

Statistical Properties of the Exponentiated Generalized Inverted Exponential Distribution

Oguntunde P. E.^{1,*}, Adejumo A. O.², Balogun O. S.³

¹Department of Mathematics, Covenant University, Ota, Ogun State, Nigeria

²Department of Statistics, University of Ilorin, Kwara State, Nigeria

³Department of Statistics and Operations Research, Modibbo Adama University, Yola, Adamawa State, Nigeria

Abstract We provide another generalization of the inverted exponential distribution which serves as a competitive model and an alternative to both the generalized inverse exponential distribution and the inverse exponential distribution. The model is positively skewed and its shape could be decreasing or unimodal (depending on its parameter values). The statistical properties of the proposed model are provided and the method of Maximum Likelihood Estimation (MLE) was proposed in estimating its parameters.

Keywords Distributions, Estimation, Exponentiated, Generalization, Inverse Exponential, Positively Skewed

1. Introduction

The Exponentiated distributions have been studied widely in statistics since 1995 and a number of authors have developed various classes of these distributions; Lemonte et al [1]. Mudholkar et al [2] introduced the Exponentiated Weibull distribution and since then, a number of authors have proposed and generalized many standard distributions based on the Exponentiated distributions. The Exponentiated Exponential distribution; Gupta and Kundu [3-5], The Exponentiated Gamma, Exponentiated Weibull, Exponentiated Gumbel and Exponentiated Frechet distributions; Nadarajah and Kotz [6], The Exponentiated Exponential-Geometric distribution; Silva et al [7], The Exponentiated Generalized Inverse Gaussian distribution Lemonte and Cordeiro [8], The Exponentiated Kumaraswamy distribution; Lemonte et al [1], The Exponentiated Inverted Weibull Distribution; Flaih et al [9] and the Exponentiated Generalized Inverse Weibull; Elbatal et al [10] distribution are examples of such in the literatures.

The Exponentiated distribution is derived by raising the cumulative density function (cdf) of an arbitrary parent distribution to an additional non-negative parameter, say ' γ '. According to Lemonte et al [1], the parameter ' γ ' characterizes the skewness, kurtosis and tails of the resulting distribution.

Let X denote a random variable from an arbitrary parent distribution G , the cumulative density function (cdf) of the

resulting Exponentiated distribution is given by;

$$F(x) = G(x)^\gamma; \gamma > 0 \quad (1)$$

where $G(x)$ is the cdf of the parent distribution.

The corresponding probability density function (pdf) is obtained by differentiating Equation (1) with respect to x to give;

$$f(x) = \gamma G(x)^{\gamma-1} g(x) \quad (2)$$

$$\text{Where } g(x) = \frac{dG(x)}{dx}$$

Once the cdf is obtained as in Equation (1), getting the pdf is quite simple; it only involves the knowledge of differentiation in calculus.

On the other hand, Cordeiro et al [11] proposed a new class of distributions as an extension of the Exponentiated type distribution which can be widely applied in many areas of biology and engineering. Given a non-negative continuous random variable X , the cdf of the Exponentiated Generalized (EG) class of distribution is defined by;

$$F(x) = \left[1 - \{1 - G(x)\}^\alpha \right]^\beta \quad (3)$$

where $\alpha, \beta > 0$ are additional shape parameters.

It was noted in their research that the model in Equation (3) has a tractability advantage over the Beta Generalized family of distributions; Eugene et al [12] since Equation (3) does not involve any special function like the incomplete beta function. Equation (3) also has mild algebraic properties for simulation purposes because its quantile function takes a

* Corresponding author:

pelueman@yahoo.com (Oguntunde P. E.)

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simple form. Hence, the corresponding pdf for Equation (3) is given by;

$$f(x) = \alpha\beta g(x) \{1 - G(x)\}^{\alpha-1} \left[1 - \{1 - G(x)\}^\alpha\right]^{\beta-1} \quad (4)$$

where $G(x)$ is the cdf of the baseline distribution and

$$g(x) = \frac{dG(x)}{dx}$$

According to Cordeiro et al [11], even if the baseline pdf; $g(x)$ is a symmetric distribution, the resulting model in Equation (4) will not be a symmetric model because the two additional parameters α and β can control the tail weights and possibly add entropy to the center of the Exponentiated Generalized family of distributions.

