Applied Mathematics 2012, 2(4): 100-115
DOI: 10.5923/j.am.20120204.02

Complete Classification of BKM Lie Superalgebras Possessing Strictly Imaginary Property

N. Sthanumoorthy*, K. Priyadharsini

Ramanujan Institute for Advanced study in Mathematics, University of Madras, Chennai , 600 005, India

Abstract In this paper, complete classifications of all BKM Lie superalgebras (with finite order and infinite order Cartan matrices) possessing Strictly Imaginary Property are given. These classifications also include, in particular, the Monster BKM Lie superalgebra.

Keywords Borcherds Kac-Moody Lie Superalgebras, Strictly Imaginary Roots, Purely Imaginary Roots

1. Introduction

In[4], the theory of Lie superalgebras was given and in[5], theory of Kac- Moody Lie superalgebras was described. Borcherds[2] initiated the study of generalized Kac-Moody algebras(GKM algebras). Wakimoto[19] introduced BKM superalgebras(BKM Lie superalgebras). The existence of special imaginary roots for Kac-Moody algebras(KM algebras) were shown in[1] and the concept of special imaginary roots was extended from KM algebras to GKM algebras in[7]. In[11], some properties of roots of GKM algebras were studied and in[12],[14], special imaginary roots of these classes were found out and finally in[15], a complete classification of GKM algebras possessing special imaginary roots was found out.

The notion of special imaginary roots of BKM algebras was generalized to BKM superalgebras in[16] and certain classes of BKM Lie superalgebras possessing special imaginary roots were found out in[16]. In[18], a complete classification of BKM Lie superalgebras possessing special imaginary roots was given. The concept of strictly imaginary roots for KM algebras was introduced by Kac([5],[6]). Casperson[3] gave a complete classification of KM algebras possessing strictly imaginary property. The concept of purely imaginary roots for KM algebras was introduced in[10] and therein the KM algebras possessing purely imaginary property were completely classified.

Again in[13], the concept of purely imaginary roots from KM algebras to GKM was extended, and the GKM algebras possessing purely imaginary property were completely classified. In[14], the properties of strictly imaginary roots and purely imaginary roots of GKM algebras were compared and using the classification of GKM algebras possessing purely imaginary property, the algebras whose purely imaginary roots are strictly imaginary roots were found. Complete classification of GKM algebras possessing special imaginary roots and strictly imaginary property were given in[15].

The concepts of strictly imaginary roots and purely imaginary roots of Borcherds Kac-Moody algebras(BKM algebras) were extended to BKM superalgebras in[17]. A complete classification of those BKM superalgebras with purely alien imaginary property and purely imaginary property were given in[17]. Moreover, the properties of strictly imaginary roots and purely imaginary roots of BKM superalgebras were compared and the BKM superalgebras whose purely imaginary roots are also strictly imaginary were found out in[17].

Aim of this paper is to give a complete classification of BKM Lie superalgebras possessing strictly imaginary property.

2. Preliminaries

2.1. Basic Definitions

In this section, we briefly recall the fundamental definitions regarding BKM Lie superalgebras, their Weyl groups and root systems as given in[19].

For the definition of Generalized Generalized Cartan matrix(GGCM) one can see[9].

Definition 2.1.1:[19].

Let $I=\{1,2,\cdots,n\}$ be a finite index set and let $A=(a_{ij})_{i,j\in I}$ be an $n\times n$ real matrix. Let ψ be a subset of I . If A satisfies the following conditions, then (A,ψ) is called a BKM super matrix.

$$(\tilde{c}1)\quad a_{ii}=2 \text{ or } a_{ii}\le 0$$

*Corresponding author:
sthanun@yahoo.com(N.Sthanumoorthy)
Published online at http://journal.sapub.org/am

$(\widetilde{c}2)$ $i \neq j \Rightarrow a_{ij} \leq 0$

$(\widetilde{c}3)$ $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$

$(\widetilde{c}4)$ if $a_{ii} = 2$ then $a_{ij} \in \mathbb{Z}$ *for all* j

$(\widetilde{c}5)$ if $i \in \psi$ and $a_{ii} = 2$ then $a_{ij} \in 2\mathbb{Z}$ *for all* j.

Define, subsets I^{re} and I^{im} of I by $I^{re} = \{i \in I; a_{ii} = 2\}$, $I^{im} = \{i \in I; a_{ii} \leq 0\}$.

Let $\underline{m} = (m_i \in \mathbb{Z}_{>0} \mid i \in I)$ be a collection of positive integers such that $m_i = 1$ for all $i \in I^{re}$. We call $\underline{m}$ a charge of A.

Also set $\psi^{re} := \{i \in \psi : a_{ii} = 2\} = I^{re} \cap \psi;$ and $\psi_0 := \{i \in \psi; a_{ii} = 0\}; \psi^- := \{i \in \psi; a_{ii} < 0\}$ $\psi^{im} := \{i \in \psi; a_{ii} \leq 0\} = \psi_0 \cup \psi^-$.

Remarks:

(1) If ψ is an empty set then the BKM super matrix coincides with the corresponding BKM matrix(or GKM matrix).

(2) For description of the quasi- Dynkin diagram, $qDyn(A)$, one can refer to[19]. A Generalized Generalized Cartan Matrix is called indecomposable if it cannot be reduced to a block diagonal form by shuffling rows and columns[8].

For the sake of completeness we repeat the following fundamentals already explained in[17].

Definition 2.1.2:[6]

Let I be an index set. (A,ψ) be an indecomposable BKM super matrix where $A = (a_{ij})_{i,j \in I}$ and $\psi \subset I$. Then one and only one of the following three possibilities holds for A.

(Fin) det $A \neq 0$; there exists $u > 0$ such that $Au > 0$ and $Au \geq 0 \Rightarrow u > 0$ or $u = 0$.

(Aff) Corank =1; there exists $u > 0$ such that $Au = 0$ and $Au \geq 0 \Rightarrow Au = 0$

(Ind) det $A \neq 0$; there exists $u > 0$ such that $Au < 0$ and $Au > 0, u > 0 \Rightarrow u = 0$

Referring to the above three cases, we say that A is of finite, affine or indefinite type respectively and write $A \in Fin, A \in Aff$ or $A \in Ind$ respectively.

Definition 2.1.3:[18]

We say that a BKM super matrix (A,ψ) is of hyperbolic type, if it is indefinite type and every principal submatrix of A is either finite or affine type BKM super matrix.

Definition 2.1.4[19]:

If a BKM super matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ decomposes as $A = DB$ where, $D = (\varepsilon_i \delta_{ij})_{1 \leq i,j \leq n}$: a diagonal matrix and $B = (b_{ij})_{1 \leq i,j \leq n}$ is a symmetric matrix, then A is said to be symmetrizable.

If A is a symmetrizable BKM supermatrix, then taking the diagonal matrix D satisfying $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n > 0$, by $a_{ij} = \varepsilon_i b_{ij}$, we have $a_{ij} \leq 0 \Leftrightarrow b_{ij} \leq 0$ and $a_{ij} \geq 0 \Leftrightarrow b_{ij} \geq 0$ for all i and j.

We assume that (A,ψ) is a symmetrizable and indecomposable BKM supermatrix.

Definition 2.1.5:[19]

For any BKM supermatrix, (A,ψ) where $A = (a_{ij})_{1 \leq i,j \leq n}$, we have a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\Pi = \{\alpha_i; i \in I\}$ and $\Pi^\vee = \{\alpha_i^\vee; i \in I\}$ satisfying the following relations:

(i) $\mathfrak{h}$ is a finite dimensional (complex) vector space such that $\dim \mathfrak{h} = 2n -$ rank A.

