Approximate Analytical Expression of Concentrations in a Kinetic Model for Biogas Generation from Banana Waste

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Abstract The initial value problem in a kinetic model for biogas generation from banana waste is discussed. The model involves the system of non-linear differential equations which has variety of non-linear rate function. Simple and approximate polynomial expressions of concentration are derived for non-linear kinetics models in biogas generation. Comparison of the approximate analytical approximation and numerical stimulations is also presented. A satisfactory agreement between theoretical predictions and numerical result is observed for times t ≤ 25 days. The concentrations are also computed for various values of parameters.

Keywords Mathematical modelling, Non-linear differential equations, Michaelis-Menten kinetics, Anaerobic digestion, Acidogenic and methanogenic steps

1. Introduction

The anaerobic respiration process is one of the attractive alternate for the economical reduction of the organic matter concentration in vegetables solids wastes [1-3]. Anaerobic digestion consists of a multitude of biochemical reaction in the series and in the parallel that occurs simultaneously [2, 4-6]. Mosey [7] and Kalyuzhni and Davlyatshina [8] developed the mathematical models which describes the kinetics of acidogenesis, ethanol degrading acetogenesis, acetogenesis, butyrate degrading acetogenesis, acetoclastic methanogenesis, hydrogenothrophic methanogenesis, bacterial decay, pH and various inhibitions of the mention steps. Mosche and Jordening [9] discussed the acetate and propionate degradation and inhibition. Masse et al. [10] studied the pre treatment of slaughterhouse wastewater at 25℃ for 5.5 h with 250mg/l of pancreatic lipase. Hu et al. [11] investigated the anaerobic digestion kinetics of ice cream water waste using monod and contois models. Aguilar et al. [12] studied the kinetic parameters of the total volatile acid degradation of source. Fang and Yu [13, 14] discussed the influence of influent concentration and temperature on the hydrolis of gelatinaceous wastewater.

Knobel and Lewis [15] developed a mathematical model in which in which the activity coefficients were calculated using the Debye-Hiicle theory and describing the anaerobic digestion of water waste with high concentration of sulphate.

Gracia et al. [16] used first and second order kinetic models to describe the anaerobic digestion of livestock manure. The result obtain showed that the second model has both statically and physical meanings in the parameter values obtained. Munch et al. [17] developed the mathematical model for volatile acid production. Valentini et al. [18] compared the Michaelis -Menten substrate first order, substrate and biomass first order and substrate first order and biomass half order equations in the anaerobic degradation of cellulose particles of known size. Zainol et al. [19] carried out the kinetic evaluation of the different steps of anaerobic digestion (hydrolysis, acidogenic and methanogenic) process of banana stem waste using two stage system. The system consist of an anaerobic sequencing batch reactor for the first stage and an anaerobic fixed bed reactor for the second stage which is operating at Hydraulic retention times of nine days. In this paper we derive an approximate analytical expression for concentrations of the non-soluble organic matter, soluble organic matter, TVA consumption and, methane formation in terms of the kinetic parameters k_1-k_7 using new Homotopy perturbation method.

2. Mathematical Formulation of the Problem

Taking into account the characteristics of banana stem waste (BSW) and the experimental results were obtained, the following hypothesis was assumed [19]: (i). The insoluble organic matter or volatile suspended solids were first transformed to soluble organic matter following first-order kinetics. (ii). the dissolved organic matter resulting from the decomposition of the insoluble organic matter and initially

present in the raw banana stem waste was transferred to volatile acids following a Michaelis-Menten kinetics model. (iii). the volatile acids resulting from the decomposition and initially present in the raw banana stem waste was transferred to methane and carbon dioxide following a Michaelis-Menten kinetics model. (iv). the disaster was assumed to be a completely mixed reactor. (v). with these assumptions, the rate equations of organic matter of the kinetic model including methane production can be expressed in the following differential equations [19]:

$$\frac{ds(t)}{dt} = -k_1 s(t) \tag{1}$$

$$\frac{du(t)}{dt} = k_1 s(t) - \frac{k_2 u(t)}{k_3 + u(t)} \tag{2}$$

$$\frac{dv(t)}{dt} = \frac{k_4 u(t)}{k_5 + u(t)} - \frac{k_6 v(t)}{k_7 + v(t)} \tag{3}$$

$$\frac{dw(t)}{dt} = \frac{k_6 v(t)}{k_7 + v(t)} \tag{4}$$

Where s is the non-soluble organic matter (g NSCOD/I day), u is soluble organic matter (g SCOD/I day), v is the TVA production (g/I day), w is the methane formation (g methane/I day). k_1 is the kinetic constant of the reaction, k_2 is the maximum removal rate of soluble organic matter, k_3 is the saturation constant, k_4 is the maximum rate of VDS uptake , k_5 is the saturation constant, k_6 is the maximum rate of TVA consumption, and k_7 is the saturation constant. These equations are solved for the

following initial conditions.