A two-parameter Generalized Inverted exponential (GIE) distribution was proposed by Abouammoh and Alshingiti [13] as a generalization of the Inverted Exponential (IE) distribution and it was shown that the Generalized Inverted Exponential distribution is better than the Inverted Exponential distribution in terms of application to a real data set when goodness of fit was assessed using the Likelihood Ratio and Kolmogorov-Smirnov tests.

The probability density function of the GIE distribution is given by;

$$f(x) = \frac{\alpha\lambda}{x^2} e^{-\frac{\lambda}{x}} \left(1 - e^{-\frac{\lambda}{x}}\right)^{\alpha-1} \quad (5)$$

$x > 0, \lambda > 0, \alpha > 0$

The corresponding cumulative density function (cdf) is given by;

$$F(x) = 1 - \left(1 - e^{-\frac{\lambda}{x}}\right)^\alpha \quad (6)$$

$x > 0, \lambda > 0, \alpha > 0$

With this understanding, this article aims at combining the works of Abouammoh and Alshingiti [13] and Cordeiro et al [11] in order to define and provide the basic statistical properties of the Exponentiated Generalized Inverted Exponential distribution.

The rest of this article is organized as follows; Section 2 defines and presents the three parameter Exponentiated Generalized Inverted Exponential (EGIE) distribution, Section 3 discusses the statistical properties of the proposed model, and then followed by the concluding remarks in Section 4.

2. The Exponentiated Generalized Inverted Exponential Distribution

The pdf and the cdf of the Inverted Exponential distribution are given respectively by;

$$g(x) = \frac{\lambda}{x^2} \exp\left(-\frac{\lambda}{x}\right) \quad (7)$$

$$G(x) = \exp\left(-\frac{\lambda}{x}\right) \quad (8)$$

where $x > 0$, the scale parameter $\lambda > 0$

Hence, the proposed Exponentiated Generalized Inverted Exponential (EGIE) distribution is derived by substituting Equations (7) and (8) into Equation (4). Therefore, if a continuous non-negative random variable X is such that; $X \sim EGIE(\alpha, \beta, \lambda)$, its pdf is given by;

$$f(x) = \alpha\beta\lambda x^{-2} e^{-\frac{\lambda}{x}} \left\{1 - e^{-\frac{\lambda}{x}}\right\}^{\alpha-1} \left[1 - \left\{1 - e^{-\frac{\lambda}{x}}\right\}^\alpha\right]^{\beta-1} \quad (9)$$

The corresponding cdf is given by;

$$F(x) = \left[1 - \left\{1 - e^{-\frac{\lambda}{x}}\right\}^\alpha\right]^\beta \quad (10)$$

where $x > 0, \alpha > 0, \beta > 0, \lambda > 0$

α and β are shape parameters

λ is a scale parameter

Special Cases:

Let X denote a non-negative random variable with the pdf in Equation (9), some other well-known distributions are found to be sub-models of the proposed distribution. For instance;

1. For $\alpha = \beta = 1$, Equation (9) reduces to the Inverse Exponential distribution which was introduced by Keller and Kamath [14]
2. For $\beta = 1$, Equation (9) reduces to the Generalized Inverse Exponential distribution which was introduced by Abouammoh and Alshingiti [13].

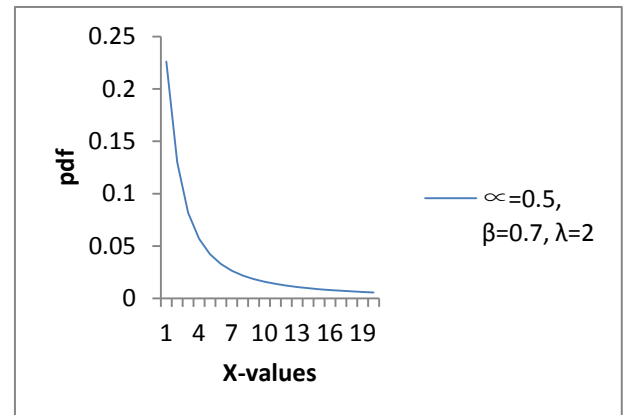


Figure 1. Plot for the pdf at $\alpha = 0.5, \beta = 0.7, \lambda = 2$

The plots for the pdf of the proposed model at various values of parameters α, β and λ is shown in Figure 1 and Figure 2.

