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The Wiener Index and Hosoya Polynomial of the Subdivision Graph of the Wheel $S(W_n)$ and the Line Graph Subdivision Graph of the Wheel $L(S(W_n))$

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Abstract Let G be a connected graph. The *Wiener index* of a graph is defined as the sum of all distances between different vertices, and the *Hosoya polynomial* of a graph G is defined as $H(G,x)=\sum_{\{u,v\}\in V(G)} x^{d(u,v)}$. In this paper, we find the Wiener index and Hosoya polynomial of the line graphs of the wheel graphs using the concept of subdivision.

Keywords Subdivision Graphs, line Graphs, Wheel W_n, Wiener Index, Hosoya polynomial

1. Introduction

Let *G=(V(G),E(G))* be a simple connected and finite graph, where *V(G)* and *E(G)* are the sets of vertices and edges, respectively. Number of elements in *V(G)*, *|V(G)|*, is called the order of the graph *G* and the number of elements in E(G), |E(G)|, is called the size of the graph *G*. We will denote the order and the size by *n* and *m*, respectively. For vertices $u,v\in V(G)$, the distance between *u* and *v* is the length of shortest *u-v* path and denoted as $d(u,v)$. The *diameter*, *d(G)*, is the largest distance between any two vertices of a graph *G*. For a vertex $u\in V(G)$, degree of *u* is the number of first neighbours of *u* in *G*.

The *subdivision graph*, *S(G)*, is the graph obtained by inserting an additional vertex into each edge of *G*, in other words by replacing each of its edge by a path of length 2. The *line graph*, *L(G)*, of a graph *G* is the graph whose vertices are the edges of *G*, two vertices *e* and *f* are incident if and only if they have a common end vertex in *G*. A *wheel graph* with *n* vertices, W_n, is a graph formed by connecting a single vertex to all vertices of a cycle. A wheel graph with *n* vertices has *2(n-1)* edges. For further details and results see [1-7].

A *topological index* is a function $Top:\Sigma\to\mathbb{R}$, where Σ is the set of finite simple graphs with the property that $Top(G)=Top(H)$ if *G* and *H* are isomorphic. There is a lot of research has been done on topological indices of different graph families so far, and is of much importance due to their chemical significance.

One of the oldest topological index which is the *Wiener index* was introduced by chemist *Harold Wiener* in 1947 [8, 9]. He introduced this index for comparing and describing the relation between Physical-Chemical properties. The definition of this index is as follows:

$$W(G)=\frac{1}{2}\sum_{u\in V(G)}\sum_{v\in V(G)} d(u,v)$$

The Wiener index of many molecular graphs has been computed, for details see [10-14]. The Hosoya polynomial was first introduced by *H. Hosoya* [15] and defines as follow:

$$H(G,x)=\frac{1}{2}\sum_{u\in V(G)}\sum_{v\in V(G)} x^{d(u,v)}$$

A lot of research on the Hosoya polynomial of many molecular graphs has been done. For more details and chemical applications and mathematical properties of these topological indices see paper series [16-28].

Let *d(G,k)* be the number of vertex pairs of *G*, the distance of which is equal to *k*. Then we can rewrite the Hosoya polynomial and the Wiener index as

$$H(G,x)=\sum_{i=1}^{d(G)} d(G,i)x^i$$

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$$W(G)=\sum_{i=1}^{d(G)} d(G,i)\times i$$

where $d(G)$ be the diameter of G and is the longest topological distance in G.

2. Main Results

Theorem 1: Let $S(W_n)$ be the subdivision graph of a wheel graph with order n. Then the Hosoya polynomial and the Wiener index of $S(W_n)$ are equals to:

$$H(S(W_n),x)=4n\,x+½n(n+9)x^2+n(n+2)x^3 +½n(3n-5)x^4+ n(n-4)\,x^5+ ½n(n-5)x^6.$$

and

$$W(S(W_n))=18n^2-33n.$$

Proof of Theorem 1: Consider the subdivision graph of a wheel graph with order n, we denotes this graph by $S(W_n)$ for all positive integer number $n\geq 3$. By the definition of the subdivision graph of graph and from Figure 1, we can see that the size of vertex set $V(S(W_n))$ is equal to $|V(W_n)|+|E(W_n)|=n+1+2n=3n+1$, where 2n vertices of the subdivision graph of the wheel graph W_n have degree two, similar to the wheel graph W_n, and only one center vertex c of the wheel graph W_n and the subdivision graph of the wheel graph $S(W_n)$ has degree n. Then all remained vertices have degree two, the vertices added to the wheel graph W_n. In other words, we can divide the vertex set $V(S(W_n))$ into several partitions V_2, V_3 and V_m, on based the degree of its members as follow:

