

Partial New Mock Theta Functions

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Abstract By using a simple identity of mine, I have proved sixty identities connecting the mock theta functions of Ramanujan with mock theta functions recently generated by Andrews and Bringmann et al, by taking their partial sums.

Keywords Mock Theta Function, Partial Mock Theta Function, Q -Hypergeometric Series

1. Introduction

In his last letter to G.H. Hardy[14], Ramanujan wrote “I discovered very interesting functions recently which I call mock theta functions”. He also listed seventeen functions—four of order three, two groups five in each group of order five and three of order seven. He did not explain what he meant by “order” of the mock theta functions. Later Watson gave three more mock theta functions of order three.

Andrews while visiting Cambridge University found notebooks of Ramanujan and called them “lost” notebooks. The “lost” notebooks contain six mock theta functions and Andrews and Hickerson[7] called them of order six. The “lost” notebook also contains eight identities for the mock theta functions and Choi[10] called these mock theta functions of order ten.

Gordon and McIntosh[13] gave eight mock theta functions and called them of order eight, though four of them are of lesser order. Hikami also gave a mock theta function and called it of order two.

Recently Andrews in his paper[6] generated some mock theta functions and found four of them interesting. Bringmann, Hikami and Lovejoy found two more mock theta functions. We have made comprehensive study of these functions in [16],[17] and also their bilateral form in[18].

In some of Ramanujan’s unpublished work, now published as “lost” notebook, a number of identities and expansion formulas for partial sums of theta functions are mentioned without proof. A study of these partial sums, identities and expansions has been made by Andrews and he wrote that some of these partial theta functions identities have interesting number theoretic interpretations. These observations motivated me to study the partial mock theta functions. The definition of a partial mock theta functions is that we take the partial sum of the infinite series representing these functions. Thus for mock theta function $\bar{\psi}_1(q)$,

$$\bar{\psi}_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_{2n}},$$

the partial mock theta function will be defined and denoted as

$$\bar{\psi}_{1,N}(q) = \sum_{n=0}^N \frac{q^{n^2}}{(-q; q)_{2n}}.$$

Moreover, by taking the partial sum, the additional advantage is that convergence of the series is not a restraint.

In this paper we use a simple summation identity and using this identity we connect the mock theta functions of Andrews among themselves and the mock theta functions of Bringmann et al among themselves. Further we show the relationship between mock theta functions of Andrews with the third, fifth, sixth and tenth order mock theta function of Ramanujan and with eighth order mock theta functions of Gordon and McIntosh.

The mock theta functions of Andrews [6]:

$$\bar{\psi}_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q; q)_{2n}}, \quad (1.1)$$

$$\bar{\psi}_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(-q; q)_{2n+1}}, \quad (1.2)$$

$$\bar{\psi}_2(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n} (q; q^2)_n}{(q^2; q^2)_n (-q; q)_{2n}}, \quad (1.3)$$

and

$$\bar{\psi}_3(q) = \sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q)_n^2}{(q; q)_{2n}}. \quad (1.4)$$

The mock theta functions of Bringmann et al [9]:

$$\bar{\phi}_0(q) = \sum_{n=0}^{\infty} q^n (-q; q)_{2n+1} \quad (1.5)$$

and

$$\bar{\phi}_1(q) = \sum_{n=0}^{\infty} q^n (-q; q)_{2n}. \quad (1.6)$$

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2. Basic Results

The following q - notations have been used
For $|q^k| < 1$,

$$(a; q^k)_n = \prod_{j=0}^{n-1} (1 - aq^{kj}), \quad n \geq 1$$

$$(a; q^k)_0 = 1,$$

$$(a; q^k)_\infty = \prod_{j=0}^{\infty} (1 - aq^{kj}),$$

$$(a)_n = (a; q)_n,$$

$$(a_1, a_2, \dots, a_m; q^k)_n = (a_1; q^k)_n (a_2; q^k)_n \dots (a_m; q^k)_n.$$

A generalized basic hypergeometric series with base q_1 is defined as

$${}_A\phi_{A-1} [a_1, a_2, \dots, a_A; b_1, b_2, \dots, b_{A-1}; q_1, z] \\ = \sum_{n=0}^{\infty} \frac{(a_1; q_1)_n \dots (a_A; q_1)_n z^n}{(b_1; q_1)_n \dots (b_{A-1}; q_1)_n (q_1; q_1)_n}, \quad |z| < 1.$$

3. Definitions of Mock Theta Functions

Third order mock theta functions of Ramanujan:

$$f(q) := \sum_{n=0}^{\infty} q^{n^2} / (-q^2)_n,$$

$$\Phi(q) := \sum_{n=0}^{\infty} q^{n^2} / (-q^2; q^2)_n,$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n^2} / (q; q^2)_n,$$

$$\gamma(q) := \sum_{n=0}^{\infty} q^{n(n+1)} / (-q; q^2)_{n+1}.$$

Fifth order mock theta functions of Ramanujan:

$$f_0(q) := \sum_{n=0}^{\infty} q^{n^2} / (-q; q)_n,$$

$$F_0(q) := \sum_{n=0}^{\infty} q^{2n^2} / (q; q^2)_n,$$

$$f_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q)_n},$$

$$F_1(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q; q^2)_{n+1}}.$$

Sixth order mock theta functions of Ramanujan:

$$\Phi_L(q) := \sum_{n=0}^{\infty} (-1)^n q^{n^2} (q; q^2)_n / (-q)_{2n},$$

$$\psi_L(q) := \sum_{n=0}^{\infty} (-1)^n q^{(n+1)^2} (q; q^2)_n / (-q)_{2n+1}.$$

To distinguish these functions from the others in notation, we have put a suffix L in their symbols.