Figure 1 shows a decreasing curve (that is, the curve decreases as the value of x increases), which means, the shape of model proposed in this article could be “decreasing”.

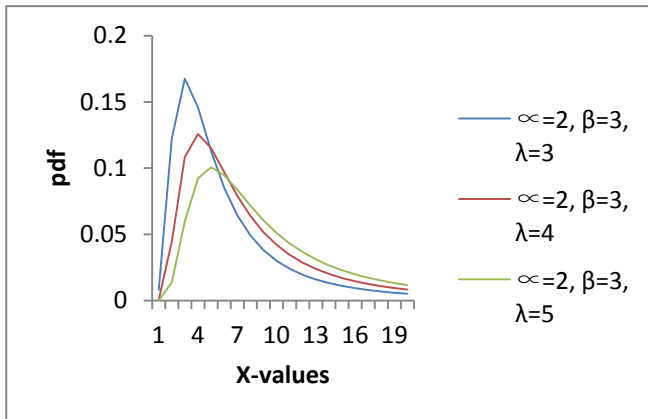


Figure 2. Plot for the pdf at constant values of α and β but at varying values of λ

Figure 2 shows that the curve increases at the initial stage and starts to decrease at some points. This indicates that the proposed model could be unimodal. Besides, the more the value of λ , the more spread out the curve becomes and the lower the value of λ , the more compacted the curve becomes. This indicates that parameter λ is a scale parameter.

The corresponding plot for the cumulative density function (cdf) at various values of α, β and λ is shown in Figure 3.

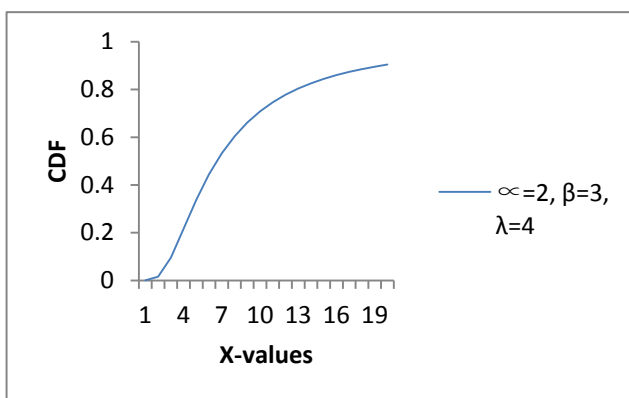


Figure 3. Plot for the cdf at $\alpha = 2, \beta = 3, \lambda = 4$

If ' b ' is a positive real non-integer and $|z| \leq 1$, we consider the power series expansion;

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} w_j z^j \quad (11)$$

Where;

$$w_j = (-1)^j \binom{b-1}{j} = \frac{(-1)^j \Gamma(b)}{j! \Gamma(b-j)}$$

Hence, the cdf in Equation (10) can be re-written is power series representation as;

$$F(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{\beta-1}{j} \binom{\alpha j}{k} e^{-\left(\frac{\lambda}{x}\right)^k} \quad (12)$$

The corresponding pdf is given by;

$$f(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{\beta-1}{j} \binom{\alpha(j+1)-1}{k} \alpha \beta \lambda x^{-2} e^{-\left(\frac{\lambda}{x}\right)^{k+1}} \quad (13)$$

In short, we can simply say;

$$f(x) = A_{jk} \alpha \beta \lambda x^{-2} e^{-\left(\frac{\lambda}{x}\right)^{k+1}} \quad (14)$$

where

$$A_{jk} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{\beta-1}{j} \binom{\alpha(j+1)-1}{k}$$

For more on the properties of the existing Exponential Models, see Anake et al [15], Oguntunde et al [16-19], Singh et al [20].

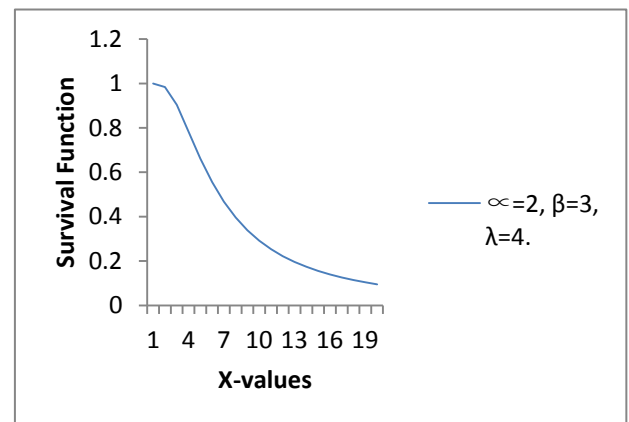


Figure 4. Plot for the survival function at $\alpha = 2, \beta = 3, \lambda = 4$.