(ii) $\Pi = \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ is linearly independent and $\Pi^\vee = \{\alpha_i^\vee\}_{i \in I} \subset \mathfrak{h}$ is linearly independent, where $\mathfrak{h}^* = \mathrm{H}om_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$.

iii) $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$, where $\langle , \rangle$ denotes a duality pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. This triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is called a realization of A.

Call an element of Π (respectively $\Pi^\vee$) a fundamental root or a simple root (respectively fundamental coroot or a simple coroot).

Moreover, set $\Pi^{re} = \{\alpha_i; i \in I^{re}\}$ and $\Pi^{im} = \{\alpha_i; i \in I^{im}\}$. We call an element of Π^{re} (resp. Π^{im}) a real simple root (resp. an imaginary simple root).

Also divide Π as $\Pi_{even} := \{\alpha_i; i \in I \setminus \psi\}$, the set of all even simple roots and $\Pi_{odd} := \{\alpha_i; i \in \psi\}$, the set of all odd simple roots.

Let $\mathbb{Z}_2$ is the residue class ring mod 2 with elements $\bar{0}$ and $\bar{1}$.

Definition 2.1.6:[19]

A $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ possessing the operation called the bracket product,

$$[,] : \mathfrak{g} \times \mathfrak{g} \ni (x,y) \xrightarrow{\text{bilinear map}} [x,y] \in \mathfrak{g},$$

is called a *Lie superalgebra* if it satisfies the following conditions:

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, (i, j \in \mathbb{Z}_2),$$
$$[y, x] = -(-1)^{|x||y|}[x, y],$$
$$[x, [y, z]] = -(-1)^{|x||y|}[y, [x, z]] + [[x, y], z],$$ for all $x, y, z \in \mathfrak{g}$.

Definition 2.1.7:[19]

The *Borcherds Kac-Moody Lie superalgebra* (abbreviated as BKM Lie superalgebra or BKM superalgebra) $\widetilde{\mathfrak{g}}(A)$ associated to a symmetrizable BKM super matrix $(A, \psi) = (a_{ij})_{1\le i,j\le n}$ is the Lie superalgebra generated by the vector space $\mathfrak{h}$ and the elements $e_i, f_i (i \in I)$ satisfying the following relations:

1. $[h, h'] = 0$ for $h, h' \in \mathfrak{h}$
2. $[h, e_i] = \alpha_i(h) e_i$ for $h \in \mathfrak{h}, 1 \le i \le n$
3. $[h, f_i] = -\alpha_i(h) f_i$ for $h \in \mathfrak{h}, 1 \le i \le n$
4. $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$ for $1 \le i, j \le n.$
5. if $i \in I^{re}, j \in I, i \ne j,$ then $(ad\, e_i)^{1-a_{ij}} e_j = 0; (ad\, f_i)^{1-a_{ij}} f_j = 0$
6. if $i, j \in I, i \ne j, a_{ij} = 0$, then $\begin{Bmatrix} [e_i, e_j] & = 0 \\ [f_i, f_j] & = 0 \end{Bmatrix}$
7. if $i \in \psi_0$ then $[e_i, e_i] = [f_i, f_i] = 0$.

Remarks:

As we are assuming that the matrix (A, ψ) is symmetrizable, the associated BKM superalgebra $\widetilde{\mathfrak{g}}(A)$ is simple (for a proof one can see[6], and also[19]),which we will denote by $\mathfrak{g}(A)$. So for a BKM supermatrix (A, ψ), $\mathfrak{g}(A)$ is called BKM Lie superalgebra or BKM superalgebra associated to (A, ψ).

In[5], Dynkin diagrams were defined for Lie superalgebras. Dynkin diagrams were already extended from KM algebras to GKM algebras in[11] and then extended to BKM Lie superalgebras in[17], which are again given below.

Definition 2.1.8:[17]

To every BKM super matrix (A, ψ), where $A = (a_{ij})_{i,j\in I}$, $\psi \subset I$, the index set I, is associated with a Dynkin diagram $S(A)$ defined as follows:

$S(A)$ has n vertices and vertices i and j are connected by $\max\{|a_{ij}|, |a_{ji}|\}$ number of lines if $a_{ij} a_{ji} \le 4$ and there is an arrow pointing towards i if $|a_{ij}| > 1$. If $|a_{ij}||a_{ji}| > 4$, i and j are connected by a bold faced edge equipped with the ordered pair $(|a_{ij}|, |a_{ji}|)$. Moreover

1. if $a_{ii} = 2$ *and* $i \notin \psi$, the i-th vertex will be denoted by a white circle.
2. if $a_{ii} = 2$ *and* $i \in \psi$, the i-th vertex will be denoted by a white circle with (od) written within parentheses and below the circle to denote the vertex corresponding to an odd simple root in this case.
3. if $a_{ii} = 0$ *and* $i \notin \psi$, the i-th vertex will be denoted by a crossed circle.
4. if $a_{ii} = 0$ *and* $i \in \psi$, the i-th vertex will be denoted by a crossed circle with (od) written within parentheses and below the circle to denote the vertex corresponding to an odd simple root in this case.
5. if $a_{ii} = -k, k > 0$ *and* $i \notin \psi$, the i-th vertex will be denoted by a white circle with $(-k)$ written within parentheses and above the circle.
6. if $a_{ii} = -k, k > 0$ *and* $i \in \psi$, the i-th vertex will be denoted by a white circle with $(-k)$ written within parentheses and above the circle with (od) written within parentheses and below the circle to denote the vertex corresponding to an odd simple root in this case.

With these definitions, the Dynkin diagrams of all BKM superalgebras can be drawn.

Some examples of Dynkin diagrams of BKM superalgebras were drawn in[17].

A BKM Lie superalgebra $\mathfrak{g}$, like a KM or BKM algebra, has the following natural root space decomposition:

$\mathfrak{g} = \oplus \mathfrak{g}_\alpha$ where

$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [h, X] = \alpha(h) X \ \forall\, h \in \mathfrak{h}\}$

is called the root space associated to α. An element $\alpha \in Q$ is called a root, if $\alpha \ne 0$ and $\mathfrak{g}_\alpha \ne 0$. The number mult $\alpha = \dim \mathfrak{g}_\alpha$ is called the multiplicity of the root α. A root α of $\mathfrak{g}(A)$ can be expressed as $\alpha = \sum_{i=1}^n m_i \alpha_i$, ($m_i \in \mathbb{Z}$), where m_i's are all ≥ 0 or all ≤ 0. Corresponding to whether m_i's are all ≥ 0 or all ≤ 0, α is called a positive root or a negative root respectively. Also $\sum_{i=1}^n m_i$ is called the height of α and is denoted by $ht(\alpha)$. We denote by Δ, Δ_+ and Δ_- the set of all roots, positive roots and negative roots respectively. Also note that $\mathfrak{g}_{\alpha_i} = \mathbb{C} e_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{C} f_i$.

Definition 2.1.9:[19]

Let $\mathfrak{G}$ be a BKM Lie superalgebra. Set $\mathfrak{h}' = \sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee$ and take a subspace $\mathfrak{h}''$ of $\mathfrak{h}$ satisfying $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$.

Define the symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ as follows:

$$(\alpha_i \mid h) := \varepsilon_i \langle \alpha_i, h \rangle (h \in \mathfrak{h}, 1 \le i \le n),$$

$(h_1'' \mid h_2'') := 0 \quad for\ h_1'', h_2'' \in \mathfrak{h}''$. Then $(\cdot|\cdot)$ is non-degenerate on $\mathfrak{h}$ and this induces the linear isomorphism, $\nu : \mathfrak{h} \to \mathfrak{h}^*$.