$V_2=\{u\in V(S(W_n))|\ d_u=2\}\rightarrow|V_2|=2n$
$V_3=\{v\in V(S(W_n))|\ d_v=3\}\rightarrow|V_3|=n$
$V_n=\{c\in V(S(W_n))|\ d_c=n\}\rightarrow|V_n|=1$

So these imply that the size of edge set $E(S(W_n))$ is equal to

$$|E(S(W_n))|=\frac{2\times 2n+3\times n+n\times 1}{2}=4n.$$

Form the definitions of the Wiener index and Hosoya polynomial of a graph G, one can see that it is enough to compute all terms $d(S(W_n),k)$ of the subdivision graph of the wheel graph $S(W_n)$, where $d(S(W_n),k)$ denote the number of unordered pairs of vertices x and y of $S(W_n)$ such that distance $d(x,y)=k$ $(1\leq k\leq d(S(W_n)))$. Also, by according to Figure 1, we can see that $d(S(W_n))=6$, obviously.

Thus we have two re-formulas of the Wiener index and Hosoya polynomial of $S(W_n)$ as follow:

$$W(S(W_n))=\sum_{k=1}^{d(S(W_n))=6} d\left(S\left(W_n\right),k\right)\times k$$

and

$$H(S(W_n),x)=\sum_{k=1}^{d(S(W_n))=6} d\left(S\left(W_n\right),k\right)x^k\ .$$

By the definitions of the Wiener index and Hosoya polynomial of G, we see that the first term of $H(S(W_n),x)$ is equal to the number of edges of $S(W_n)$¸ thus $d(S(W_n),1)=|E(S(W_n))|=4n$.

By Figure 1 and the definition of the subdivision graph of G, we see that there are n 2-edges paths between the Centre vertex c and the members of V_3 and n 2-edges paths between all members of V_3. Also, for vertices from V_2 we see that there are $½[2n+n(n-1)+4n]= ½(n^2+5n)$ 2-edges paths between them. Thus, the second term $d(S(W_n),2)$ is equal to $½(n^2+9n)$.

Now, for the third terms of the Wiener index and Hosoya polynomial of $S(W_n)$, we start from the Centre vertex c, we have n 3-edges paths until the members of V_2. Also there are $n(n-1)+2n=n(n+1)$ 3-edges paths start from a vertex v of V_3 and end to a vertex u of V_2 and there is not any 3-edges path between all pairs of vertices (V_2, V_2), (V_2, V_n) and (V_3, V_3). These imply that $d(S(W_n),3)$ is equal to n^2+2n.

From Figure 1, we see that there is not any 4-edges path with the Centre vertex c and between all members of V_2 and V_3. But there are $n(n-2)+n= n^2-n$ 4-edges paths between all vertices from member of V_2, and there are $½n(n-3)$ 4-edges paths between all vertices from V_3. Thus, the number of the 4-edges paths of the subdivision graph of the wheel graph $S(W_n)$, is $n^2-n+½n(n-3)$ and $d(S(W_n),4)$ is equal to $½n(3n-5)$.