$$s = s_0,\ u = u_0,\ v = v_0,\ w = w_0,\ at \quad t = 0 \tag{5}$$

3. Approximate Analytical Expression of the Concentration of Non-Soluble Organic Matter, Soluble Organic Matter, Tva Consumption and, Methane Formation Using New Homotopy Perturbation Method (Hpm)

Non-linear phenomenon plays an important role in applied mathematical and physical science. Explicit solutions of the non-linear equations are the fundamental importance. Various methods for obtaining explicit solutions of non-linear equations have been proposed. Recently many authors have used the HPM for solving various problems and demonstrated the efficiency of the HPM for solving the non-linear problems in various physics and engineering problems [19-21]. This method is the combination of topology and classical perturbation techniques. He used the HPM to solve the Lighthill equation [22], the Diffusing equation [23] and the Blasius equation [24]. The idea has been used to solve non-linear boundary value problems, integral equations and many other problems. The Homotopy perturbation method [25-27] is very effective and simple. The HPM is unique in its applicability, accuracy, efficiency uses the imbedding parameter p as a small parameter and only a few iterations are needed to search for asymptotic solutions. Using this method, we can obtained the following approximate solution for the equations (1)-(4) (ref Appendix-A).

$$s(t) = s_0 e^{-k_1 t} \tag{6}$$

$$u(t) = Be^{-k_1 t} + (u_0 - B)e^{-At} \tag{7}$$

$$v(t) = \frac{k_4}{k_5 + u_0}\left[\frac{(D-A)B(e^{-k_1 t} - e^{-Dt}) + (D-k)(u_0 - B)(e^{-At} - e^{-Dt})}{(D-A)(D-k)}\right] + ve^{-Dt} \tag{8}$$

$$w(t) = k_4 k_6 t\left[\frac{(D-A)B(e^{-k_1 t} - e^{-Dt}) + (D-k_1)(u_o - B)(e^{-At} - e^{-Dt})}{(D-A)(D-k_1)(k_5 + w_0)(k_7 + v_o)}\right] + \frac{t\ k_6 v_0 e^{-Dt}}{(k_7 + v_o)} + w_0 \tag{9}$$

where the constant

$$A = \frac{k_2}{k_3 + u_0}, \quad B = \frac{k_1 s_0}{A - k_1}, \quad D = \frac{k_6}{k_7 + v_0} \tag{10}$$

Equations (6)-(9) satisfy the initial conditions (5). Equations (6)-(8) represent the new approximate analytical expression of concentration of hydrolysis reactor effluent $s(t)$, methanogenesis reactor effluent $u(t)$, acetic acid effluent from methanogenesis reactor $v(t)$ and methane formation $w(t)$ in terms of kinetic parameters.

4. Numerical Simulation

The non-linear differential equations (1)-(4), are also solved using numerical methods. The function pdex4 in Scilab software which is the function of solving the initial value problems for ordinary differential is used to solve this equation. Its numerical solution is compared with our approximate analytical result in Figure-1 and it gives a satisfactory agreement for all experimental values of the parameters ($k_1 = 0.0037$, $k_2 = 0.0241$, $k_3 = 0.0236$, $k_4 = 0.0086$, $k_5 = 0.0189$, $k_6 = 0.0092$, $k_7 = 0.0003$) and time $t \leq 25$ days. The Scilab program is also given in Appendix-B.
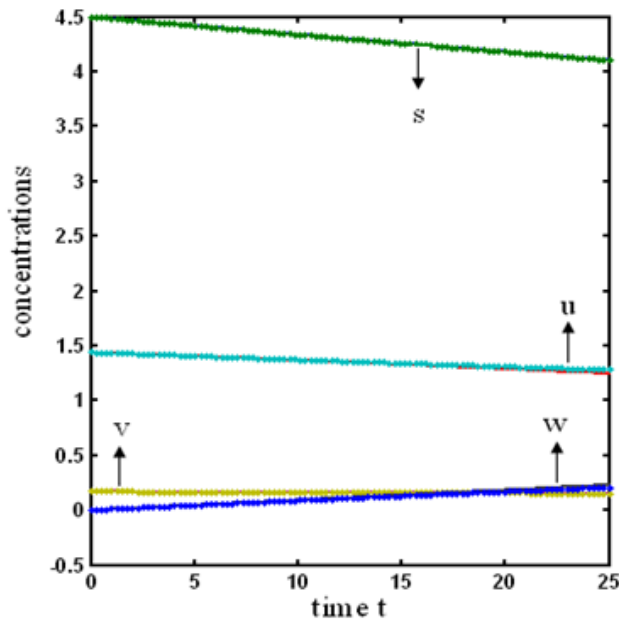
5. Discussion