Seventh order mock theta functions of Ramanujan:

$$\mathfrak{F}_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1}; q)_n}.$$

Eighth order mock theta functions of Gordon and McIntosh:

$$S_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q^2)_n}{(-q^2; q^2)_n},$$

$$S_1(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+2)} (-q; q^2)_n}{(-q^2; q^2)_n},$$

$$V_0^*(q) := (V_0(q) + 1) / 2 = \sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q^2)_n}{(q; q^2)_n} \\ = \sum_{n=0}^{\infty} \frac{q^{2n^2} (-q^2; q^4)_n}{(q; q^2)_{2n+1}}.$$

Tenth order mock theta functions of Ramanujan:

$$X(q) := \sum_{n=0}^{\infty} (-1)^n q^{n^2} / (-q; q)_{2n},$$

$$\chi(q) := \sum_{n=0}^{\infty} (-1)^n q^{(n+1)^2} / (-q; q)_{2n+1}.$$

Second order mock theta function

$$A(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)^2} (-q; q^2)_n}{(q; q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{n+1} (-q^2; q^2)_n}{(q; q^2)_{n+1}},$$

$$B(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n} (-q^2; q^2)_n}{(q; q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^n (-q; q^2)_n}{(q; q^2)_{n+1}},$$

$$\mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(-q^2; q^2)_n^2}.$$

Alternative definitions of second order mock theta functions

$$A(q) = \frac{q(q^2; q^2)_\infty (-q^3; q^2)_\infty}{(q; q^2)_\infty^2} {}_2\phi_1(q, q; -q^3; q^2, q^2),$$

$$B(q) = \frac{(q^4; q^4)_\infty}{(q; q^2)_\infty^2} {}_2\phi_1(q, q; -q^2; q^2, q^2),$$

$$\mu(q) = \frac{(-q; q^2)_\infty}{(-q^2; q^2)_\infty} {}_2\phi_1(-1, q; -q^2; q^2, -q).$$

4. Main Summation Theorem

We give a proof of the following identity

Theorem

$$\sum_{r=0}^p \alpha_r \beta_r = \beta_{p+1} \sum_{r=0}^p \alpha_r + \sum_{m=0}^p (\beta_m - \beta_{m+1}) \sum_{r=0}^m \alpha_r. \quad (4.1)$$

Proof

The proof is given in my paper[15]. However it is very simple, so I repeat the proof here. It is a simple rearrangement of the series.

$$\sum_{m=0}^p (\beta_m - \beta_{m+1}) \sum_{r=0}^m \alpha_r = (\beta_0 - \beta_1) \alpha_0 + (\beta_1 - \beta_2) \sum_{r=0}^1 \alpha_r \\ + (\beta_2 - \beta_3) \sum_{r=0}^2 \alpha_r + \dots + (\beta_p - \beta_{p+1}) \sum_{r=0}^p \alpha_r$$

$$= \alpha_0 \beta_0 + \alpha_1 \beta_1 + \dots + \alpha_p \beta_p - \beta_{p+1} (\alpha_0 + \alpha_1 + \dots + \alpha_p) \\ = \sum_{r=0}^p \alpha_r \beta_r - \beta_{p+1} \sum_{r=0}^p \alpha_r,$$

which proves the theorem.

5. Relations of Partial New Mock Theta Functions with Mock Theta Functions of Ramanujan

5.1. Relations among Partial Mock Theta Functions of Andrews and Partial Mock Theta Functions of Bringmann et al

We give relations between partial mock theta functions of Andrews and partial mock theta functions of Bringmann et al. In the limit we have the mock theta functions expressed as a series in partial mock theta functions.

By definition

$$\bar{\psi}_{0,p}(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2} (1+q^{2n+1})}{(-q; q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q; q)_{2n+1}} + q \bar{\psi}_{1,p}(q) \quad (5.1)$$

(i) Taking $\alpha_n = q^{2n^2} / (-q; q)_{2n+1}$, $\beta_n = q^{2n+1}$ in (4.1), we have

$$q \bar{\psi}_{1,p}(q) = q^{2p+3} \sum_{n=0}^p \frac{q^{2n^2}}{(-q; q)_{2n+1}} + \sum_{m=0}^p (q^{2m+1} - q^{2m+3}) \sum_{n=0}^m \frac{q^{2n^2}}{(-q; q)_{2n+1}} \\ = q^{2p+3} [\bar{\psi}_{0,p}(q) - q \bar{\psi}_{1,p}(q)] + (1-q^2) \sum_{m=0}^p (q^{2m+1} [\bar{\psi}_{0,m}(q) - q \bar{\psi}_{1,m}(q)]) \quad (5.2)$$

we have used (5.1) in the penultimate step.