We completely identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via this map ν and omit the symbol ν in the following results. The proofs of these results are in[19].

Lemma 2.1.10:[19]

For $1 \le i, j \le n$, one has the following:

1. $\alpha_i^\vee = \varepsilon_i \alpha_i$,
2. $(\alpha_i \mid \alpha_j) = b_{ij}$,
3. $(\alpha_i^\vee \mid \alpha_j^\vee) = \varepsilon_i \varepsilon_j b_{ij}$,

4.If $a_{ii} \ne 0$, then $\dfrac{2(\alpha_i \mid \lambda)}{(\alpha_i \mid \alpha_i)} = \dfrac{2\langle \alpha_i^\vee, \lambda \rangle}{a_{ii}}, \lambda \in \mathfrak{h}^*$, in particular if $a_{ii} = 2$, then $\dfrac{2(\alpha_i \mid \lambda)}{(\alpha_i \mid \alpha_i)} = \langle \alpha_i^\vee, \lambda \rangle (\lambda \in \mathfrak{h}^*)$.

Remark:

For $i \in I^{im}, j \in I \Rightarrow (\alpha_i \mid \alpha_j) \le 0$. In terms of inner product $(\cdot|\cdot)$, we have $\Pi^{re} = \{\alpha_i; (\alpha_i \mid \alpha_i) > 0\}$ and $\Pi^{im} = \{\alpha_i; (\alpha_i \mid \alpha_i) \le 0\}$.

Definition 2.1.11:[17]

For each $i \in I^{re}$, we define the simple reflection $r_i \in \mathfrak{h}^*$ by

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \lambda \in \mathfrak{h}^*.$$

The Weyl group W of $\mathfrak{g}(A)$ is the subgroup of $GL(\mathfrak{h}^*)$ generated by the r_i's $(i \in I^{re})$. Note that $(W, \{r_i ; i \in I^{re}\})$ is a coxeter system. So for a real root $\alpha = w(\alpha_i) \quad (w \in W, \alpha_i \in \Pi^{re})$, we define the reflection r_α of $\mathfrak{h}^*$ with respect to α by

$$r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \ (\lambda \in \mathfrak{h}^*),$$

where $\alpha^\vee = w(\alpha_i^\vee) \in \mathfrak{h}$ is the dual real root of α. Note that $r_\alpha = w r_i w^{-1} \in W$.

Lemma 2.1.12:[19].

The bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ and $\mathfrak{h}^*$ is invariant under the action of the Weyl group.

In particular, we have $(\alpha_i \mid \alpha_j) = \varepsilon_i a_{ij}$ for $1 \le i, j \le n$.

Definition 2.1.13:[17]

The set of all real roots of a BKM Lie superalgebra is defined as

$$\Delta^{re} = W(\Pi^{re}) \cup W(\{2\alpha_i; i \in \psi^{re}\}).$$

Then the set of all imaginary roots is $\Delta^{im} = \Delta \setminus \Delta^{re}$. We have, $\alpha \in \Delta^{re} \Rightarrow -\alpha \in \Delta^{re}$ and $\alpha \in \Delta^{im} \Rightarrow -\alpha \in \Delta^{im}$.

Definition 2.1.14:[19]

Let $Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i, Q_+ = \sum_{i=1}^{n} \mathbb{Z}_{\ge 0}\alpha_i$, then we have $\Delta_+ \subset Q_+$. Q is called the root lattice and Q_+ is called the positive root lattice.The root lattice Q becomes a (partially) ordered set by putting $\alpha \ge \beta \Leftrightarrow \alpha - \beta \in Q_+$ for $\alpha, \beta \in Q$.

Now, for $\alpha = \sum m_i \alpha_i \in Q$, support of α is defined as $supp(\alpha) = \{i \in I; m_i \ne 0\}$. If $supp(\alpha)$ is a connected subset of the Dynkin diagram of A, we say that $supp(\alpha)$ is connected.

Definition 2.1.15:[19]

Imaginary roots of BKM superalgebras are basically of two types, domestic-type and alien-type.

Domestic-type imaginary root:

An imaginary root which is conjugate to a fundamental root under the action of the Weyl group is called domestic-type imaginary root. We denote by $\Delta^{dom \cdot im}$, the set of all domestic-type imaginary roots.

Alien-type imaginary root:

An imaginary root which is not a conjugate to a fundamental root under the action of the Weyl group is called alien-type imaginary root. We denote by $\Delta^{ali \cdot im}$, the set of all alien-type imaginary roots.

Since an imaginary root is either conjugate or not conjugate to a fundamental root under the action of Weyl group, each imaginary root is either domestic imaginary or alien imaginary.

We have $\Delta^{im} = \Delta^{dom.im.} \cup \Delta^{ali.im.}$.

Lemma 2.1.16:[19]

1. Δ_+^{im} is invariant under the action of the Weyl group.
2. If $\alpha \in \Delta_+^{im}$, then there exists $w \in W$ satisfying $w\alpha \in -C^\vee$.
3. For $\alpha \in \Delta_+, \alpha \in \Delta_+^{im} \Leftrightarrow (\alpha \mid \alpha) \le 0$.

Theorem 2.1.17:[19]

For a symmetrizable BKM supermatrix (A, ψ), if we set

$$\Delta^{re} := W(\Pi^{re}) \cup W(\{2\alpha_i; i \in \psi^{re}\}),$$

$$\Delta^{im} := \Delta \setminus \Delta^{re},$$
$$\Delta^{dom\cdot im} := W(\pm\Pi^{im}) \cup W(\{\pm 2\alpha_i; i \in \psi_-\}),$$
$$\Delta^{ali\cdot im} := \Delta^{im} \setminus \Delta^{dom\cdot im},$$

then concerning Δ^{im} and $\Delta^{ali\cdot im}$ we have the following results: 1. $\Delta_+^{ali\cdot im} = W(\overset{\circ}{K}) = \cup_{w\in W} w(\overset{\circ}{K})$.

2. $\Delta_+^{im} = W(\overset{\circ}{K}) \cup W(\Pi^{im}) \cup W(\{2\alpha_i; i \in \psi_-\})$.

where

$K = \{\alpha \in Q_+ \,|< \alpha, \alpha_i^\vee > \leq 0$ *and* $supp(\alpha)$ *is connected* $\}$

and $\overset{\circ}{K} = \{\alpha \in K \,\|\, supp(\alpha) | \geq 2\}$.

Also by notation $\mathfrak{h}_{\mathbb{R}}' = \sum_{i=1}^n \mathbb{R}\alpha_i^\vee = \sum_{i=1}^n \mathbb{R}\alpha_i$,

$C^\vee = \{h \in \mathfrak{h}_{\mathbb{R}}'; \langle \alpha_i^\vee, h\rangle \geq 0 (i \in I^{re})\}$ and

$$\overset{\circ}{C^\vee} = \{h \in \mathfrak{h}_{\mathbb{R}}'; \langle \alpha_i^\vee, h\rangle > 0 (i \in I^{re})\}.$$

Lemma 2.1.18:[19]

For $\alpha \in \Delta$ *and* $i \in I^{re}$, *one has the following:*

1. *The set* $\{j \in \mathbb{Z}; \alpha + j\alpha_i \in \Delta\}$ *is a finite set. Let* p *be the minimum contained in this set, and let* q *be the maximum in this set. Then,*

(a) $p + q = -\langle \alpha_i^\vee, \alpha\rangle$

(b) $\{j \in \mathbb{Z}; \alpha + j\alpha_i \in \Delta\} = \{j \in \mathbb{Z}; p \leq j \leq q\}$,

(c) *the sequence* $\{mult(\alpha + j\alpha_i)\}_{p\leq j\leq q}$ *is bilaterally symmetric, and the left half of this sequence is monotone nondecreasing. Namely,*

1. $j + k = p + q \Rightarrow mult(\alpha + j\alpha_i) = mult\,(\alpha + k\alpha_i)$

2. $p \leq j \leq k \leq \dfrac{p+q}{2} \Rightarrow mult\,(\alpha + j\alpha_i) \leq mult(\alpha + k\alpha_i)$.