Figure 1. Plot of concentrations of hydrolysis reactor effluent $s(t)$, methanogenesis reactor effluent $u(t)$, acetic acid effluent from methanogenesis reactor $v(t)$, methane formation $w(t)$ verses time (day) using the Eqs. (5)-(8)
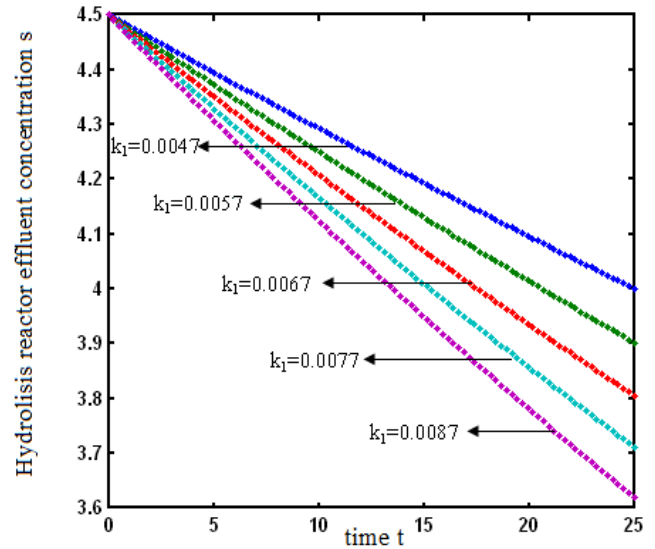

Figure 2. Plot of concentration of hydrolysis reactor effluent s verses time t (day) for various values of parameters k_1 using the Eq. 5. (a) $k_1 = 0.0047$; (b) $k_1 = 0.0057$; (c) $k_1 = 0.0067$; (d) $k_1 = 0.0077$; (a) $k_1 = 0.0087$
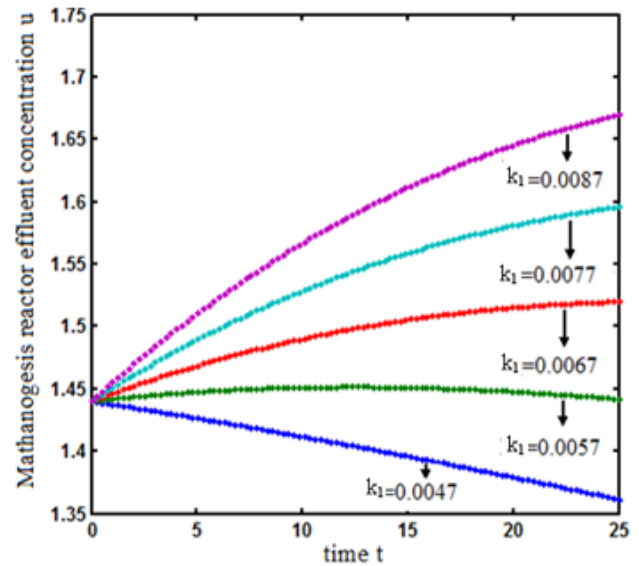

Figure 3. Plot of concentration of Mathanogesis reactor effluent u verses time t (day) for various values of parameters k_1 using the Eq.6. (a) $k_1 = 0.0047$; (b) $k_1 = 0.0057$; (c) $k_1 = 0.0067$; (d) $k_1 = 0.0077$; (a) $k_1 = 0.0087$

Equations (6)-(9) are simple and closed-form of approximate analytical expression of the concentrations of hydrolysis reactor effluent $s(t)$, methanogenesis reactor effluent $u(t)$, acetic acid effluent from methanogenesis reactor $v(t)$, methane formation $w(t)$ for various kinetic parameters $k_1 - k_7$. In Figs.1-11, the concentrations of hydrolysis reactor effluent $s(t)$, methanogenesis reactor