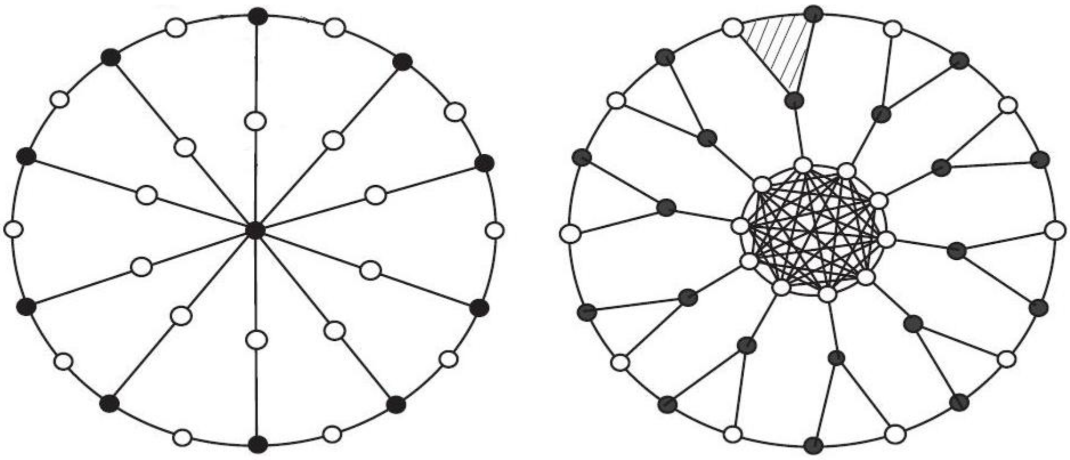

Figure 1. [1, 2] The subdivision graph of the wheel Wn and the line graph subdivision graph of the wheel $L(S(W_n))$ for all positive integer number $n\geq 3$

For $d(S(W_n),5)$, from Figure 1, one can see that there is not any 5-edges path between all pairs of vertices (V_2, V_2), (V_2, V_n), (V_3, V_3), and (V_3, V_n) obviously. And we have only $n(n-4)$ 5-edges paths by start from a vertex v of V_3 and end to a vertex u of V_2 and the last term of the Wiener index and Hosoya polynomial of $S(W_n)$ is $d(S(W_n),5)=n(n-4)$.

Finally, from the structure of the subdivision graph of the wheel graph $S(W_n)$, we see that there are only $½n(n-5)$ 6-edges paths between all pairs of vertices from the vertex partition V_2, thus $d(S(W_n),6)= ½n(n-5)$.

Here, by these above mentions results, we have the clearly formulas for the Wiener index and the Hosoya polynomial of the subdivision graph of the wheel graph $S(W_n)$ as:

$$H(S(W_n),x)=\Sigma_{d=1}^{6} d(S(W_n),d)x^d$$
$$=4n\,x+½n(n+9)x^2+n(n+2)x^3+½n(3n-5)x^4$$
$$+\ n(n-4)\,x^5+\ ½n(n-5)x^6.$$

And also,

$$W(S(W_n)\ )=\ \frac{\partial}{\partial x}\ H(S(W_n),x)$$
$$=4n\times 1+½n(n+9)\times 2+n(n+2)\times 3+½n(3n-5)\times 4$$
$$+\ n(n-4)\times 5+½n(n-5)\times 6$$
$$=18n^2-33n.$$

Also, by definition of the Hosoya polynomial of an arbitrary graph G as order n ($|V(G)|=n$), it is obvious that

$$H\left(G,1\right)=\sum_{i=1}^{d(G)} d(G,i)=\binom{n}{2}=\frac{n(n-1)}{2}.$$

And, we can check

$$H(S(W_n),x) = \Sigma_{d=1}^{5}\ d(S(W_n),d)$$
$$= 4n+½n(n+9)+n(n+2)\ +½n(3n-5)\ +\ n(n-4)\ +\ ½n(n-5)$$
$$=\binom{3n+1}{2}=\frac{(3n+1)(3n)}{2}=\frac{9n^2+3n}{2}.$$

And this complete the proof of theorem 1.

Theorem 2. Consider the line graph of the subdivision graph of the wheel graph $L(S(W_n))$ for all positive integer number $n\geq 3$. Then,

- The Hosoya polynomial of $L(S(W_n))$ is equal to

$$H(L(S(W_n)),x)=½n(n+9)x+n(n+5)x^2$$
$$+½n(5n+3)x^3+2n(n-2)x^4+\ n(n-9)x^5.$$

- The Wiener polynomial of $L(S(W_n))$ is equal to

$$W(L(S(W_n))\ )=17n^2-34n.$$

Proof of Theorem 2: Let $L(S(W_n))$ be the line graph of the subdivision graph of the wheel graph (Figure 1). The number of vertices in this graph is equal to the number of edge of the subdivision graph of the wheel graph $S(W_n)$ and $|V(L(S(W_n))\ )|=4n$, where n vertices of the line graph of the subdivision graph of the wheel graph $L(S(W_n))$ have degree $n=|V(K_{n-1})|+1$ and all other vertices have degree three. So, we divide the vertex set of $L(S(W_n))$ into two partitions V_3 and V_m and obviously

$$V_n=\{u\in V(L(S(W_n))\ )|\ d_u=n\}\rightarrow |V_n|=n$$

$$V_3=\{v\in V(L(S(W_n))\ )|d_v=3\}\rightarrow |V_3|=3n$$

Here, we named all members of the vertex partition V_n as the Centre vertices.

