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On the Solution of Volterra-Fredholm and Mixed Volterra-Fredholm Integral Equations Using the New Iterative Method

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Abstract In this paper, new iterative method proposed by Daftadar-Gejji and Jafari (2006) was used in solving both linear and nonlinear Volterra-Fredholm and Mixed Volterra-Fredholm integral equations. The method yields a series with faster convergence. Finally, some concrete examples are given to illustrate the validity of the method.

Keywords Volterra-Fredholm Integral equation, New iterative method, Mixed Volterra-Fredholm integral equation, Convergence Analysis

1. Introduction

Volterra-Fredholm Integral equations have received considerable interest in the Mathematical Physics, engineering, biology and contact problems in the theory of elasticity (see [1-5]). The solution of these integral equations can be obtained both analytically and numerically using one of the following method: Homotopy perturbation method [6], Rationalized Haar functions [7], Taylor polynomial method [8], Adomian decomposition method [9] and Minggen et al [10], used the representation of the exact solution for the nonlinear Volterra-Fredholm integral equations in the reproducing kernel space. The exact solution is given by the series form. In [1], Abdou used orthogonal polynomial to solve Fredholm-Volterra integral equations. Also, Yusufoglu and Erbas presented the method based on interpolation in solving linear Volterra-Fredholm integral equations [12]. Maleknejad and Sohrabi [13] solved nonlinear Volterra-Fredholm Hammerestein integral equations in terms of Legendre polynomials. Yousefi and Razzaghi in [14] applied legendre wavelets to special type of nonlinear Volterra-Fredholm integral equations of the form

$$x(s) = y(s) + \lambda_1 \int_0^s k_1(s,t)F\big(x(t)\big)dt + \lambda_2 \int_0^1 k_2(s,t)G\big(x(t)\big)dt\,,\ 0 \le s,t \le 1 \tag{1}$$

where $y(s)$, $k_1(s,t)$ and $k_2(s,t)$ are assumed to be in $L^2(R)$ on the interval $0 \le s,t \le 1$.

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2. Review on New Iterative Method

Recently, Daftardar-Gejji and Jafari [15] proposed a new technique for solving linear and nonlinear functional equations known as the New Iterative Method. This method has proven useful for solving a variety of nonlinear equations such as algebraic equations, integral equations, ordinary and partial differential equations of integer and fractional order and systems of equations as well. Bhalekar and Daftardar-Gejji [16] applied the new iterative method to Fractional-order logistic equation. The obtained results were compared with the Adomian decomposition and Homotopy perturbation method. They concluded that the new iterative method converges faster to the approximate solutions. Ambreen *et al.* [17] solved time-fractional Schrödinger equations using the New Iterative Method (NIM). The technique is fully compatible with the complexity of these problems and obtained results are highly encouraging. Numerical results coupled with graphical representations explicitly reveal the complete reliability of the suggested algorithm. Daftardar-Gejji and Bhalekar [19] employed the New Iterative Method to find solutions of linear and nonlinear fractional diffusion-wave equations. The results obtained were free from rounding off errors since it does not involve discretization.

Ramadan and Al-luhaibi [20] applied the new iterative method (NIM) to find approximate analytical solution of the Fornberg-Whitham equation. A comparison is made between the NIM results, homotopy perturbation transform method (HPTM) and Adomian decomposition method (ADM). It was discovered that the new iterative method solved nonlinear problems without using Adomian's polynomials and He's polynomials. Ibrahim and Ayoo [21] Srivastava and Rai [22]) proposed a new mathematical model, namely a multi-term fractional diffusion equation, for oxygen delivery through a capillary of tissues. They used the new iterative method (NIM) and modified Adomian decomposition method (MADM) to solve the multi-term fractional diffusion equation for different conditions. The results thus obtained are compared and presented graphically. It was observed that the order of the diffusion equation affects the delivery of oxygen significantly. The basic difference between the methods is that, the New Iterative Method (NIM) is direct and straightforward and it avoids the volume of calculations resulting from computing the Adomain polynomials.

Hemeda [23] obtained the solution of fractional difference equations using the new iterative method (NIM). The obtained results confirmed the power of the method in reducing the size of calculations compared with other traditional methods. Kocak and Yildrim [24] applied new iterative method in finding the exact solutions of nonlinear time-fractional partial differential equations. The fractional derivatives are described in the Caputo sense. Yaseen *et al.* [25] used the iterative method to find an analytic treatment for Laplace equation with Dirichlet and Neumann boundary conditions. The obtained results show that the present approach is highly accurate and requires reduced amount of calculations compared with the existing iterative methods.