effluent $u(t)$, acetic acid effluent from methanogenesis reactor $v(t)$, methane formation $w(t)$ verses time t have been plotted for some fixed values of kinetic parameters. In Fig-1, our approximate analytical results (s, u, v, w) are compared with the numerical results [19]. Our approximate analytical result fit very well with the experimental result. Fig. 2 shows the concentration hydrolysis reactor effluent$s$ versus time t using Eq.6 for various values of kinetic constant k_1. From this figure, we can see that the value of concentration decreases when time t increases with increasing values of k_1.
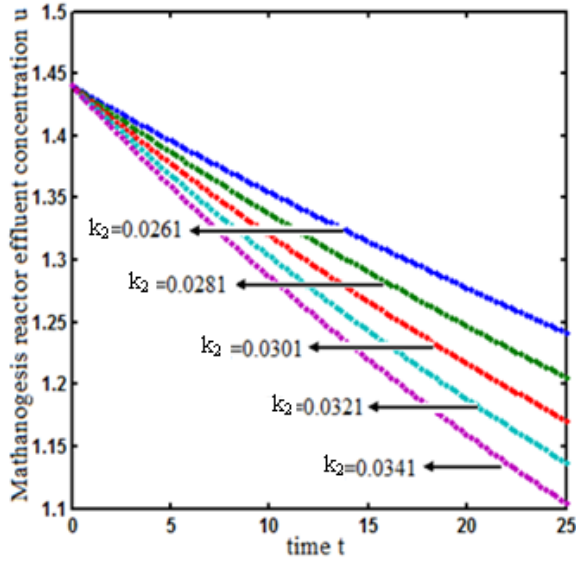
Figs. 3- 5 represent concentration methanogenesis reactor effluent u at the time t using Eq.7. From the Fig 3, it is observed that the concentration u increases when k_1 increases. In Fig. 4 and 5 the concentrations are plotted for various values of parameters k_2 and k_3 respectively. From these figuresit is inferred that value of concentration decreases when k_2 and k_3 increases.

In Figs. 6-9 we present the concentration of methanogenesis reactor v at time t using Eq.8. From Figs. 6 and 9 it is deducted that values of concentration increases when parameters k_4 and k_6 increases. From Fig.7 and 8 it is inferred that value of concentration decreases when parameters k_5 and k_7 increases.
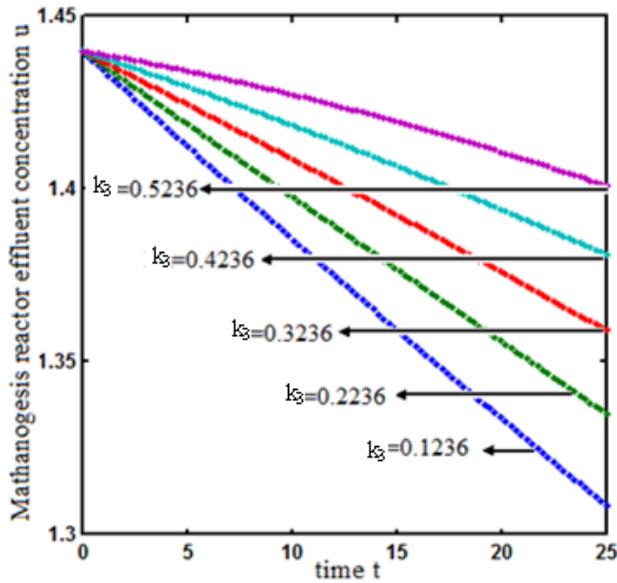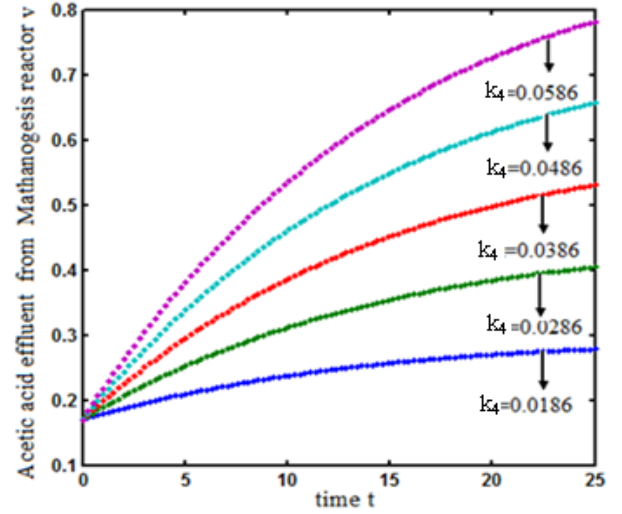

Figure 4. Plot of concentration of Mathanogesis reactor effluent u verses time t (day) for various values of parameters k_2 using the Eq. 6. (a) k_2 = 0.0261; (b) k_2=0.0281; (c) k_2=0.0301; (d) k_2=0.0321; (e) k_2=0.0341

