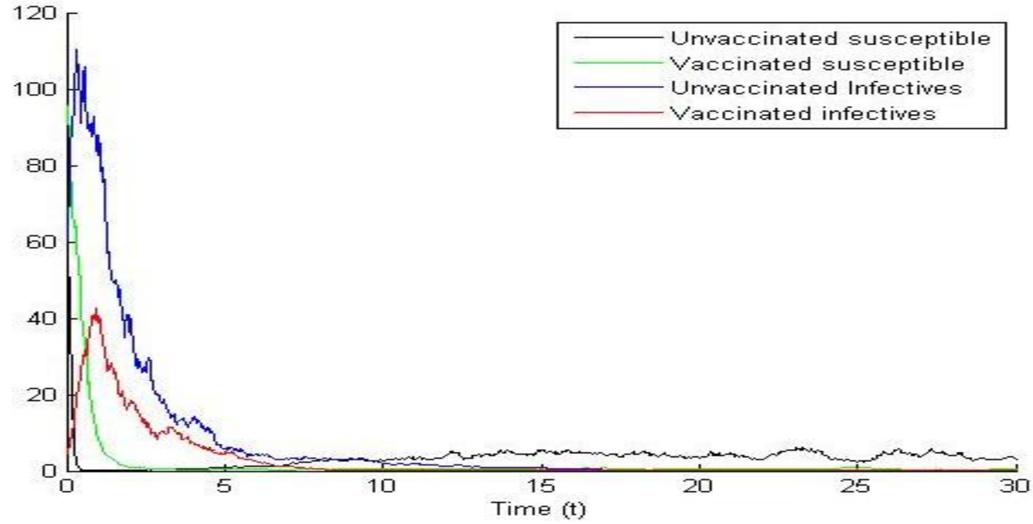

Figure 2b.

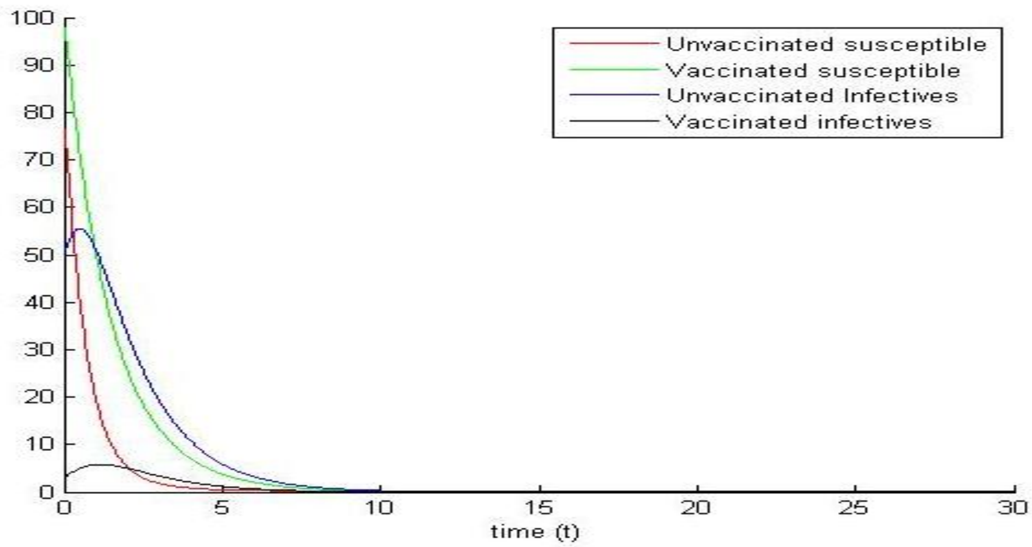

Figure 3a.

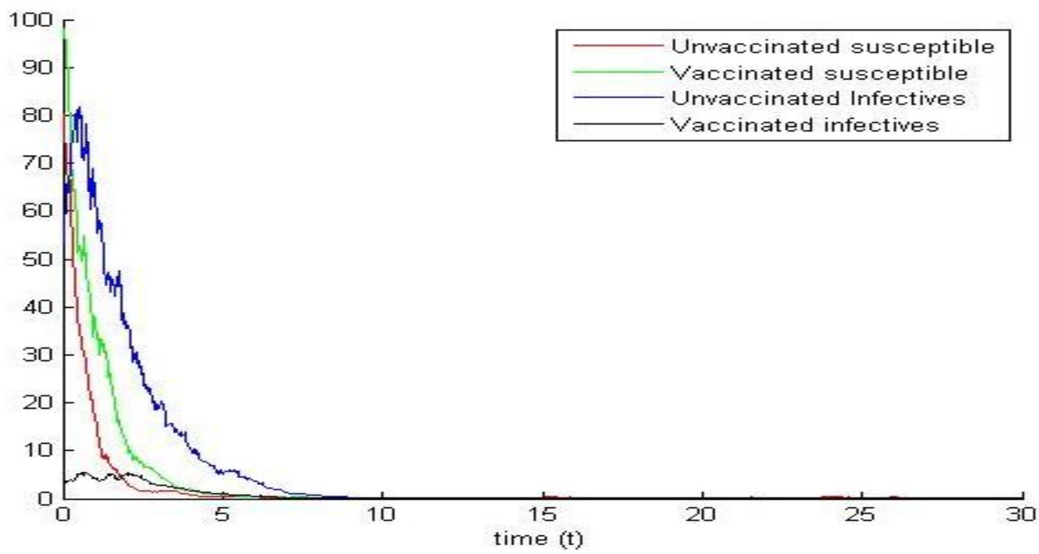

Figure 3b.

5. Conclusions

In this paper, the dynamics of deterministic SI vaccination model and its stochastic variant are presented. The stability analyses of the deterministic model were investigated. Suitable Lyapunov functions were constructed for the global stability of the two equilibria. We constructed the stochastic version of the model by employing the idea of Mukhere et al [4, 9, 10].

Our main purpose of the study was to investigate the asymptotic stability behavior of the endemic equilibrium E^* of the stochastic version of the deterministic SI model proposed by Gardon et al [7]. The numerical simulation for the models shows that the trajectories of the stochastic plots were the same as the trajectories of the deterministic model. Further, from our stochastic plots, the simulation shows an initial random fluctuation of the stochastic trajectories, until they eventually approach asymptotic level. We have demonstrated that our stochastic system is globally asymptotically stable in probability when the densities of white noise are less than certain threshold parameters. However, if these densities of white noise are zero, it means there are no stochastic environmental factors on the population and hence no stochastic perturbation. Hence theorem 3.2 conditions would be reduced to the condition $R_0 > 1$, which implies a nonlinear stability condition for the deterministic system (1) .In our future research, we would consider how control strategies may be devised for the model.

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