It can be concluded that the New Iterative method (NIM) is a useful technique for solving both linear and nonlinear problems, most especially, in sciences and engineering.

In this paper, we consider the linear Volterra-Fredholm and mixed Volterra-Fredholm integral equation of the form

$$y(x) = f(x) + \int_0^1 k_1(x,t)y(t)dt + \lambda_2 \int_a^b k_2(x,t)y(t)dt \quad (2)$$

$$y(x) = f(x) + \int_0^x \int_a^b k(,t)\, y(t)\, dtd \qquad (3)$$

respectively, where $f(x)$ and $k_1(x,t)$, $k_2(x,t)$ and $k(,t)$ are analytic functions. $y(x)$ is the unknown function. We also consider nonlinear Fredholm-Volterra integral equation is the type

$$y(x) = f(x) + \int_a^x k_1(x,t,y(t))dt + \int_a^b k_2(x,t,y(t))dt \quad for -\infty < a \le x \le b < \infty \qquad (4)$$

Where $y,\ k_1,\ k_2, f$ are ^{n}, the n-dimensional Euclidean space with appropriate norm denoted by $|.|$.

3. Basic Idea of NIM

Consider the Nonlinear function equation

$$y = f + L(y) + N(y) \qquad (5)$$

where f is a given function, L and N are linear and nonlinear operators respectively. It is assumed that the NIM solution for eqn (5) has the form

$$y = \sum_{i=0}^{\infty} y_i \qquad (6)$$

The convergence of series (6) is proved in [26] and described in section 3.

Since L is linear,

$$L\left(\sum_{i=0}^{\infty} y_i\right) = \sum_{i=0}^{\infty} L(y_i) \qquad (7)$$

The nonlinear operator N in eqn (5) is decomposed by Dafterder-Gejj and Jafari [15] as below:

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_i\right) - N\left(\sum_{j=0}^{i-1} y_i\right)\right\} \quad (8)$$
$$= \sum_{i=0}^{\infty} G_i$$

where $G_0 = N(y_0)$ and

$$G_i = \left\{N\left(\sum_{j=0}^{i} y_i\right) - N\left(\sum_{j=0}^{i-1} y_i\right)\right\}, \quad i \ge 1$$

Using eqns (6), (7), (8) in eqn (5), we get

$$\sum_{i=0}^{\infty} y_i = f + \sum_{i=0}^{\infty} L(y_i) + \sum_{i=0}^{\infty} G_i \qquad (9)$$

If N is a contraction, i.e.

$$\|N(x) - N(y)\| \le k\|x - y\|, 0 < k < 1,$$

then

$$\|y_{m+1}\| = \|N(y_0 + y_1 + \cdots + y_m) - N(y_0 + y_1 + \cdots + y_{m-1})\|$$
$$\le k\|y_m\| \le \cdots \le k^n\|y_0\|, \qquad m = 0,1,2, \ldots,$$

and the series $\sum_{i=0}^{\infty} y_i$ and $\sum_{i=0}^{\infty} G_i$ are absolutely and uniformly converges to solution of Eq. (5) [29]. Which is unique, in view of Banach fixed point theorem [30]. The k-term approximate solution of Eq. (2-4) is given by $\sum_{i=0}^{k-1} y_i$.

4. Condition of NIM

The following convergence result for NIM is described by Daftardar-Gejji and Bhalekar.

Theorem 1. [26] if N is $C^{(\infty)}$ in a neighborhood of y_0 and $\left||N^n(y_0)\right|| \le L$ for any n and for some real $L > 0$ and $\left||y_i\right|| \le M < \frac{1}{e}$, $\quad i = 1, 2, 3, \ldots$, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent to N and moreover

$$\left||G_n\right|| \le LM^n e^{n-1}(e-1), \quad n = 1, 2, \ldots$$

Theorem 2. [26] if N is $C^{(\infty)}$ and $\left||N^n(y_0)\right|| \le M \le e^{-1}$, $\forall\, n$, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent to N.

5. Numerical Examples

In this section, we present the numerical result of some problems solved by the proposed method of this article.