3. $(\alpha|\alpha_i) > 0 \Rightarrow \alpha - \alpha_i \in \Delta_+$,

$(\alpha|\alpha_i) < 0 \Rightarrow \alpha + \alpha_i \in \Delta_+$.

4. $\alpha + \alpha_i \notin \Delta \Rightarrow (\alpha|\alpha_i) \geq 0$,

$\alpha - \alpha_i \notin \Delta \Rightarrow (\alpha|\alpha_i) \leq 0$.

Strictly domestic type imaginary roots, strictly alien type imaginary roots, strictly imaginary roots, purely imaginary roots, purely domestic type imaginary roots and purely alien imaginary root were already explained in[17]. We repeat the following definitions which we need here.

Definition 2.1.19:[`17]

A domestic-type imaginary root γ in a BKM super algebra is said to be *strictly domestic-type imaginary*, if for every $\alpha \in \Delta^{re}$, either $\alpha + \gamma$ or $\alpha - \gamma$ is a root. Let $\Delta^{s\cdot dom\cdot im}, \Delta_+^{s\cdot dom\cdot im}, \Delta_-^{s\cdot dom\cdot im}$ denote the set of all strictly domestic-type imaginary roots, positive and negative strictly domestic-type imaginary roots respectively.

Definition 2.1.20:[17]

An alien-type imaginary root γ in a BKM super algebra is said to be *strictly alien-type imaginary*, if for every $\alpha \in \Delta^{re}$, either $\alpha + \gamma$ or $\alpha - \gamma$ is a root. Let $\Delta^{s\cdot ali\cdot im}, \Delta_+^{s\cdot ali\cdot im}, \Delta_-^{s\cdot ali\cdot im}$ denote the set of all strictly alien-type imaginary roots, positive and negative strictly alien-type imaginary roots respectively.

Definition 2.1.21:[17]

An imaginary root γ in a BKM super algebra is said to be *strictly imaginary*, if for every $\alpha \in \Delta^{re}$, either $\alpha + \gamma$ or $\alpha - \gamma$ is a root. The set of all strictly imaginary roots is denoted by Δ^{sim}. Let $\Delta^{sim}, \Delta_+^{sim}, \Delta_-^{sim}$ denote the set of all strictly imaginary roots, positive and negative strictly imaginary roots respectively.

Remark:

As it was noticed in[5],

(1)If $\alpha \in \Delta_+^{sim}$, $\beta \in \Delta_+^{im}$, then $\alpha + \beta \in \Delta_+^{im}$

(2) Δ_+^{sim} is a semigroup.

Definition 2.1.22:[17]

A BKM super matrix (A, ψ) is said to have *strictly imaginary property*, if

$$\Delta_+^{sim}(A) = \Delta_+^{im}(A).$$

If a BKM supermatrix satisfies strictly imaginary property, we say that corresponding BKM Lie superalgebra satisfies strictly imaginary property.

Purely alien imaginary roots, purely domestic imaginary roots were already explained in Sthanumoorthy et al.(2009).

Definition 2.1.23:[17]

Let $\alpha \in \Delta_+^{im}$, we say that α is *purely imaginary*, if for any $\beta \in \Delta_+^{im}, \alpha + \beta \in \Delta_+^{im}$. We say that the BKM super algebra $\mathfrak{g}(A)$ has the *purely imaginary property*, if (A, ψ) satisfies this property. We have, $\Delta^{im} = \Delta^{dom\cdot im} \cup \Delta^{ali\cdot im}$.

Similarly we say that a negative root $\gamma \in \Delta_-^{im}$ is purely imaginary if $-\gamma$ is a purely imaginary root. Denote by

$$\Delta_+^{pim}(A) = \Delta_+^{pim} = \{\alpha \in \Delta_+^{im} \mid \alpha \text{ is } purely\ imaginary\}$$

and

$$\Delta_-^{pim}(A) = \Delta_-^{pim} = \{\alpha \in \Delta_-^{im} \mid \alpha \text{ is } purelyimaginary\}.$$

Then, the set of all purely imaginary roots is

$$\Delta^{pim} = \Delta_+^{pim} \cup \Delta_-^{pim}.$$

We omit the proof of the following theorem for BKM Lie superalgebras which can be directly verified using the proof for KM algebras already proved in[3].

Theorem 2.1.24:[17]

For BKM Lie superalgebras, the following results are true:

(a) *If* $\alpha \in \Delta_+$ *and* $\langle \alpha_i^\vee, \alpha \rangle < 0$ *for all* $i \in I^{re}$, *then* $\alpha \in \Delta_+^{s.ali.im}$.

(b) *If* $\alpha \in \Delta_+^{ali\cdot im}, r_\gamma(\alpha) \neq \alpha$ *for all* $\gamma \in \Delta^{re}$, *then* $\alpha \in \Delta_+^{s.ali.im}$.

(c) *If* $\alpha \in \Delta_+^{s.ali.im}$ *and* $\langle \alpha_i^\vee, \alpha \rangle \le 0$ *for all* $i \in I^{re}$, *then* $\alpha + \beta \in \Delta_+$ *for all* $\beta \in \Delta_+$.

(d) *If* $\alpha \in \Delta_+^{s.ali.im}, \beta \in \Delta_+^{ali\cdot im}$, *then* $\alpha + \beta \in \Delta_+^{ali\cdot im}$.

(e) $\Delta_+^{s.ali.im}$ *is a semigroup.*

In addition to the above results, we prove the following results for BKM Lie superalgebras.

Theorem 2.1.25:

a) If $\alpha \in \Delta^{dom.im} \setminus \{\alpha_i\}$ and $supp(\alpha + \alpha_i)$ is connected, then $\alpha \in \Delta^{im}$.

b) If $\alpha \in \Delta^{dom.im} \setminus \{\alpha_i\}$, $r_\gamma(\alpha) \neq \alpha$ for all $\gamma \in \Delta^{re}$, then $\alpha \in \Delta^{s.ali.im.}$

Proof:

a) Let $\alpha \in \Delta^{dom.im} \setminus \{\alpha_i\}$ and $supp(\alpha + \alpha_i)$ be connected. Then $\alpha + \alpha_i \in \Delta_+$.

So by lemma 2.1.18. $\alpha \in \Delta^{sim}$.

b) Let $\alpha \in \Delta^{dom.im} \setminus \{\alpha_i\}$. If $r_\gamma(\alpha) \neq \alpha$ for all $\alpha_\gamma \in \Delta^{re}$, then $\alpha - <\alpha, \alpha_\gamma> \alpha_\gamma \neq \alpha$ $\Rightarrow <\alpha, \alpha_\gamma> \neq 0$ $\Rightarrow$ $<\alpha, \alpha_\gamma> < 0$ or $<\alpha, \alpha_\gamma> > 0.$

So, by lemma 2.1.18., we have $\alpha + \alpha_\gamma \in \Delta_+$ or $\alpha - \alpha_\gamma \in \Delta_+$. Hence α is a strictly imaginary root.