Figure 6. Plot of concentration of Acetic acid effluent from Mathanogesis reactor v verses time t (day) for various values of parameters k_4 using the Eq. 7. (a) k_4 = 0.0186; (b) k_4 = 0.0286; (c) k_4 = 0. 0386; (d) k_4 = 0.0486; (e) k_4 = 0.0586
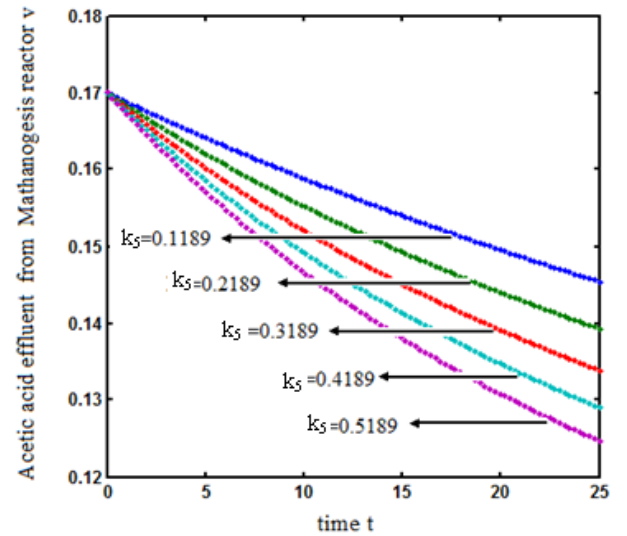

Figure 5. Plot of concentration of Mathanogesis reactor effluent u verses time t (day) for various values of parameters k_3 using the Eq. 6. (a) k_3=0.1236; (b) k_3 = 0.2236; (c) k_3 = 0.3236; (d) k_3= 0. 4236; (e) k_3 = 0.5236

Figure 7. Plot of concentration of Acetic acid effluent from Mathanogesis reactor v verses time t (day) for various values of parameters k_5 using the Eq. 7. (a) k_5=0.1189; (b) k_5 =0.2186; (c) k_5 =0.3189; (d) k_5 =0.4189; (e) k_5 = 0.5189
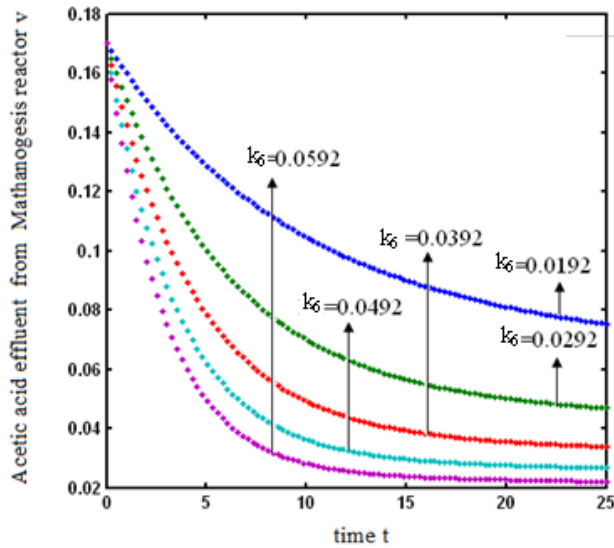
Figure 8. Plot of concentration of Acetic acid effluent from Mathanogesis reactor v verses time t (day) for various values of parameters k_6 using the Eq. 7. (a) $k_6 = 0.0192$; (b) $k_6 = 0.0292$; (c) $k_6 = 0.0392$; (d) $k_6 = 0.0492$; (e) $k_6 = 0.0592$
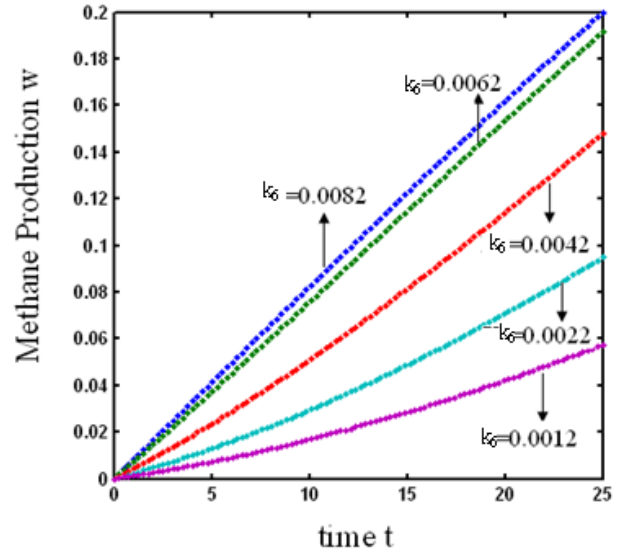