3. Properties of the Proposed Model

In this section, we present the statistical properties of the Exponentiated Generalized Inverted Exponential distribution.

3.1. Asymptotic Behavior

We seek to investigate the behavior of the proposed model as given in Equation (9) as $x \rightarrow 0$ and as $x \rightarrow \infty$. This involves considering $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow \infty} f(x)$.

As $x \rightarrow 0$;

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left[\alpha \beta \lambda x^{-2} e^{-\frac{\lambda}{x}} \left\{ 1 - e^{-\frac{\lambda}{x}} \right\}^{\alpha-1} \left[1 - \left\{ 1 - e^{-\frac{\lambda}{x}} \right\}^{\alpha} \right]^{\beta-1} \right]$$

$$= 0$$

As $x \rightarrow \infty$;

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left[\alpha \beta \lambda x^{-2} e^{-\frac{\lambda}{x}} \left\{ 1 - e^{-\frac{\lambda}{x}} \right\}^{\alpha-1} \left[1 - \left\{ 1 - e^{-\frac{\lambda}{x}} \right\}^{\alpha} \right]^{\beta-1} \right]$$

$$= 0$$

These results confirm further that the proposed distribution has a mode.

3.2. Reliability Analysis

The reliability (survival) function is given by;

$$S(x) = 1 - F(x)$$

Hence, we present the reliability function of the Exponentiated Generalized Inverse Exponential distribution as;

$$S_{EGIE}(x) = 1 - \left[1 - \left\{ 1 - e^{-\frac{\lambda}{x}} \right\}^{\alpha} \right]^{\beta} \quad (15)$$

In the same way, the probability that a system having age ' x ' units of time will survive up to ' $x+t$ ' units of time for $x > 0, \alpha > 0, \beta > 0, \lambda > 0$ and $t > 0$ is given by;

$$S_{EGIE}(t|x) = \frac{S_{EGIE}(x+t)}{S_{EGIE}(x)}$$

$$S_{EGIE}(t|x) = \frac{1 - \left[1 - \left\{ 1 - e^{-\frac{\lambda}{(x+t)}} \right\}^{\alpha} \right]^{\beta}}{1 - \left[1 - \left\{ 1 - e^{-\frac{\lambda}{x}} \right\}^{\alpha} \right]^{\beta}} \quad (16)$$

Hazard function (Failure rate) is given by;

$$h(x) = \frac{f(x)}{1 - F(x)}$$

We thus present the hazard function for the proposed distribution as;

$$h_{EGIE}(x) = \frac{\alpha\beta\lambda x^{-2} e^{-\frac{\lambda}{x}} \left\{1 - e^{-\frac{\lambda}{x}}\right\}^{\alpha-1} \left[1 - \left\{1 - e^{-\frac{\lambda}{x}}\right\}^{\alpha}\right]^{\beta-1}}{1 - \left[1 - \left\{1 - e^{-\frac{\lambda}{x}}\right\}^{\alpha}\right]^{\beta}} \quad (17)$$

The plots for the reliability (survival) and hazard functions are shown in Fig.4 and Fig. 5 respectively;

Figure 5 indicates that the hazard function increases at the initial stage but started decreasing at some points. The hazard function has an inverted bathtub shape (or unimodal).