Thus, the size of edge set $E(L(S(W_n))\ )$ is equal to

$$|E(L(S(W_n))\ )|=½[n\times n+3\times 3n]\ =½n(n+9).$$

Or

$$|E(L(S(W_n))\ )|=5n+|E(Kn)|=5n+½n(n-1)\ =½n(n+9).$$

Of course by definition of the line graph of an arbitrary graph G, it is obvious

$$|E(L(G))|\ =\ d(G,2),$$
$$|E(L(G))|\ =\ d(L(G),1).$$

New, similar to above proof of Theorem 1, to achieve our aims it is enough, we compute all terms $d(L(S(W_n)),i)$ for all $i=1,..., d(L(S(W_n))\ )$ (which from Figure 1, one can see that the diameter of the line graph of the subdivision graph of the wheel graph $L(S(W_n))$ $d(L(S(W_n)))$ is equal to 5).

Obviously, by above equations, we know that

$$d(L(S(W_n)),1)\ =|E(L(S(W_n))\ )|=\ d(S(W_n)),2)\ =½n(n+9).$$

On the other hand from the structure of $L(S(W_n))$ (Figure 1), $d(L(S(W_n)),2)\ =n^2+5n$. Since, there are $½[2n+2n]+2n$ 2-edges paths between all vertices of the vertex partition V_3. There are $n+n+n(n-1)$ Centre vertices $c\in V_n\subset V(L(S(W_n))\ )$ and vertices from V_3.

From Figure 1, one can see that $d(L(S(W_n)),3)\ =½n(5n+3)$, because $2n(n-1)$ 3-edges paths start from the Centre vertices of V_n and end to the vertices of the vertex partition V_3. Also, there are $2n+½n(n-1)+2n=½n(n+7)$ 3-edges paths between all members of V_3.

For $d(L(S(W_n)),4)$, we see that there is not any 4-edges path started from the Centre vertices c and between all members of V_n and V_3. And only there are $2n+2n(n-3)=2n^2-4n$ 4-edges paths between all member of V_3. Thus, the number of the 4-edges paths of the line graph of the subdivision graph of the wheel graph $L(S(W_n))$ is equal to $2n(n-2)$.

Finally, from the structure of $L(S(W_n))$ (Figure 1), we see that there are $½(2n)(n-9)$ 5-edges paths between all vertices from V_3. Therefor the last terms of the Hosoya polynomial of $L(S(W_n))$ is equal to $d(L(S(W_n)),5)\ =n(n-9)$.

Now, these above computations imply that the Hosoya polynomial of the line graph of the subdivision graph of the wheel graph $L(S(W_n))$ will be

$$H(L(S(W_n)),x)=\Sigma_{d=1}^{5}\ d(L(S(W_n)),\ d)x^d$$
$$=½n(n+9)x+n(n+5)x^2+½n(5n+3)x^3+2n(n-2)x^4+n(n-9)x^5.$$

And obviously

$$W(L(S(W_n))\ )=\ \frac{\partial}{\partial x}\ H(L(S(W_n)),\times)$$
$$=½n(n+9)\times 1+n(n+5)\times 2+½n(5n+3)\times 3+2n(n-2)\times 4$$
$$+\ n(n-9)\times 5$$
$$=17n^2-34n.$$

Here the proof of theorem is completed.

3. Conclusions

In this paper, we discuss one of the oldest and thoroughly studied distance based topological index, the Wiener index, and the Hosoya polynomial. We found the Wiener index and Hosoya polynomial of the subdivision graph of the wheel Wn and the line graph subdivision graph of the wheel $L(S(W_n))$ for all positive integer number $n\geq3$.

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