Remark:

From the property (d) of the Theorem (2.1.24), we have

$$\Delta_+^{sim} + \Delta_+^{im} \subset \Delta_+^{im}.$$

3. Complete Classification of BKM Lie Superalgebras Possessing Strictly Imaginary Property

Remark:

In[17], a complete classification of BKM Lie superalgebras possessing purely imaginary property was given.

3.1. First we Give the following Results from[3].

Definition 3.1.1:[3]

We say that the generalized Cartan matrix A has the property SIM(more briefly: $A \in SIM$) if $\Delta^{sim}(A) = \Delta^{im}(A)$.

Definition 3.1.2:[3]

A is said to satisfy NC1, if there exists no subsets $S, T \subset \{1, \cdots, n\}$ such that $A|S$ is affine or indefinite type, and $A|S \cup T$ is decomposable.

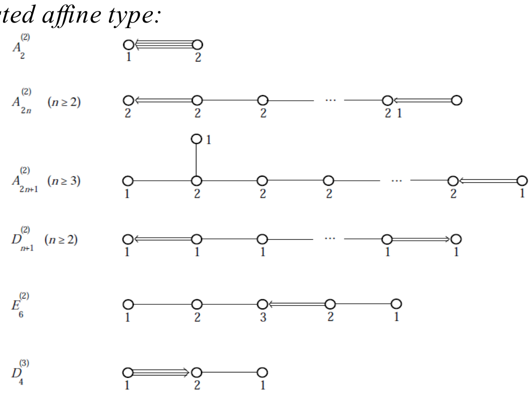
The following theorem proved by Casperson (1994) is for the indefinite Kac-Moody algebras possessing strictly imaginary property. Casperson (1994) gave a complete classification of Kac-Moody algebras possessing strictly imaginary property.

Theorem 3.1.3:[3]

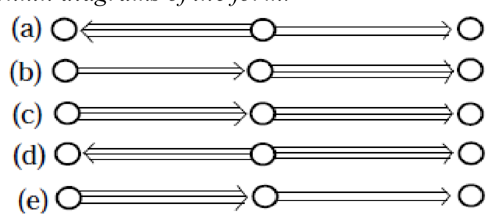
A GCM *lies in* SIM *if and only if it satisfies the condition* NC1 *and has no principal submatrix contained in the following list:*

1. *The* 2×2 *matrices of the form* $\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ *with* $ab \ge 4$ *and* $a = 1$ *or* $b = 1$.

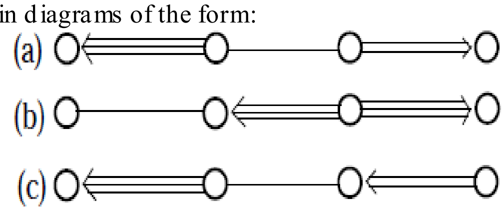
2. *The matrices of the following Dynkin diagrams of twisted affine type:*



3. *The strictly hyperbolic* 3×3 *matrices associated with the Dynkin diagrams of the form:*



4. The hyperbolic 4×4 matrices associated with the Dynkin diagrams of the form:



From[3], we can conclude the following for affine Kac-Moody algebras:

5. If an algebra is affine, we have that $\Delta^{sim} = \{nk\delta \mid n \in \mathbb{Z} \setminus \{0\}\}$ and hence $\Delta^{im} \setminus \Delta^{sim} = \{n\delta \mid n \in \mathbb{Z} \setminus k\mathbb{Z}\}$, where δ is the unique minimal positive imaginary root and k is the order of the diagram automorphism used to construct the algebra.

Again for the case of 2×2 matrices, the following theorem gives a complete classification of the non-strictly imaginary roots:

Theorem 3.1.4:[3]

Suppose, for the GCM $A=\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$, $(a,b\in \mathbb{Z}_+)$, *that* $\Gamma_1(A)=\{\gamma\in K_A:\gamma\pm\alpha_1\notin\Delta(A)\}\neq\varnothing.$

Then, either

1. $A=\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ and $\Gamma_1(A)=\{(2n+1)(\alpha_1+2\alpha_2)\mid n\geq 0\}$ (or)
2. $A=\begin{pmatrix} 2 & -1 \\ -b & 2 \end{pmatrix}$ where $b>4,$ and $\Gamma_1(A)=\{\alpha_1+2\alpha_2\}$.

Corollary 3.1.5:[3]

The GCM $\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ is not in SIM if and only if $ab\geq 4$ *and either* $a=1$ *or* $b=1$.

Proposition 3.1.6:[3]

A GCM of affine type is a member of SIM if and only if it is of non-twisted affine type.

Remarks:

From[17], the set of all strictly imaginary roots of any BKM superalgebra is a subset of set of all purely imaginary roots, that is , $\Delta^{pim}\setminus\Delta^{sim}$ may be an empty set or non-empty set depending upon the algebras. So, for the BKM superalgebras possessing purely imaginary property, we verify whether the set $\Delta^{pim}\setminus\Delta^{sim}$ is empty or not. In the case where $\Delta^{pim}\setminus\Delta^{sim}=\varnothing(\Delta^{pim}=\Delta^{sim})$ all purely imaginary roots are strictly imaginary as $\Delta^{sim}\subset\Delta^{pim}$ is always true. BKM superalgebras which satisfy the condition $\Delta^{pim}\setminus\Delta^{sim}=\varnothing$ will be in the class of BKM algebras possessing strictly imaginary property.Hence the condition $\Delta^{pim}=\Delta^{sim}$ is equivalent to $\Delta^{sim}=\Delta^{im}$, which is equivalent to SIM property.

3.2. As in the Cases of Special And Purely Imaginary Roots We Divide The Classes of BKM Superalgebras Into Two Categories. We Divide these BKM Lie Superalgebras into Two Categories.

Category 1: BKM Lie superalgebras **without odd roots**(GKM algebras only)

Category 2: BKM Lie superalgebras **with a non-empty set of odd roots:**

We discuss category 1 below.

Category 1: BKM Lie superalgebras **without odd root:**(GKM algebras only) Complete classification of GKM algebras possessing Strictly imaginary property was already given in[17].

Category 2: BKM Lie superalgebras with a non-empty set of odd roots:

We divide this category 2 into two classes, which are

Category2: Class(I): BKM Lie superalgebras of finite order Cartan matrices and with a non-empty set of odd roots

Category2: Class(II): BKM Lie superalgebras of infinite order Cartan matrices and with a non-empty set of odd roots

We discuss below these two classes separately .

Category 2:Class(I): BKM Lie superalgebras **with a non-empty set of odd roots**

We classify these BKM superalgebras into three subclasses (i),(ii) and (iii).

(i).BKM superalgebras with all simple roots being real **with a non-empty set of odd roots**:

These are BKM superalgebras which do not have any imaginary root. So this set of BKM superalgebras do not possess strictly imaginary property.

(ii).BKM superalgebras all whose simple roots are imaginary **with a non-empty set of odd roots:**

These are BKM superalgebras whose supermatrices do not appear as the extensions of KM matrices. So all the diagonal elements are negative. Hence there is no real simple root and all the roots are imaginary and also strictly imaginary.

(iii).BKM superalgebras with finite (non-zero) number of real simple roots and finite (non-zero) number of imaginary simple roots **with a non-empty set of odd roots:**

Remark:

Hereafter we denote by GGX, a Generalized Generalized Cartan matrix (BKM super matrix or BKM matrix). We prove the following theorem for this case.