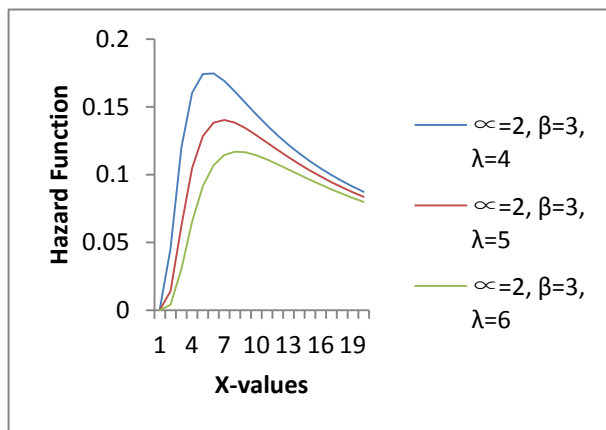


Figure 5. Plot for the hazard function at varying values of α, β and λ

3.3. Quantile Function and Median

The Quantile function is given by;

$$Q(u) = F^{-1}(u)$$

Therefore, the corresponding quantile function for the proposed model is given by;

$$Q(u) = \lambda \left[-\log \left[1 - \left\{ 1 - u^{\frac{1}{\beta}} \right\}^{\frac{1}{\alpha}} \right] \right]^{-1} \quad (18)$$

where U has the uniform $U(0,1)$ distribution. We obtain the median by substituting $u = 0.5$. Hence, the median of the proposed model is given by;

$$F^{-1} = \lambda \left[-\log \left[1 - \left\{ 1 - 0.5^{\frac{1}{\beta}} \right\}^{\frac{1}{\alpha}} \right] \right]^{-1} \quad (19)$$

To simulate the Exponentiated Generalized Inverted Exponential distribution is quite straight forward. Using the means of inverse transformation method, the random variable X is given by;

$$X = \lambda \left[-\log \left[1 - \left\{ 1 - u^{\frac{1}{\beta}} \right\}^{\frac{1}{\alpha}} \right] \right]^{-1} \quad (20)$$

3.4. Moments

The r -th moment of a continuous random variable X is given by;

$$\mu_r = E[X^r] = \int_0^{\infty} x^r f(x) dx$$

If a continuous random variable X is such that; $X \sim EGIE(\alpha, \beta, \lambda)$, using Equation (14),

$$\begin{aligned} \mu_r &= A_{jk} \alpha \beta \lambda \int_0^{\infty} x^{r-2} e^{-\left(\frac{\lambda}{x}\right)^{\alpha}} dx \\ \mu_r &= A_{jk} \alpha \beta \lambda^r (k+1)^{r-1} \Gamma(1-r) \end{aligned} \quad (21)$$

where

$$A_{jk} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{\beta-1}{j} \binom{\alpha(j+1)-1}{k}$$

3.5. Parameter Estimation and Inference

We estimate the parameters of the proposed distribution using the method of maximum likelihood estimation (MLE) as follows;

Let X_1, X_2, \dots, X_n be a random sample of 'n' independently and identically distributed random variables each having an Exponentiated Generalized Inverse Exponential distribution defined in Equation (9), the likelihood function L is given by;

$$L\left(\tilde{X} | \alpha, \beta, \lambda\right) = \prod_{i=1}^n \left[\alpha \beta \lambda x^{-2} e^{-\frac{\lambda}{x}} \left\{ 1 - e^{-\frac{\lambda}{x}} \right\}^{\alpha-1} \left[1 - \left\{ 1 - e^{-\frac{\lambda}{x}} \right\}^{\alpha} \right]^{\beta-1} \right]$$

Let $l = \log L\left(\tilde{X} | \alpha, \beta, \lambda\right)$

$$l = n \log \alpha + n \log \beta + n \log \lambda - 2 \sum_{i=1}^n \log(x_i) - \sum_{i=1}^n \left(\frac{\lambda}{x_i} \right) + (\alpha-1) \sum_{i=1}^n \log \left[1 - e^{-\left(\frac{\lambda}{x_i} \right)} \right] + (\beta-1) \sum_{i=1}^n \log \left[1 - \left\{ 1 - e^{-\left(\frac{\lambda}{x_i} \right)} \right\}^{\alpha} \right]$$