(ii) Letting $p \rightarrow \infty$ in (5.2), we have

$$q \bar{\psi}_1(q) = (1-q^2) \sum_{m=0}^{\infty} q^{2m+1} (\bar{\psi}_{0,m}(q) - q \bar{\psi}_{1,m}(q)). \quad (5.3)$$

In further relations the values of α_n and β_n will be indicated in parenthesis.

$$(iii) \quad \bar{\psi}_{1,p}(q) = \frac{q^{2p+2}}{1+q^{2p+3}} \bar{\psi}_{0,p}(q) + (1-q^2) \sum_{m=0}^p \frac{q^{2m}}{(1+q^{2m+1})(1+q^{2m+3})} \bar{\psi}_{0,m}(q) \quad (5.4)$$

$$(\alpha_n = q^{2n^2} / (-q; q)_{2n}, \quad \beta_n = q^{2n} / (1+q^{2n+1}) \text{ in (4.1)})$$

$$(iv) \quad \bar{\psi}_1(q) = (1-q^2) \sum_{m=0}^{\infty} \frac{q^{2m}}{(1+q^{2m+1})(1+q^{2m+3})} \bar{\psi}_{0,m}(q). \quad (5.5)$$

($p \rightarrow \infty$ in (5.4))

$$(v) \quad \bar{\psi}_{2,p}(q) = \frac{q^{2p+2}(q; q^2)_{p+1}}{(q^2; q^2)_{p+1}} \bar{\psi}_{0,p}(q) \\ + \sum_{m=0}^p \frac{q^{2m}(q; q^2)_m (1-q^2 - q^{2m+2} + q^{2m+3})}{(q^2; q^2)_{2m+1}} \bar{\psi}_{0,m}(q). \quad (5.6)$$

$$(\alpha_n = q^{2n^2} / (-q; q)_{2n}, \quad \beta_n = q^{2n}(q; q^2)_n / (q^2; q^2)_n \text{ in (4.1)})$$

$$(vi) \quad \bar{\psi}_2(q) = \sum_{m=0}^{\infty} \frac{q^{2m}(q; q^2)_m (1-q^2 - q^{2m+2} + q^{2m+3})}{(q^2; q^2)_{2m+1}} \bar{\psi}_{0,m}(q). \quad (5.7)$$

($p \rightarrow \infty$ in (5.6))

$$(vii) \quad \bar{\phi}_{0,p}(q) = (1+q^{2p+3}) \bar{\phi}_{1,p}(q) + (1-q^2) \sum_{m=0}^p q^{2m+1} \bar{\phi}_{1,m}(q). \quad (5.8)$$

$$(\alpha_n = q^n(-q; q)_{2n}, \quad \beta_n = 1+q^{2n+1} \text{ in (4.1)})$$

$$(viii) \quad \bar{\phi}_0(q) = \bar{\phi}_1(q) + (1-q^2) \sum_{m=0}^{\infty} q^{2m+1} \bar{\phi}_{1,m}(q). \quad (5.9)$$

($p \rightarrow \infty$ in (5.8))

5.2. Relations of Mock Theta Functions of Andrews with Third Order Mock Theta Functions of Ramanujan

$$(i) \quad \bar{\psi}_{0,p}(q) = \frac{(-q^2; q^2)_{p+1}}{(-q; q^2)_{p+1}} f_p(q^2) + (1-q) \sum_{m=0}^p \frac{q^{2m+1}(-q^2; q^2)_m}{(-q; q^2)_{m+1}} f_m(q^2) \quad (5.10)$$

$$(\alpha_n = q^{2n^2} / (-q^2; q^2)_n, \quad \beta_n = (-q^2; q^2)_n / (-q; q^2)_n \text{ in (4.1)})$$

$$(ii) \quad \bar{\psi}_0(q) = \frac{(-q^2; q^2)_{\infty}}{(-q; q^2)_{\infty}} f(q^2) + (1-q) \sum_{m=0}^{\infty} \frac{q^{2m+1}(-q^2; q^2)_m}{(-q; q^2)_{m+1}} f_m(q^2) \quad (5.11)$$

($p \rightarrow \infty$ in (5.10))

$$(iii) \quad \bar{\psi}_{0,p}(q) = -\frac{(-1)^p q^{(p+1)^2}}{(-q^2; q^2)_{p+1}} \psi_p(-q) \\ + \sum_{m=0}^p \frac{(-1)^m q^{m^2} (1-q^{2m+1} + q^{2m+2})}{(-q^2; q^2)_{m+1}} \psi_m(-q). \quad (5.12)$$

$$(\alpha_n = (-1)^n q^{n^2} / (-q; q^2)_n, \quad \beta_n = (-1)^n q^{n^2} / (-q^2; q^2)_n \text{ in (4.1)})$$

$$(iv) \quad \bar{\psi}_0(q) = \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2} (1-q^{2m+1} + q^{2m+2})}{(-q^2; q^2)_{m+1}} \psi_m(-q). \quad (5.13)$$

($p \rightarrow \infty$ in (5.12))

$$(v) \quad \bar{\psi}_{0,p}(q) = \frac{q^{p^2+p}}{(-q^2; q^2)_{p+1}} \gamma_p(q) \\ + \sum_{m=0}^p \frac{q^{m^2-m} (1-q^{2m} + q^{2m+2})}{(-q^2; q^2)_{m+1}} \gamma_m(q). \quad (5.14)$$

$$(\alpha_n = q^{n^2+n} / (-q; q^2)_n, \quad \beta_n = q^{n^2-n} / (-q^2; q^2)_n \text{ in (4.1)})$$

$$(vi) \quad \bar{\psi}_0(q) = \sum_{m=0}^{\infty} \frac{q^{m^2-m} (1-q^{2m} + q^{2m+2})}{(-q^2; q^2)_{m+1}} \gamma_m(q). \quad (5.15)$$