Figure 10. Plot of concentration of Methane production w verses time t (day) for various values of parameters k_6 using the Eq. 8. (a) $k_6 = 0.0082$; (b) $k_6 = 0.0062$; (c) $k_6 = 0.0042$; (d) $k_6 = 0.0022$; (e) $k_6 = 0.0012$
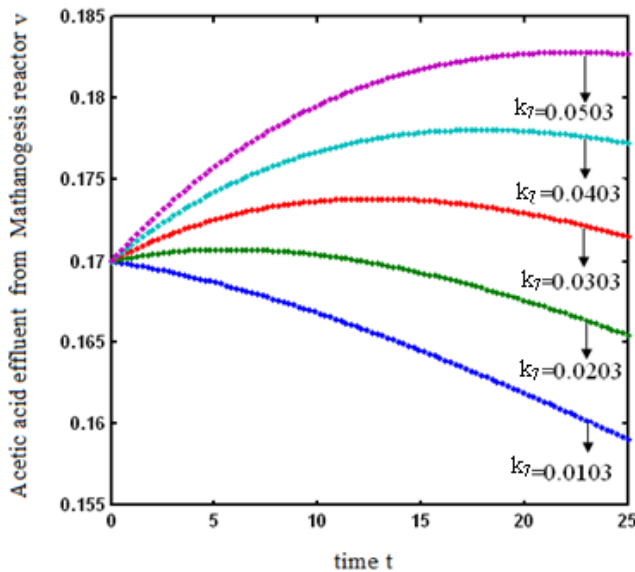

Figure 9. Plot of concentration of Acetic acid effluent from Mathanogesis reactor v verses time t (day) for various values of parameters k_7 using the Eq. 7. (a) $k_7 = 0.0103$; (b) $k_7 = 0.0203$; (c) $k_7 = 0.0303$; (d) $k_7 = 0.0403$; (e) $k_7 = 0.0503$

Figure 11. Plot of concentration of Methane production w verses time t (day) for various values of parameters k_7 using the Eq. 5. (a) $k_7 = 0.0103$; (b) $k_7 = 0.0503$; (c) $k_7 = 0.1003$; (d) $k_7 = 0.1503$; (e) $k_7 = 0.2003$
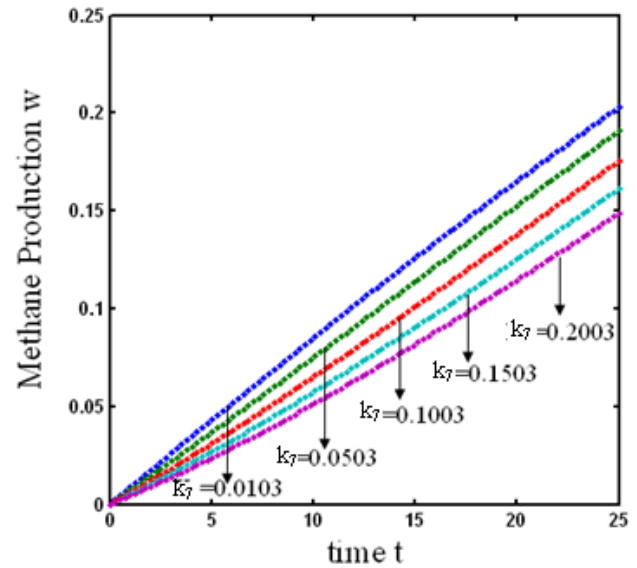
Figs. 10 and 11 represents concentration w for various values of k_6 and k_7 respectively. From these figures, it is observed that concentration methane formation w increases when saturation constant k_6 and k_7 increases.

6. Conclusions

A nonlinear time independent equation has been solved analytically using new Homotopy perturbation method. In this paper we have presented approximate analytical

expression of the concentration of hydrolysis reactor effluent, methanogenesis reactor effluent, acetic acid effluent from methanogenesis reactor, methane formation for time $t \leq 25$ days. The influence of the saturation constant on concentrations is also discussed.

ACKNOWLEDGEMENTS

Nomenclature

SYMPOL	MEANING	UNIT	NUMERICL VALUES [19]
s	Concentration of hydrolysis reactor effluent	g NSCOD /I day	1.45
u	Concentration of methanogenesis reactor effluent	(g SCOD /I day)	1.44
v	Concentration TVA production,	g/I	0.17
w	Methane formation	(g methane /I day)	0
k_1	Maximum removal rate of soluble organic matter	(g SCOD /I day)	0.0037
k_2	Maximum removal rate of soluble organic matter	(g SCOD /I day)	0.0241
k_3	Saturation constant	(g SCOD /I day)	0.0236
k_4	Maximum rate of VDS uptake	(g SCOD /I day)	0.0086
k_5	Saturation constant	(g SCOD /I day)	0.0189
k_6	Maximum rate of TVA consumption	(g/I)	0.0092
k_7	Saturation constant	(g TVA /I)	0.0003