Example 1: Consider the linear Volterra-Fredholm integral equation [27]

$$y(x) = x - \frac{1}{3}x^3 + \int_0^x t y(t)dt + \int_{-1}^1 t^2 y(t)dt \quad (10)$$

Using NIM, we get an iterative scheme

$u_0 = x - \frac{1}{3}x^3, \;\; u_1 = \frac{1}{15}x^5 + \frac{1}{3}x^3,$

$u_2 = \frac{1}{105}x^7 + \frac{1}{15}x^5, \qquad u_3 = -\frac{1}{945}x^9 + \frac{1}{105}x^7,$

$u_4 = 0, \ldots$

Thus, the solution (10) is $y(x) = x - \frac{1}{945}x^9$

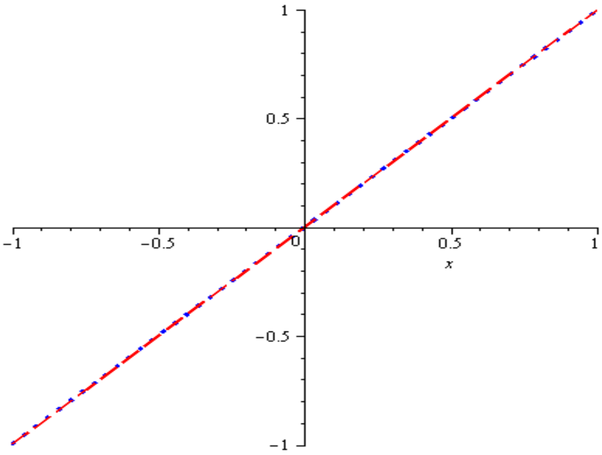

Figure 1. Exact and Approximate Solution of Example 1, where dash and dotted lines represents the approximate and exact solution respectively

Example 2: Consider the linear mixed Volterra-Fredholm integral equations,

$$y(x) = xe^x - \frac{1}{2}x^2 + \int_0^x \int_0^1 y(t)dtd \qquad (11)$$

This NIM leads to

$y_0 = xe^x - \frac{1}{2}x^2, \quad y_1 = \frac{5}{12}x^2,$

$y_2 = \frac{5}{72}x^2, \quad y_3 = \frac{5}{432}x^2,$

$y_4 = \frac{5}{2592}x^2 \quad y_5 = \frac{5}{15552}x^2, \quad y_6 = \frac{5}{93312}x^2, \ldots$

Thus the solution of (11) is

$$y(x) = xe^x - \left(\frac{1}{3359232}\right)x^2$$

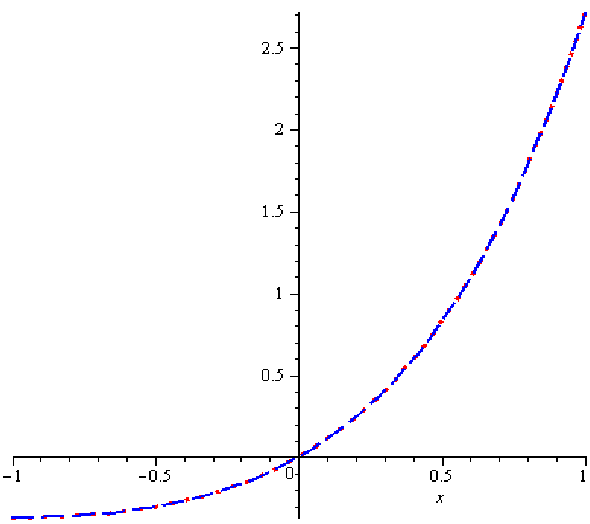

Figure 2. Exact and Approximate Solution of Example 2, where dash and dotted lines represents the approximate and exact solution respectively

Example 3: Consider the nonlinear Volterra-Fredholm integral equation [28]

$$y(x) = 2x - \frac{1}{12}x^4 - \frac{5}{3} + \frac{1}{4}\int_0^x (x-t)[y(t)]^2 dt$$

$$+ \int_0^1 (1+t)y(t)dt, \quad x \in [0,1] \qquad (12)$$

with exact solution $y(x) = 2x$

Applying NIM to (12), we get

$y_0 = 2x - \frac{1}{12}x^4 - \frac{5}{3}$

$y_1 = -\frac{311}{360} + \frac{25}{72}x^2 - \frac{5}{18}x^3 + \frac{1}{12}x^4 + \ldots$