($p \rightarrow \infty$ in (5.14))

$$(vii) \quad \bar{\psi}_{0,p}(q) = \frac{q^{(p+1)^2}}{(-q^2; q^2)_{p+1}} \Phi_p(q) + \sum_{m=0}^p \frac{q^{m^2}}{(-q; q^2)_{m+1}} \Phi_m(q). \quad (5.16)$$

$$(\alpha_n = q^{n^2} / (-q^2; q^2)_n, \quad \beta_n = q^{n^2} / (-q; q^2)_n \text{ in (4.1)})$$

$$(viii) \quad \bar{\psi}_0(q) = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(-q; q^2)_{m+1}} \Phi_m(q). \quad (5.17)$$

($p \rightarrow \infty$ in (5.16))

$$(ix) \quad \bar{\psi}_{1,p}(q) = \frac{q^{p^2+4p+3}}{(-q; q^2)_{p+2}} \Phi_p(q) + \sum_{m=0}^p \frac{q^{m^2+2m}}{(-q; q^2)_{m+2}} \Phi_m(q). \quad (5.18)$$

$$(\alpha_n = q^{n^2} / (-q^2; q^2)_n, \quad \beta_n = q^{n^2+2n} / (-q^3; q^2)_n \text{ in (4.1)})$$

$$(x) \quad \bar{\psi}_1(q) = \sum_{m=0}^{\infty} \frac{q^{m^2+2m}}{(-q; q^2)_{m+2}} \Phi_m(q). \quad (5.19)$$

($p \rightarrow \infty$ in (5.18))

5.3. Relations of Mock Theta Functions of Andrews with Fifth Order Mock Theta Functions of Ramanujan

$$(i) \quad \bar{\psi}_{0,p}(q) = \frac{1}{(-q; q^2)_{p+1}} f_{0,p}(q^2) + \sum_{m=0}^p \frac{q^{2m+1}}{(-q; q^2)_{m+1}} f_{0,m}(q^2) \quad (5.20)$$

$$(\alpha_n = q^{2n^2} / (-q^2; q^2)_n, \quad \beta_n = 1 / (-q; q^2)_n \text{ in (4.1)})$$

$$(ii) \quad \bar{\psi}_0(q) = \frac{1}{(-q; q^2)_\infty} f_0(q^2) + \sum_{m=0}^\infty \frac{q^{2m+1}}{(-q; q^2)_{m+1}} f_{0,m}(q^2) \quad (5.21)$$

$$(p \rightarrow \infty \text{ in (5.20)})$$

$$(iii) \quad \bar{\psi}_{1,p}(q) = \frac{q^{2p+2}}{(-q; q^2)_{p+2}} f_{0,p}(q^2) + \sum_{m=0}^p \frac{q^{2m}(1 - q^2 + q^{2m+3})}{(-q; q^2)_{m+2}} f_{0,m}(q^2). \quad (5.22)$$

$$(\alpha_n = q^{2n^2} / (-q^2; q^2)_n, \quad \beta_n = q^{2n} / (-q; q^2)_{n+1} \text{ in (4.1)})$$

$$(iv) \quad \bar{\psi}_1(q) = \sum_{m=0}^\infty \frac{q^{2m}(1 - q^2 + q^{2m+3})}{(-q; q^2)_{m+2}} f_{0,m}(q^2) \quad (5.23)$$

$$(p \rightarrow \infty \text{ in (5.22)})$$

$$(v) \quad \bar{\psi}_{2,p}(q) = \frac{(q; q^2)_{p+1} q^{2p+2}}{(-q; q^2)_{p+1} (q^2; q^2)_{p+1}} f_{0,p}(q^2) + \sum_{m=0}^p \frac{(q; q^2)_m q^{2m}(1 - q^2 + q^{2m+1} - q^{2m+2} + q^{2m+3})}{(-q; q^2)_{m+1} (q^2; q^2)_{m+1}} f_{0,m}(q^2) \quad (5.24)$$

$$(\alpha_n = q^{2n^2} / (-q^2; q^2)_n, \quad \beta_n = q^{2n} (q; q^2)_n / (-q; q^2)_n (q^2; q^2)_n \text{ in (4.1)})$$

$$(vi) \quad \bar{\psi}_2(q) = \sum_{m=0}^\infty \frac{(q; q^2)_m q^{2m}(1 - q^2 + q^{2m+1} - q^{2m+2} + q^{2m+3})}{(-q; q^2)_{m+1} (q^2; q^2)_{m+1}} f_{0,m}(q^2) \quad (5.25)$$

$$(p \rightarrow \infty \text{ in (5.24)})$$

$$(vii) \quad \bar{\psi}_{0,p}(q) = \frac{1}{(-q^2; q^2)_{p+1}} F_{0,p}(-q) + \sum_{m=0}^p \frac{q^{2m+2}}{(-q^2; q^2)_{m+1}} F_{0,m}(-q). \quad (5.26)$$

$$(\alpha_n = q^{2n^2} / (-q; q^2)_n, \quad \beta_n = 1 / (-q^2; q^2)_n \text{ in (4.1)})$$