Theorem 3.2.1:

Let $A=(a_{ij})_{i,j=1}^{n+r}$ *(the symmetrizable GGX)*

$$=\begin{pmatrix} -k_1 & -a_1 & -a_2 & \cdots & -a_{r-1} & -a_r & \cdots & \cdots & -a_{n+r-1} \\ -b_1 & -k_2 & -c_2 & \cdots & -c_{r-1} & -c_r & \cdots & \cdots & -c_{n+r-1} \\ -b_2 & -d_2 & -k_3 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & & \ddots & & & & & \\ -b_{r-1} & -d_{r-1} & \cdots & \cdots & \cdots & -k_r & x_{r+1} & \cdots & x_{n+r-1} \\ -b_r & -d_r & \vdots & \vdots & & w_r & & & \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & GX & \\ -b_{n+r-1} & -d_{n+r-1} & & & & w_{n+r-1} & & & \end{pmatrix}.$$

Here $k_i(1\leq i\leq r)\in\mathbb{Z}_{\geq 0}$, a_i,b_i,c_i,d_i are positive integers . Moreover, GX is KM matrix of finite, affine or indefinite type of order $n\geq 1$ and GGX is a supermatrix of finite, affine or indefinite type with r simple imaginary roots added to that of GX. Then the following results are true for BKM superalgebras with odd roots.

1. GX is of finite type:

(a) If $\alpha=\sum_{i=1}^{n+r}l_i\alpha_i\in W(\overset{\circ}{K})$ with $\sum_{i=1,i\neq j}^{n+r}l_i a_{ji}<2l_j$ (for all $j\in I^{re}$) is true for all $\alpha,$ then the corresponding BKM superalgebra satisfies SIM property.

(b)If $\alpha \in W(\Pi^{im} \setminus \psi_0) \cup W(\psi_0) \cup W\{2\alpha_i \mid i \in \psi^-\}$ $(\psi_0 = \{i \in \psi^{im} \mid k_i = 0\}\ and\ \psi^- = \{i \in \psi^{im} \mid k_i < 0\})$, with $b_i, d_i, ..., x_i, w_i > 1$ in the above GGX for $r+1 \le i \le n+r$ is true for all α, then the corresponding BKM superalgebra satisfies SIM property.

Here in (a) and (b), $\psi = \psi_0 \cup \psi^-$ is the set of all odd roots and $\Delta_+^{im} = \bigcup_{w \in W} w(\overset{\circ}{K}) \bigcup W(\Pi^{im}) \bigcup W\{2\alpha_i \mid \alpha_i \in \psi^-\}$ is the set of all positive imaginary roots.

2. GX is of untwisted affine type:

(a) If $\alpha = \sum_{i=1}^{n+r} l_i \alpha_i \in W(\overset{\circ}{K})$ with $\sum_{i=1, i\neq j}^{n+r} l_i a_{ji} < 2l_j$ (for all $j \in I^{re}$) is true for all α, then the corresponding BKM Lie superalgebra satisfies SIM property.

(b)If $\alpha \in W(\Pi^{im} \setminus \psi_0) \cup \psi_0 \cup W\{2\alpha_i \mid i \in \psi^-\}$ with $b_i, d_i, ..., x_i, w_i > 1$ in the above GGX for $r+1 \le i \le n+r$ is true for all α, then the corresponding BKM Lie superalgebra satisfies SIM property.

3. If GX is of twisted affine type, then Strictly imaginary property does not hold.

4. If GX is of indefinite type, then Strictly imaginary property does not always hold.

Proof:

In the usual notation, let $I = \{1,2,...,n+r\}$ with $I^{im} = \{1,2,...,r\}$ and $I = \{r+1, r+2, ..., n+r\}$. $\Pi = \{\alpha_1, \alpha_2, ..., \alpha_{n+r}\}$ is the set of all simple roots with $\Pi^{im} = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ is the set of all simple imaginary roots and $\Pi^{re} = \{\alpha_{r+1}, \alpha_{r+2}, ..., \alpha_{n+r}\}$ is the set of all simple real roots.

Ingeneral $< \alpha_i, \alpha_j >= a_{ij}, 1 \le i, j \le n+r$ and

$\psi = \{i \mid i \in I^{im}\}\ (or)$

$\{i \mid i \in I^{im}\} \cup \{j \mid a_{jk} \in 2\mathbb{Z}\ \forall\ k\ \ and\ \ j \in I^{re}\}$

$(or)\ \{j \mid a_{jk} \in 2\mathbb{Z}\ \forall\ k\ \ and\ \ j \in I^{re}\}$.

(1) Let GX be of finite type GCM and α be an positive imaginary root. Then

$$\alpha \in \Delta_+^{im} = \bigcup_{w \in W} w(\overset{\circ}{K}) \bigcup W(\Pi^{im}) \bigcup W\{2\alpha_i \mid \alpha_i \in \psi^-\}.$$

Here

$$\overset{\circ}{K} = \{\alpha \in Q_+ \mid < \alpha, \alpha_i^\vee > \le 0 (i \in I^{re})\ and\ |supp(\alpha)| > 2\}.$$

We discuss below Case(a), Case(b) and Case(c) separately.

Case(a): If $\alpha = \sum_{i=1}^{n+r} l_i \alpha_i \in W(\overset{\circ}{K})$ then we have

$$< \alpha, \alpha_j^\vee > = < \sum_{i=1}^{n+r} l_i \alpha_i, \alpha_j^\vee >, for\ r+1 \le j \le r+n$$

$$= \sum_{i=1}^{n+r} < l_i \alpha_i, \alpha_j^\vee > = \sum_{i=1}^{n+r} l_i a_{ji}.$$

By Theorem 2.1.24.(a), it is clear that if $\alpha \in \Delta_+^{im}$ and $\sum_{i=1}^{n+r} l_i a_{ji} < 0$, then Strictly imaginary property holds.

Case(b): If $\alpha \in W(\Pi^{im} \setminus \psi_0) \cup \psi_0$, then α can be written as

$$\alpha = \prod_{i=r+1}^{r+n} r_i(\alpha_i), for\ all\ i \in \Pi^{im}.$$

Here $r_{r+1}(\alpha_i) = \alpha_i - a_{r+1,i}\alpha_{r+1}$

$$r_{r+2} r_{r+1}(\alpha_i) = r_{r+2}(\alpha_i - a_{r+1,i}\alpha_{r+1})$$

$$= \alpha_i - \sum_{j=r+1}^{r+2} a_{j,i}\alpha_j + \prod_{j=r+1}^{r+2} a_{j,i} a_{j+1,j} \alpha_{r+2}$$

$r_{r+3} r_{r+2} r_{r+1}(\alpha_i)$

$$= \alpha_i - \sum_{j=r+1}^{r+3} a_{j,i}\alpha_j + (a_{r+1,i} \sum_{j=r+2}^{r+3} a_{j,r+1}\alpha_j + a_{r+2,i} a_{r+3,r+2} \alpha_{r+3})$$

$$+ \prod_{j=r+1}^{r+3} a_{j,i} a_{j+1,j} \alpha_{r+3}.$$

Finally,

$r_{r+n} r_{r+n-1} .... r_{r+3} r_{r+2} r_{r+1}(\alpha_i)$

$$= \alpha_i - \sum_{k=r+1}^{r+n} a_{ki}\alpha_k + \Bigg(a_{r+1,i} \sum_{k=r+2}^{r+n} a_{k,r+1}\alpha_k + a_{r+2,i} \sum_{k=r+3}^{r+n} a_{k,r+2}\alpha_k$$

$$+ ... + a_{r+n-1,i} a_{r+n,r+n-1} \alpha_{r+n} \Bigg) + ... + \prod_{j=r+1}^{n+r} a_{j,i} a_{j+1,j} \alpha_{r+n}.$$

We divide this case(b) into Case(b)(1), Case(b)(2) (Case(b)(2)(i),Case(b)(2)(ii)) and Case(b)(3) separately.