Differentiating l with respect to each of the parameters; α, β and λ gives;

$$\frac{dl}{d\alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log \left[1 - e^{-\left(\frac{\lambda}{x_i} \right)} \right] + (\beta-1) \sum_{i=1}^n \frac{\left\{ 1 - e^{-\left(\frac{\lambda}{x_i} \right)} \right\}^{\alpha} \log \left[1 - e^{-\left(\frac{\lambda}{x_i} \right)} \right]}{\left[1 - \left\{ 1 - e^{-\left(\frac{\lambda}{x_i} \right)} \right\}^{\alpha} \right]} \quad (22)$$

$$\frac{dl}{d\beta} = \frac{n}{\beta} + \sum_{i=1}^n \log \left[1 - \left\{ 1 - e^{-\left(\frac{\lambda}{x_i} \right)} \right\}^{\alpha} \right] \quad (23)$$

$$\frac{dl}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^n \left(\frac{1}{x_i} \right) + (\alpha-1) \sum_{i=1}^n \frac{\left(\frac{1}{x_i} \right) e^{-\left(\frac{\lambda}{x_i} \right)}}{\left[1 - e^{-\left(\frac{\lambda}{x_i} \right)} \right]} + \alpha(\beta-1) \sum_{i=1}^n \frac{\left(\frac{1}{x_i} \right) e^{-\left(\frac{\lambda}{x_i} \right)} \left\{ 1 - e^{-\left(\frac{\lambda}{x_i} \right)} \right\}^{\alpha-1}}{\left[1 - \left\{ 1 - e^{-\left(\frac{\lambda}{x_i} \right)} \right\}^{\alpha} \right]} \quad (24)$$

Solving the nonlinear system of equations of $\frac{dl}{d\alpha} = 0$, $\frac{dl}{d\beta} = 0$ and $\frac{dl}{d\lambda} = 0$ gives the maximum likelihood estimates of α, β and λ respectively.

We obtain the 3 x 3 observed information matrix through;

$$\begin{pmatrix} \hat{\lambda} \\ \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \sim \begin{pmatrix} \lambda \\ \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \hat{V}_{\lambda\lambda} & \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\beta} \\ \hat{V}_{\alpha\lambda} & \hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\beta} \\ \hat{V}_{\beta\lambda} & \hat{V}_{\beta\alpha} & \hat{V}_{\beta\beta} \end{pmatrix}$$

$$V^{-1} = -E \begin{pmatrix} V_{\lambda\lambda} & V_{\lambda\alpha} & V_{\lambda\beta} \\ V_{\alpha\lambda} & V_{\alpha\alpha} & V_{\alpha\beta} \\ V_{\beta\lambda} & V_{\beta\alpha} & V_{\beta\beta} \end{pmatrix}$$

where

$$V_{\lambda\lambda} = \frac{\partial^2 l}{\partial \lambda^2}, V_{\alpha\alpha} = \frac{\partial^2 l}{\partial \alpha^2}, V_{\beta\beta} = \frac{\partial^2 l}{\partial \beta^2}$$

$$V_{\lambda\alpha} = V_{\alpha\lambda} = \frac{\partial^2 l}{\partial \lambda \partial \alpha}, V_{\lambda\beta} = V_{\beta\lambda} = \frac{\partial^2 l}{\partial \lambda \partial \beta}, V_{\alpha\beta} = V_{\beta\alpha} = \frac{\partial^2 l}{\partial \alpha \partial \beta}$$

The solution of the above inverse dispersion matrix yields the asymptotic variance and covariance of the maximum likelihood estimators $\hat{\lambda}, \hat{\alpha}, \hat{\beta}$. Hence, the approximate $100(1-\alpha)\%$ confidence intervals for λ, α, β are given respectively by;

$$\hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{\lambda\lambda}}, \hat{\alpha} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{\alpha\alpha}}, \hat{\beta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{\beta\beta}}$$

where $Z_{\frac{\alpha}{2}}$ is the α -th percentiles of the standard normal distribution.

4. Conclusions

We define a three-parameter Exponentiated Generalized Inverted Exponential Distribution as a generalization of the Inverse Exponential distribution. The model is positively skewed, its shape could be decreasing or unimodal (depending on the values of the parameters) and it has an inverted bathtub failure rate. We provide explicit expressions for the quantile function, reliability function, failure rate and the r-th moment. The Generalized Inverse Exponential Distribution and the Inverse Exponential Distribution are found to be sub-models of the proposed model. The behavior of the failure rate shows that the proposed model is applicable in situations where the Generalized Inverse Exponential Distribution and the Inverse Exponential Distribution are used and can as well serve as an alternative to both of them. We propose the use of the new model in situations where the risk is low at the initial stage, increases

with time and then decreases (for example breast cancer, bladder cancer). Further research would involve comparing the performance of this model to the Beta, Kumaraswamy, Generalized and Transmuted counterpart distributions with the aid of a real data set.

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