$$(viii) \quad \bar{\psi}_0(q) = \frac{1}{(-q^2; q^2)_\infty} F_0(-q) + \sum_{m=0}^\infty \frac{q^{2m+2}}{(-q^2; q^2)_{m+1}} F_{0,m}(-q) \quad (5.27)$$

$$(p \rightarrow \infty \text{ in (5.26)})$$

$$(ix) \quad \bar{\psi}_{1,p}(q) = \frac{1}{(-q; q^2)_{p+2}} f_{1,p}(q^2) + \sum_{m=0}^p \frac{q^{2m+3}}{(-q; q^2)_{m+2}} f_{1,m}(q^2). \quad (5.28)$$

$$(\alpha_n = q^{2n^2+2n} / (-q^2; q^2)_n, \quad \beta_n = 1 / (-q^3; q^2)_n \text{ in (4.1)})$$

$$(x) \quad \bar{\psi}_1(q) = \frac{1}{(-q; q^2)_\infty} f_1(q^2) + \sum_{m=0}^\infty \frac{q^{2m+3}}{(-q; q^2)_{m+2}} f_{1,m}(q^2) \quad (5.29)$$

$$(p \rightarrow \infty \text{ in (5.28)})$$

$$(xi) \quad \bar{\psi}_{2,p}(q) = \frac{(q; q^2)_{p+1}}{(q^2; q^2)_{p+1} (-q; q^2)_{p+1}} f_{1,p}(q^2) + \sum_{m=0}^p \frac{(q; q^2)_m q^{2m+1} (2 - q - q^{2m+2})}{(-q; q^2)_{m+1} (q^2; q^2)_{m+1}} f_{1,m}(q^2). \quad (5.30)$$

$$(\alpha_n = q^{2n^2+2n} / (-q^2; q^2)_n, \quad \beta_n = (q; q^2)_n / (q^2; q^2)_n (-q; q^2)_n$$

in (4.1))

$$(xii) \quad \bar{\psi}_2(q) = \frac{(q; q^2)_\infty}{(-q; q^2)_\infty (q^2; q^2)_\infty} f_1(q^2) + \sum_{m=0}^\infty \frac{(q; q^2)_m q^{2m+1} (2 - q - q^{2m+2})}{(-q; q^2)_{m+1} (q^2; q^2)_{m+1}} f_{1,m}(q^2). \quad (5.31)$$

$$(p \rightarrow \infty \text{ in (5.30)})$$

$$(xiii) \quad \bar{\psi}_{1,p}(q) = \frac{1}{(-q^2; q^2)_{p+1}} F_{1,p}(-q) + \sum_{m=0}^p \frac{q^{2m+2}}{(-q^2; q^2)_{m+1}} F_{1,m}(-q). \quad (5.32)$$

$$(\alpha_n = q^{2n^2+2n} / (-q; q^2)_{n+1}, \quad \beta_n = 1 / (-q^2; q^2)_n \text{ in (4.1)})$$

$$(xiv) \quad \bar{\psi}_1(q) = \frac{1}{(-q^2; q^2)_\infty} F_1(-q) + \sum_{m=0}^\infty \frac{q^{2m+2}}{(-q^2; q^2)_{m+1}} F_{1,m}(-q). \quad (5.33)$$

$$(p \rightarrow \infty \text{ in (5.32)})$$

5.4. Relations of Mock Theta Functions of Andrews with Sixth Order Mock Theta Functions of Ramanujan

$$(i) \quad \bar{\psi}_{0,p}(q) = -\frac{(-1)^p q^{(p+1)^2}}{(q; q^2)_{p+1}} \Phi_{L,p}(q) + \sum_{m=0}^p \frac{(-1)^m q^{m^2}}{(q; q^2)_{m+1}} \Phi_{L,m}(q). \quad (5.34)$$

$$(\alpha_n = (-1)^n q^{n^2} (q; q^2)_n / (-q; q^2)_{2n}, \quad \beta_n = (-1)^n q^{n^2} / (q; q^2)_n \text{ in (4.1)})$$

$$(ii) \quad \bar{\psi}_0(q) = \sum_{m=0}^\infty \frac{(-1)^m q^{m^2}}{(q; q^2)_{m+1}} \Phi_{L,m}(q). \quad (5.35)$$

$$(p \rightarrow \infty \text{ in (5.34)})$$

$$(iii) \quad q\bar{\psi}_{1,p}(q) = -\frac{(-1)^p q^{(p+1)^2}}{(q; q^2)_{p+1}} \Psi_{L,p}(q) + \sum_{m=0}^p \frac{(-1)^m q^{m^2}}{(q; q^2)_{m+1}} \Psi_{L,m}(q). \quad (5.36)$$

$$(\alpha_n = (-1)^n q^{(n+1)^2} (q; q^2)_n / (-q; q^2)_{2n+1}, \quad \beta_n = (-1)^n q^{n^2} / (q; q^2)_n \text{ in (4.1)})$$

$$(iv) \quad q\bar{\psi}_1(q) = \sum_{m=0}^\infty \frac{(-1)^m q^{m^2}}{(q; q^2)_{m+1}} \Psi_{L,m}(q). \quad (5.37)$$