Case(b)(1): Let $\alpha = \alpha_i \in \Delta_+^{im} (n = 0)$.

We have $(\alpha_i \mid \alpha_j) = \varepsilon_i a_{ji}$ for $j \in I^{re}$.

Since a_{ji}'s are always negative integers and ε_i's are always positive, by theorem 2.1.25., it is clear that $\alpha_i \in \Delta^{sim}$.

Case(b)(2): Let $\alpha = r_{r+1}(\alpha_i)(n = 1)$.

Case(b)(2)(i): Let $\alpha = r_{r+1}(\alpha_i))(n = 1)$ with $r+1 \neq j$.

If $r_j(\alpha_i - a_{r+1,i}\alpha_j) \neq \alpha_i - a_{r+1,i}\alpha_j$, by the theorem 2.1.25, $\alpha_i - a_{r+1,i}\alpha_j \in \Delta^{sim}$.

On the contrary, if $r_j(\alpha_i - a_{r+1,i}\alpha_j) = \alpha_i - a_{r+1,i}\alpha_j$. Then

$$\alpha_i - a_{r+1,i}\alpha_i - a_{ji}\alpha_j + a_{r+1,i}a_{j,r+1}\alpha_j = \alpha_i - a_{r+1,i}\alpha_i.$$

$$\Rightarrow a_{ji} - a_{r+1,i}a_{j,r+1} = 0\,(or)\,\alpha_j = 0.$$

As α_j is a real simple root, $\alpha_j \neq 0$, $a_{ji} - a_{r+1,i}a_{j,r+1} = 0$. This is also not true, because $a_{ji}, a_{r+1,i}, a_{j,r+1}$ are negative integers. So by theorem 2.1.25., $\alpha_i - a_{r+1,i}\alpha_i \in \Delta^{sim}$.

Case(b)(2)(ii): Let $\alpha = r_{r+1}(\alpha_i)(n=1)$ with $r+1 = j$. Then

$$r_j(\alpha_i - a_{ji}\alpha_i) = \alpha_i - a_{ji}\alpha_j - a_{ji}\alpha_j + a_{ji}a_{jj}\alpha_j$$

$$= \alpha_i - 2a_{ji}\alpha_j + 2\alpha_j = \alpha_i.$$

But

$$r_j(\alpha_i - a_{ji}\alpha_j) = \alpha_i - a_{ji}\alpha_j.$$

$$\Rightarrow \alpha_i = \alpha_i - a_{ji}\alpha_j \Rightarrow a_{ji}\alpha_j = 0.$$

$$\Rightarrow a_{ji} = 0 \text{ or } \alpha_j = 0.$$

Here $\alpha_j \neq 0$ because α_j is a simple real root and $a_{ij} \neq 0$ for $j \in \psi^{re}$ with $a_{ji} \in 2Z$ and $i \neq j$. So $r_j(\alpha_i - a_{ji}\alpha_j) \neq \alpha_i - a_{ji}\alpha_j \Rightarrow r_j(\alpha) \neq \alpha$ with $|a_{ji}| > 1$. Hence by theorem 2.1.25., $\Rightarrow \alpha \in \Delta^{sim}$ if $|a_{ji}| > 1$.

Case(b)(3): Let

$$\alpha = r_{r+n}r_{r+n-1}....r_{r+3}r_{r+2}r_{r+1}(\alpha_i)(n>1).$$

We have

$$r_{r+n}r_{r+n-1}....r_{r+3}r_{r+2}r_{r+1}(\alpha_i)$$

$$= \alpha_i - \sum_{k=r+1}^{r+n} a_{ki}\alpha_k + a_{r+1,i}\sum_{k=r+2}^{r+n} a_{k,r+1}\alpha_k$$

$$+ a_{r+2,i}\sum_{k=r+3}^{r+n} a_{k,r+2}\alpha_{r+3} + ... + a_{r+n-1,i}a_{r+n,r+n-1}$$

$$+ .... + \prod_{j=r+1}^{r+n} a_{j,i}a_{j+1,j}\alpha_{n+r} \neq \alpha_i.$$

As all the a_{ji}'s are negative integers, by theorem 2.1.25.,

$$r_{r+n}r_{r+n-1}....r_{r+3}r_{r+2}r_{r+1}(\alpha_i) \in \Delta^{sim}.$$

Case(c): If $\alpha \in W\{2\alpha_i \mid i \in \psi^-\}$, then α can be written as

$$\alpha = \prod_{i=r+1}^{r+n} r_i(2\alpha_i), \textit{ for all } i \in \psi^-.$$

Here

$$r_{r+1}(2\alpha_i) = 2\alpha_i - 2a_{r+1,i}\alpha_{r+1}$$

$$r_{r+2}r_{r+1}(2\alpha_i) = r_{r+2}(2\alpha_i - 2a_{r+1,i}\alpha_{r+1})$$

$$= 2\alpha_i - \sum_{j=r+1}^{r+2} 2a_{j,i}\alpha_j + \prod_{j=r+1}^{r+2} 2a_{j,i}a_{j+1,j}\alpha_{r+2}$$

$$r_{r+3}r_{r+2}r_{r+1}(2\alpha_i)$$

$$= 2\alpha_i - \sum_{j=r+1}^{r+3} 2a_{j,i}\alpha_j + (2a_{r+1,i}\sum_{j=r+2}^{r+3} a_{j,r+1}\alpha_j + 2a_{r+2,i}a_{r+3,r+2}\alpha_{r+3})$$

$$+ \prod_{j=r+1}^{r+3} 2a_{j,i}a_{j+1,j}\alpha_{r+3}.$$

Finally,

$$r_{r+n}r_{r+n-1}....r_{r+3}r_{r+2}r_{r+1}(2\alpha_i)$$

$$= 2\alpha_i - \sum_{k=r+1}^{r+n} 2a_{ki}\alpha_k + \Bigg(2a_{r+1,i}\sum_{k=r+2}^{r+n} a_{k,r+1}\alpha_k +$$

$$2a_{r+2,i}\sum_{k=r+3}^{r+n} a_{k,r+2}\alpha_k + ... + 2a_{r+n-1,i}a_{r+n,r+n-1}\alpha_{r+n}\Bigg).$$

$$+ .... + \prod_{j=r+1}^{n+r} 2a_{j,i}a_{j+1,j}\alpha_{r+n}.$$

We discuss below Case(c)(1), Case(c)(2) and Case(3) separately in Case(c).

Case(c)(1): Let $\alpha = 2\alpha_i \in \Delta_+^{im}(n=0)$.

We have $(2\alpha_i \mid \alpha_j) = 2\varepsilon_i a_{ji}$ for $j \in I^{re}$.

Since a_{ji}'s are always negative integers and ε_i's are always positive, by theorem 2.1.25., it is clear that $2\alpha_i \in \Delta^{sim}$.

Case(c)(2): Let $\alpha = r_{r+1}(2\alpha_i)(n=1)$.

Case(c)(2)(i): Let $\alpha = r_{r+1}(2\alpha_i)(n=1)$ with $r+1 \neq j$.

If $r_j(2\alpha_i - 2a_{r+1,i}\alpha_j) \neq 2\alpha_i - 2a_{r+1,i}\alpha_j$, then $2\alpha_i - 2a_{r+1,i}\alpha_j \in \Delta^{sim}$.