Appendix A: Approximate Analytical Solutions of the Equation (1)-(4) Using New Homotopy perturbation Method

In this appendix, we indicate how equations (6)-(9) in this paper are derived. Furthermore a new Homotopy was constructed to determine the solutions of (1)-(4) and the initial conditions are as follows:

Solving Eq.1, we get

$$s(t) = s_0 e^{(-k_1 t)} \tag{A1}$$

Substitute the values of Eq. A2 in Eq. 2 we get

$$\frac{du}{dt} + \frac{k_2 u}{k_3 + u} = k_1 s_0 e^{(-k_1 t)} \tag{A2}$$

The Homotopy ofEqu.2 can be constructed as follows:

$$(1-p)[\frac{du}{dt} + \frac{k_2 u}{k_3 + u} - k_1 s_0 e^{(-k_1 t)}]$$
$$+p[\frac{du}{dt} + \frac{k_2 u}{k_3 + u} - k_1 s_0 e^{(-k_1 t)}] = 0 \tag{A3}$$

The zeroth iteration of the above equation can be written as

$$\frac{du}{dt} + \frac{k_2 u}{k_3 + u_0} = k_1 s_0 e^{(-k_1 t)} \tag{A4}$$

Solving the above equation using the initial condition, we get

$$u(t) = Be^{(-k_1 t)} + (u_0 - B)e^{(-At)} \tag{A5}$$

where the constant

$$A = \frac{k_2}{k_3 + u_0} \quad \text{and} \quad B = \frac{k_1 s_0}{A - k_1} \tag{A6}$$

Substitute the values of Eq.A5 in Eq.3 and Using new HPM approach we get the zeroth iteration equation as

$$\frac{dv}{dt} + Dv = \frac{k_4 B e^{(-k_1 t)} + k_4 (u_0 - B)e^{(-At)}}{k_5 + u_0} \tag{A7}$$

Where the constant

$$D = \frac{k_6}{k_7 + v_0} \tag{A8}$$

Solving the above equation using the initial condition we get

$$v(t) = \frac{k_4}{k_5 + u_0} \left[\frac{(D-A)B(e^{-k_1 t} - e^{-Dt}) + (D-k_1)(u_0 - B)(e^{-At} - e^{-Dt})}{(D-A)(D-k_1)} \right] + v e^{-Dt} \tag{A9}$$

Substitute the values of Eq.A9 in Eq.3 and using new HPM approach, we get the zeroth iteration equation as

$$\frac{dw}{dt} = \frac{k_6 \dfrac{k_4}{k_5 + u_0} \left[\dfrac{(D-A)B(e^{-k_1 t} - e^{-Dt}) + (D-k_1)(u_0 - B)(e^{-At} - e^{-Dt})}{(D-A)(D-k_1)} \right] + ve^{-Dt}}{k_7 + v_0} \qquad (A10)$$

Solving the above equation using the initial condition, we get

$$w(t) = k_4 k_6 t \left[\frac{(D-A)B(e^{-k_1 t} - e^{-Dt}) + (D-k_1)(u_o - B)(e^{-At} - e^{-Dt})}{(D-A)(D-k_1)(k_5 + w_0)(k_7 + v_o)} \right] + \frac{k_6 v_0 e^{(-Dt)} t}{(k_7 + v_o)} + w_0 \qquad (A11)$$

Appendix B: Matlab Program to Find the Numerical Solution of Eqs.(1)-(4)

```
function main1
options= odeset('RelTol',1e-6,'Stats','on');
Xo = [4.5;1.44;0.17;0];
tspan = [0,50];
tic
[t,X] = ode45(@TestFunction,tspan,Xo,options);
toc
figure
hold on
%plot(t, X(:,1))
%plot(t, X(:,2))
%plot(t, X(:,3))
plot(t, X(:,4))
return
function [dx_dt]= TestFunction(t,x)
k1=.0037;k2=0.0241;k3=0.0236;k4=0.0086;k5=0.0189;
k6=0.0092;k7=0.0003;
dx_dt(1)=-k1*x(1);
dx_dt(2)=k1*x(1)-((k2*x(2))/(k3+x(2)));
dx_dt(3)=((k4*x(2))/(k5+x(2)))-((k6*x(3))/(k7+x(3)));
dx_dt(4)=((k6*x(3))/(k7+x(3)));
dx_dt = dx_dt';
return
```

REFERENCES

[1] Borja R, Rinco´n B, Raposo F, Alba J, Martı´n A. A study of anaerobic digestibility of two-phases olive mill solid waste (OMSW) at mesophilic temperature. Process Biochem 2002; 38:733–42.