$$(p \rightarrow \infty \text{ in (5.36)})$$

5.5. Relations of Mock Theta Functions of Andrews with Seventh Order Mock Theta Functions of Ramanujan

$$(i) \quad \bar{\psi}_{3,p}(q) = \frac{(-q; q^2)_{p+1}}{(q; q^2)_{p+1}} \mathfrak{F}_{0,p}(q) - \sum_{m=0}^p \frac{q^{m+1} (-q; q^2)_m^2 (3 + q^{m+1})}{(q; q^2)_{m+1}} \mathfrak{F}_{0,m}(q) \quad (5.38)$$

$$(\alpha_n = q^{n^2} (q; q)_n / (q; q)_{2n}, \beta_n = (-q; q)_n^2 / (q; q)_n \text{ in (4.1)})$$

$$(ii) \quad \bar{\psi}_3(q) = \frac{(-q; q)_\infty^2}{(q; q)_\infty} \mathfrak{F}_0(q) - \sum_{m=0}^{\infty} \frac{q^{m+1} (-q; q)_m^2 (3 + q^{m+1})}{(q; q)_{m+1}} \mathfrak{F}_{0,m}(q) \quad (5.39)$$

($p \rightarrow \infty$ in (5.38))

5.6. Relations of Mock Theta Functions of Andrews with Eighth Order Mock Theta Functions of Gordon and McIntosh

$$(i) \quad \bar{\psi}_{2,p}(q) = -\frac{(-1)^p q^{(p+1)^2}}{(q^2; q^2)_{p+1} (-q; q^2)_{p+1}} S_{1,p}(-q) + \sum_{m=0}^p \frac{(-1)^m q^{m^2} (1 + 2q^{m+1} - q^{2m+2} - q^{4m+3})}{(-q; q^2)_{m+1} (q^2; q^2)_{m+1}} S_{1,m}(-q) \quad (5.40)$$

$$(\alpha_n = (-1)^n q^{n^2+2n} (q; q^2)_n / (-q^2; q^2)_n, \beta_n = (-1)^n q^{n^2} / (q^2; q^2)_n (-q; q^2)_n \text{ in (4.1)})$$

$$(ii) \quad \bar{\psi}_2(q) = \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2} (1 + 2q^{m+1} - q^{2m+2} - q^{4m+3})}{(-q; q^2)_{m+1} (q^2; q^2)_{m+1}} S_{1,m}(-q) \quad (5.41)$$

($p \rightarrow \infty$ in (5.40))

$$(iii) \quad \bar{\psi}_{2,p}(-q) = \frac{q^{p^2+4p+3}}{(q^4; q^4)_{p+1}} V_{0,p}^*(q) + \sum_{m=0}^p \frac{q^{m^2+2m} (1 - q^{2m+3} - q^{4m+4})}{(q^4; q^4)_{m+1}} V_{0,m}^*(q) \quad (5.42)$$

$$(\alpha_n = q^{n^2} (-q; q^2)_n / (q; q^2)_n, \beta_n = q^{n^2+2n} / (q^2; q^2)_n (-q^2; q^2)_n \text{ in (4.1)})$$

$$(iv) \quad \bar{\psi}_2(-q) = \sum_{m=0}^{\infty} \frac{q^{m^2+2m} (1 - q^{2m+3} - q^{4m+4})}{(q^4; q^4)_{m+1}} V_{0,m}^*(q) \quad (5.43)$$

($p \rightarrow \infty$ in (5.42))

$$(v) \quad \bar{\psi}_{0,p}(q) = \frac{q^{(p+1)^2}}{(-q; q^2)_{p+1}^2} S_{0,p}(q) + \sum_{m=0}^p \frac{q^{m^2} (1 + q^{2m+1} + q^{4m+2})}{(-q; q^2)_{m+1}^2} S_{0,m}(q) \quad (5.44)$$

$$(\alpha_n = q^{n^2} (-q; q^2)_n / (-q^2; q^2)_n, \beta_n = q^{n^2} / (-q; q^2)_n^2 \text{ in (4.1)})$$

$$(vi) \quad \bar{\psi}_0(q) = \sum_{m=0}^{\infty} \frac{q^{m^2} (1 + q^{2m+1} + q^{4m+2})}{(-q; q^2)_{m+1}^2} S_{0,m}(q). \quad (5.45)$$

($p \rightarrow \infty$ in (5.44))

$$(vii) \quad \bar{\psi}_{1,p}(q) = \frac{q^{(p+1)^2}}{(-q; q^2)_{p+1} (-q; q^2)_{p+2}} S_{1,p}(q) + \sum_{m=0}^p \frac{q^{m^2} (1 + q^{2m+3} + q^{4m+4})}{(-q; q^2)_{m+1} (-q; q^2)_{m+2}} S_{1,m}(q). \quad (5.46)$$

$$(\alpha_n = q^{n^2+2n} (-q; q^2)_n / (-q^2; q^2)_n, \beta_n = q^{n^2} / (-q; q^2)_n (-q; q^2)_{n+1} \text{ in (4.1)})$$

$$(viii) \quad \bar{\psi}_1(q) = \sum_{m=0}^{\infty} \frac{q^{m^2} (1 + q^{2m+3} + q^{4m+4})}{(-q; q^2)_{m+1} (-q; q^2)_{m+2}} S_{1,m}(q) \quad (5.47)$$

($p \rightarrow \infty$ in (5.46))