$y_2 = -\frac{56885107}{47900160} + \frac{469921}{1036800}x^2 - \frac{311}{2160}x^3 - \frac{4555}{124416}x^4 + \ldots$

$y_3 = -\frac{104171156499283}{11919092661312000}x^4 - \frac{276648261943}{23883818522624000}x^{10} + \ldots$

Thus the solution of (12) is

$$y(x) = 2x - \frac{147808202259238}{1191909261312000}x^4 + \ldots$$

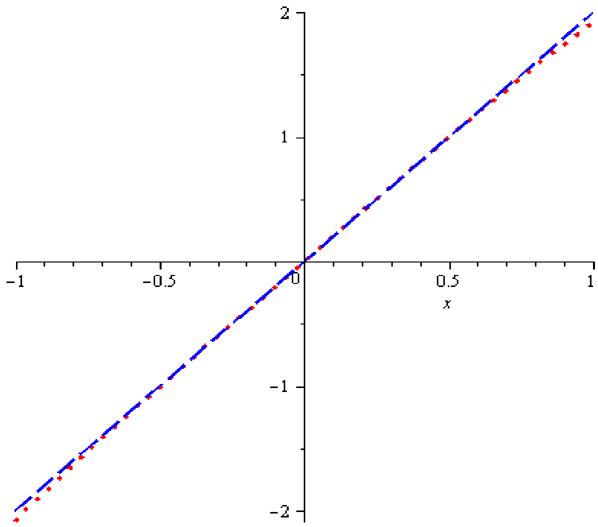

Figure 3. Exact and Approximate Solution of Example 3, where dash and dotted lines represents the approximate and exact solution respectively

Table 1. Results for Example 1

x_i	y-exact	y-approx NIM	y-error NIM	y-error ADM
-1	-1.0000	-0.9989	1.1E-2	1.234E-3
-0.8	-0.8000	-0.7999	1.0E-3	1.112E-4
-0.6	-0.6000	-0.6000	0	0.12E-4
-0.4	-0.4000	-0.4000	0	0
-0.2	-0.2000	-0.2000	0	0
0	0	0	0	0
0.2	0.2000	0.2000	0	0
0.4	0.4000	0.4000	0	0
0.6	0.6000	0.6000	0	0.23E-4
0.8	0.8000	0.7999	1.0E-3	1.223E-4
1.0	1.0000	0.9989	1.1E-2	1.236E-3

Table 1 shows the computed error of the proposed method and compares the results with the Adomian decomposition method. Table 1 shows that our results are considerable accurate as the error of the NIM is lesser than that of ADM.

Table 2. Results for Example 2

x_i	y-exact	y-approx NIM	y-error NIM	y-error ADM
-1	-0.3679	-0.3679	0	1.0E-3
-6	-0.3293	-0.3293	0	1.23E-2
-2	-0.1637	-0.1637	0	0.21E-3
0.2	0.2443	0.2443	0	0
0.6	1.0933	1.0933	0	0.21E-3
1.0	2.7183	2.7183	0	1.0E-3

In Table 2, computed error shows that the value of the exact and approximated solutions of the proposed method are the same, comparing this error with that of ADM, we see that, the proposed method converge accurately to the exact solution of Example 2.

Table 3. Results for Example 3

x_i	y-exact	y-approx NIM	y-error NIM
-1	-2.000	-2.7702	7.702E-2
-6	-1.2000	-1.2998	9.98E-2
-2	-0.4000	-0.4012	1.2E-2
2	0.4000	0.3988	1.2E-2
6	1.2000	1.1002	9.98E-2
1	2.000	1.2298	7.702E-2

6. Conclusions

We have successfully utilized new iterative method (NIM) to obtain semi-analytical solutions of both Volterra-Fredholm and mixed Volterra-Fredholm integral equations. The solution obtained and exact solutions are plotted using Maple 13 software. It is observed that the present method reduces the computational difficulties of other traditional methods and all the calculation can be made in simple manipulations. The accuracy of the obtained solutions can be improved by taking more iteration in the solutions. In many cases, the series solution obtained with NIM can be written in exact closed form. The solutions obtained are highly in agreement with the exact solutions.

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