[2] Borja R, Rinco´n B, Raposo F, Alba J, Martı´n A. Kinetics of mesophilicanaerobic digestion of the two-phase olive mill solid waste. Biochem Eng J 2003;13:139–45.

[3] Beccari M, Bonemazzi F, Majone M, Riccardi C. Interaction between acidogenesis and methanogenesis in the anaerobic treatment of oliveoil mill effluents. Water Res1996;30(1): 183-9.

[4] Gujer W, Zehnder AJB. Conversion process in anaerobic digestion. Water SciTechnol1983; 15:127–67.

[5] McCarty PL, Mosey FE. Modeling of anaerobic digestion processes (A discussion of concepts). Water SciTechnol 1991;24 (8):17–33.

[6] Pavlostathis SG, Giraldo E. Kinetics of anaerobic treatment. Water Sci Technol 1991;24 (8): 35–9.

[7] Mosey FE. Mathematical modelling of the anaerobic digestion processregulatory mechanisms for the formation of short-chain volatileacids from glucose. Water Sci Technol 1983;15:209–32.

[8] Kalyuzhnyi SV, Davlyatshina MA. Batch anaerobic digestion of glucose and its mathematical modeling. I. Kinetic investigations. Bioresour Technol 1997;59 (1):73–80.

[9] Mosche M, Jo¨rdening HJ. Comparison of different models of substrate and product inhibition in anaerobic digestion. Water Res 1999;33(11):2545–54.

[10] Masse L, Masse DI, Kennedy KJ. Effect of hydrolysis pretreatment on fat degradation during anaerobic digestion of slaughterhouse wastewater. Process Biochem 2003;38 (9):1365–72.

[11] Hu WC, Thayanithy K, Forster CF. A kinetic study of the anaerobicdigestion of ice-scream wastewater. Process Biochem 2002;37(9):965–71.

[12] Aguilar A, Casas C, Lema JM. Degradation of volatile fatty acids by differently enriched methanogenic cultures: Kinetics and inhibition. Water Res 1995;29 (2):505–9.

[13] Fang HHP, Yu H. Mesophilic acidification of gelatinaceous wastewater. J Biotechnol 2002;93 (2):99–108.

[14] Yu HQ, Fang HHP. Acidogenic of gelatine-rich wastewater in an upflow anaerobic reactor: influence of pH and temperature. Water Res2003;37 (1):55–66.

[15] Knobel AN, Lewis AE. A mathematical model of a high sulfatewastewater anaerobic treatment system. Water Res 2002;36(1):265–75.

[16] Garcı´a F, Santos VE, Naval L, Guardiola E, Lo´pez B. Kinetic modelfor anaerobic digestion of livestock manure. Enzyme MicrobialTechnol 1999;25 (1-2):55–60.

[17] Mu¨nch VE, Keller J, Lant P, Newell R. Mathematical model of prefermenters-I. Model development and verification. Water Res1999;33 (12):2757–68.

[18] Valentini A, Garuti G, Rozzi A, Tilche A. Anaerobic degradationkinetics of particulate organic matter: a new approach. Water SciTechnol 1997;36 (6-7):239–46.

[19] Zainol. N, Salihon J. and Abdul-Rahman R. (2012) A. Öchsner et al. (eds.), Analysis and Design of Biological Materials and Structures, Advanced Structured Materials 14,

DOI: 10.1007/978-3-642-22131-6_14.

[20] Li. S.J and Liu. y.x (2006) An improved Approach to nonlinear dynamical system identification using PID neural networks", International Journal of Nonlinear sciences and Numerical simulation, 7, 177-182.

[21] Mousa. M.M., Ragab.S.F. and Nturforsch. Z. (2008) Applications of the Homotopy perturbation method to linear and nonlinear Schrodinger equation, Zeitsehriftfiir naturforschung, 63,142-144.

[22] He. J.H., Homotopy perturbation technique, Computer methods in Applied Mechanics and Engineering, 178,(1999), 257-262.

[23] He. J.H., (2003) HPM: a new nonlinear analytic Technique.

Applied Mathematics and Computations, 135.73-79.

[24] He. J.H., (2003) A simple perturbation approach to Blasius Equation", Applied Mathematics and Computations, 140, 2-3, 217-222.

[25] He. J.H., (2006) Some asymptotic methods for strongly nonlinear equations. International journal of Modern Physics B,20, 1141-1199.

[26] He. J.H., (2003) A coupling method of Homotopy technique and a perturbation technique for nonlinear problems. International Journal of Nonlinear Mechanics, 35,37-43.

[27] Ganji, D.D., Amini,. M. and Kolahdooz, A. (2008) Analytical investigation of hyperbolic equations via He,s methods. American Journal of Engineering and Applied science.