5.7. Relations of Mock Theta Functions of Andrews with Tenth Order Mock Theta Functions of Ramanujan

$$(i) \quad \bar{\psi}_{0,p}(q) = -(-1)^p q^{(p+1)^2} X_p(q) + \sum_{m=0}^p (-1)^m q^{m^2} (1 + q^{2m+1}) X_m(q) \quad (5.48)$$

$$(\alpha_n = (-1)^n q^{n^2} / (-q; q)_{2n}, \beta_n = (-1)^n q^{n^2} \text{ in (4.1)})$$

$$(ii) \quad \bar{\psi}_0(q) = \sum_{m=0}^{\infty} (-1)^m q^{m^2} (1 + q^{2m+1}) X_m(q). \quad (5.49)$$

($p \rightarrow \infty$ in (5.48))

$$(iii) \quad q\bar{\psi}_{1,p}(q) = -(-1)^p q^{(p+1)^2} \chi_p(q) + \sum_{m=0}^p (-1)^m q^{m^2} (1 + q^{2m+1}) \chi_m(q) \quad (5.50)$$

$$(\alpha_n = (-1)^n q^{(n+1)^2} / (-q; q)_{2n+1}, \beta_n = (-1)^n q^{n^2} \text{ in (4.1)})$$

$$(iv) \quad q\bar{\psi}_1(q) = \sum_{m=0}^{\infty} (-1)^m q^{m^2} (1 + q^{2m+1}) \chi_m(q). \quad (5.51)$$

($p \rightarrow \infty$ in (5.50))

5.8. Relations of Mock Theta Functions of Andrews and Mock Theta Functions of Bringmann et al with Second Order Mock Theta Functions

$$(i) \quad \bar{\phi}_{0,p}(q) = (-1)^{p+1} q^{p+1} (1 + q^{2p+3}) / (-q^2; q^2)_{p+1}^2 \mu_p(q) + \sum_{m=0}^p (-1)^{m+1} q^m (-q^2; q^2)_m^2 [q(1 + q^{2m+3})(1 + q^{2m+2})^2 - (1 + q^{2m+1})] \mu_m(q) \quad (5.52)$$

$$(\alpha_n = 2(-1)^n (-q; q^2)_n / (-q^2; q^2)_n, \beta_n = (-1)^n q^n (-q^2; q^2)_n^2 (-q^3; q^2)_n / (-q; q^2)_n \text{ in (4.1)})$$

$$(ii) \quad \bar{\phi}_0(q) = \sum_{m=0}^p (-1)^{m+1} q^m (-q^2; q^2)_m^2 [q(1 + q^{2m+3})(1 + q^{2m+2})^2 - (1 + q^{2m+1})] \mu_m(q). \quad (5.53)$$

($p \rightarrow \infty$ in (5.52))

$$(iii) \quad \bar{\phi}_{1,p}(q) = (-1)^{p+1} q^{p+1} (-q^2; q^2)_{p+1}^2 \mu_p(q) + \sum_{m=0}^p (-1)^{m+1} q^m (-q^2; q^2)_m^2 [1 + q(1 + q^{2m+2})^2] \mu_m(q) \quad (5.54)$$

$$(\alpha_n = (-1)^n (-q; q^2)_n / (-q^2; q^2)_n, \beta_n = (-1)^n q^n (-q^2; q^2)_n^2 \text{ in (4.1)})$$

$$(iv) \quad \bar{\phi}_1(q) = \sum_{m=0}^{\infty} (-1)^{m+1} q^m (-q^2; q^2)_m^2 [1 + q(1 + q^{2m+2})^2] \mu_m(q) \quad (5.55)$$

($p \rightarrow \infty$ in (5.54))

$$(v) \quad \frac{(q^4; q^4)_\infty}{(q; q^2)_\infty} \bar{\psi}_{2,p}(q) = \frac{q^{2(p+1)^2}}{(q^2; q^4)_{p+1}} B_p(q) - \sum_{m=0}^p \frac{q^{2m^2} (1 + 2q^{4m+2})}{(q^2; q^4)_{m+1}} B_m(q) \quad (5.56)$$

$$(\alpha_n = (q^4; q^4)_\infty (q; q^2)_n^2 q^{2n} / (q; q^2)_\infty^2 (q^2; q^2)_n (-q^2; q^2)_n, \beta_n = q^{2n^2} / (q; q^2)_n (-q; q^2)_n \text{ in (4.1)})$$

$$(vi) \frac{(q^4; q^4)_\infty}{(q; q^2)_\infty^2} \bar{\psi}_2(q) = \sum_{m=0}^{\infty} \frac{q^{2m^2} (1 + 2q^{4m+2})}{(q^2; q^4)_{m+1}} B_m(q) \quad (5.57)$$

$$(p \rightarrow \infty \text{ in (5.56)})$$

$$(vii) \frac{q(-q^2; q^2)_\infty}{(1+q)(q; q^2)_\infty} \bar{\psi}_{1,p}(q) = \frac{(-1)^p q^{2(p+1)^2}}{(q^2; q^4)_{p+1} (-q^2; q^2)_{p+1}} A_p(-q) + \sum_{m=0}^p \frac{(-1)^m q^{2m^2} (1 - q^{4m+1} + q^{2m+2} - q^{4m+2} + q^{6m+4})}{(q^2; q^4)_{m+1} (-q^2; q^2)_{m+1}} A_m(-q) \quad (5.58)$$

$$(\alpha_n = \frac{q(-q^2; q^2)_\infty}{(1+q)(-q; q^2)_\infty} \frac{(-1)^{n+1} q^{2n^2} (q^2; q^4)_n}{(-q^3; q^2)_n}, \beta_n = \frac{(-1)^n q^{2n^2}}{(q^2; q^4)_n (-q^2; q^2)_n} \text{ in (4.1)})$$

$$(viii) \frac{q(-q^2; q^2)_\infty}{(1+q)(q; q^2)_\infty} \bar{\psi}_1(q) =$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{2m^2} (1 - q^{4m+1} + q^{2m+2} - q^{4m+2} + q^{6m+4})}{(q^2; q^4)_{m+1} (-q^2; q^2)_{m+1}} A_m(-q) \quad (5.59)$$

$$(p \rightarrow \infty \text{ in (5.58)})$$

$$(ix) \frac{(q^4; q^4)_\infty}{(q^2; q^4)_\infty^2} \bar{\psi}_{0,p}(q) = \frac{q^{2p^2+2p}}{(q^2; q^4)_{p+1} (q; q^2)_{p+1}} B_p(q) + \sum_{m=0}^p \frac{q^{2m^2-2m} [q^{4m} - (1 - q^{4m+2})(1 - q^{2m+1})]}{(q^2; q^4)_{m+1} (q; q^2)_m} B_m(q) \quad (5.60)$$

$$(\alpha_n = (q^4; q^4)_\infty (q; q^2)_n^2 q^{2n} / (q; q^2)_\infty^2 (-q^2; q^2)_n, \beta_n = q^{2n^2-2n} / (q; q^2)_n^2 (-q; q^2)_n \text{ in (4.1)})$$

$$(x) \frac{(q^4; q^4)_\infty}{(q^2; q^4)_\infty^2} \bar{\psi}_0(q) = \sum_{m=0}^{\infty} \frac{q^{2m^2-2m} [q^{4m} - (1 - q^{4m+2})(1 - q^{2m+1})]}{(q^2; q^4)_{m+1} (q; q^2)_m} B_m(q) \quad (5.61)$$

$$(p \rightarrow \infty \text{ in (5.60)})$$

6. Conclusions

By the help of a simple summation formula identities connecting various partial mock theta functions are given. The consideration of partial sums of the mock theta functions is helpful as there is no constraint about convergence of the series. These partial identities may be helpful in giving number theoretic interpretations.

REFERENCES

- [1] R. P. Agarwal, Certain basic hypergeometric identities associated with mock theta functions, *Quart. J. Math.* (Oxford) 20 (1968), 121-128.
- [2] R. P. Agarwal, Mock Theta Functions- an analytical point of view, *Proc. Nat. Acad. Sci.*, 64 (1994), 95-107.
- [3] K. Alladi, A partial theta identity of Ramanujan and its number-theoretic interpretation, *Ramanujan J.* 20 (2009), 329-339.
- [4] K. Alladi, A combinatorial study and comparison of partial theta identities of Andrews and Ramanujan, *Ramanujan J.* 23 (2010), 227-241.
- [5] G. E. Andrews, On basic hypergeometric series, mock theta functions and identities-I, *Quart. J. Math.* (Oxford) 17 (1966), 64-80.
- [6] G. E. Andrews, q -orthogonal polynomials, Rogers-Ramanujan identities, and mock theta functions, preprint.
- [7] G. E. Andrews, D. Hickerson, Ramanujan's "lost" notebook VII: The sixth order mock theta functions, *Adv. in Math.* 89 (1991), 60-105.
- [8] B. Berndt, B. Kim and A. Yee, Ramanujan's lost notebook: combinatorial proofs of identities associated with Heine's transformation or partial theta functions. *J. Combin. Theory Ser. A* 117 (2010), 857-973.
- [9] K. Bringmann, K. Hikami, J. Lovejoy, On the modularity of the unified WRT invariants of certain Seifert manifolds, *Adv. Appl. Math.*, to appear.
- [10] Youn-Seo Choi, Tenth order mock theta functions in Ramanujan's "lost" notebook IV, *Trans. Amer. Math. Soc.* 354(2) (2002), 705-733.
- [11] N.J. Fine, Basic Hypergeometric series and application. Mathematical surveys and Monographs, *Amer. Math. Soc.* 27 (1998).
- [12] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge (1990).
- [13] B. Gordon and R.J. McIntosh, Some eighth order mock theta functions, *J. London Math. Soc.* 62(2) (2000), 321-335.
- [14] S. Ramanujan, *Collected Paper*, Cambridge University Press 1927, reprinted by Chelsea New York, (1962).
- [15] Bhaskar Srivastava, Ramanujan's Mock Theta Functions, *Math. J. Okayama Univ.* 47 (2005), 163-174.
- [16] Bhaskar Srivastava, A generalization and study of four new mock theta functions, (Communicated).
- [17] Bhaskar Srivastava, Some new mock theta functions, accepted in *Math. Sci. Research J.*
- [18] Bhaskar Srivastava, A study of Bilateral new mock theta functions, (Communicated).
- [19] G.N. Watson, The final problem: An account of the mock theta functions, *J. London Math. Soc.* 11 (1936), 55-80.
- [20] G.N. Watson, The Mock Theta Function II, *Proc. London Math. Soc.* (2) 42 (1937), 272-304.
- [21] D. Zagier, Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann), *S'eminare Bourbaki*. Vol. 2007/2008. Ast'erisque No. 326 (2009), Exp. No. 986, vii-viii, 143-164 (2010).