We have,

$$r_j(2\alpha_i - 2a_{r+1,i}\alpha_j)$$

$$= 2\alpha_i - 2a_{r+1,i}\alpha_i - 2a_{ji}\alpha_j + 2a_{r+1,i}a_{j,r+1}\alpha_j.$$

But

$$2\alpha_i - 2a_{r+1,i}\alpha_i - 2a_{ji}\alpha_j + 2a_{r+1,i}a_{j,r+1}\alpha_j$$

$$= 2\alpha_i - 2a_{r+1,i}\alpha_i$$

$$\Rightarrow 2a_{ji} - 2a_{r+1,i}a_{j,r+1} = 0\,(or)\,2\alpha_j = 0.$$

As α_j is a real simple root with $\alpha_j \neq 0$, $a_{ji} - a_{r+1,i} a_{j,r+1} = 0$. This is not true, because $a_{ji}, a_{r+1,i}, a_{j,r+1}$ are negative integers. So by theorem 2.1.25., $\alpha_i - a_{r+1,i}\alpha_i \in \Delta^{sim}$.

Case(c)(2)(ii): Let $\alpha = r_{r+1}(2\alpha_i)(n=1)$ with $r+1 = j$. Then

$$\begin{aligned} & r_j(2\alpha_i - 2a_{ji}\alpha_i) \\ & = 2\alpha_i - 2a_{ji}\alpha_j - 2a_{ji}\alpha_j + 2a_{ji}a_{jj}\alpha_j \\ & = 2\alpha_i - 4a_{ji}\alpha_j + 4a_{ji}\alpha_j = 2\alpha_i. \end{aligned}$$

But

$$r_j(2\alpha_i - 2a_{ji}\alpha_j) = 2\alpha_i - 2a_{ji}\alpha_j$$

$$\Rightarrow 2\alpha_i = 2\alpha_i - 2ae_{ji}\alpha_j \Rightarrow 2a_{ji}\alpha_j = 0$$

$$\Rightarrow a_{ji} = 0 \text{ or } \alpha_j = 0.$$

Here $\alpha_j \neq 0$ because α_j is a simple real root and $a_{ij} \neq 0$ for $j \in \psi^{re}$ with $a_{ji} \in 2\mathbf{Z}$ and $i \neq j$. So $r_j(2\alpha_i - 2a_{ji}\alpha_j) \neq 2\alpha_i - 2a_{ji}\alpha_j$. $r_j(\alpha) \neq \alpha$ with $|a_{ji}| > 1$. By theorem 2.1.25., $\Rightarrow \alpha \in \Delta^{sim}$ if $|a_{ji}| > 1$.

Case(c)(3): Let $\alpha = r_{r+n} r_{r+n-1} \ldots r_{r+3} r_{r+2} r_{r+1}(2\alpha_i)(n>1)$. We have

$$\begin{aligned} & r_{r+n} r_{r+n-1} \ldots r_{r+3} r_{r+2} r_{r+1}(2\alpha_i) \\ & = 2\alpha_i - \sum_{k=r+1}^{r+n} a_{ki} 2\alpha_k + 2a_{r+1,i} \sum_{k=r+2}^{r+n} a_{k,r+1}\alpha_k \\ & + 2a_{r+2,i} \sum_{k=r+3}^{r+n} a_{k,r+2}\alpha_{r+3} + \ldots + 2a_{r+n-1,i} a_{r+n,r+n-1} \\ & + \ldots + \prod_{j=r+1}^{r+n} 2a_{j,i} a_{j+1,j} \alpha_{n+r} \neq 2\alpha_i. \end{aligned}$$

As all the a_{ji}'s are negative integers, by theorem 2.1.25., $r_{r+n} r_{r+n-1} \ldots r_{r+3} r_{r+2} r_{r+1}(2\alpha_i) \in \Delta^{sim}$.

(2) Let GX be of untwisted affine type. For KM algebras of untwisted affine type, SIM property holds as per Casperson(1994). For BKM algebras with odd roots which we get as extensions of KM algebras untwisted affine type, the proof is exactly same to case(1) and hence SIM property holds.

(3) Let GX be of twisted affine type. As per Casperson(1994) mentioned above, SIM property does not hold for KM algebras and the same is true for BKM Lie superalgebras which appear as extension of KM algebras of twisted affine type. Hence SIM property does not hold.

(4) Let GX be of indefinite type. As far as indefinite BKM Lie superalgebras are concerned, extension of finite and untwisted affine type of KM algebras will hold SIM property, where as other algebras do not hold.

The following example will illustrate the above theorem.

Example: Extension of finite type

Let $A = (a_{ij})_{i,j=1}^{n+r}$ (the symmetrizable GGX)

$$= \begin{pmatrix} -k & -a_1 & -a_2 \\ -b_1 & 2 & -1 \\ -b_2 & -1 & 2 \end{pmatrix}.$$

This is a BKM supermatrix of indefinite type denoted by $SBGA_2$, which is an extension of finite type A_2.

If $k > 0$, $a_1 b_1 > 4$ and $a_2 = b_2 = 2$, then the Dynkin diagram can be drawn as follows:

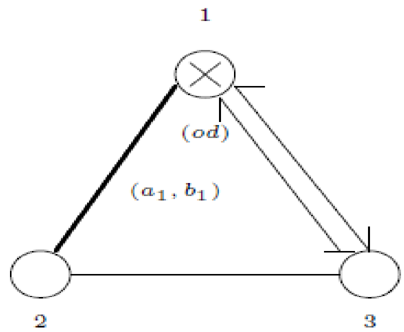


Figure 1. Dynkin diagram of $SBGA_2$

The Weyl group for corresponding BKM Lie superalgebra is $W = \{1, r_2, r_3, r_2r_3, r_3r_2, r_2r_3r_2\}$.

$$\Delta_+^{im} = \bigcup_{w \in W} W(\overset{\circ}{K}) \cup W(\Pi^{im}) \cup W\{2\alpha_i \mid i \in \psi^-\}.$$

Here

$$\bigcup_{w \in W} W(\overset{\circ}{K})$$

$$= \left\{ \begin{aligned} & p_1\alpha_1 + q_1\alpha_2 + r_1\alpha_3 \mid (p_1, q_1, r_1) = \\ & (k_1, k_2, k_3) \, or \, (k_1, k_1b_1 - k_2 + k_3, k_3), \\ & or \, (k_1, k_2, k_1b_2 + k_2 - k_3) \, or \\ & (k_1, k_1(b_2 + b_1) - k_3, k_1b_2 + k_2 - k_3) \, or \\ & or (k_1, k_1(b_1 + b_2) - k_3, k_1(b_1 + b_2) - k_2) \\ & with \ 2k_3 - k_1b_2 \le k_2 \le \frac{k_1b_1 + k_3}{2} \end{aligned} \right\}.$$

$$W(\Pi^{im})$$

$$= \left\{ \begin{aligned} & p_2\alpha_1 + q_2\alpha_2 + r_2\alpha_3 \mid (p_2, q_2, r_2) \\ & = (1,0,0) \, or \, (1, b_1, 0) \, or \, (1, 0, b_2) \, or \\ & (1, (b_1 + b_2), 1) \, or \, (1, (b_1 + b_2), (b_1 + b_2)) \\ & or \, (1, b_1, (b_1 + b_2)) \end{aligned} \right\}.$$

$$W\{2\alpha_i \mid i \in \psi^-\} = \begin{cases} p_3\alpha_1 + q_3\alpha_2 + r_3\alpha_3 \mid (p_3, q_3, r_3) \\ = (2,0,0) \ or \ (2,2b_1,0) \ or \ (2,0,2b_2) \ or \\ (2,2(b_1+b_2),2) \ or \ (2,2(b_1+b_2),2(b_1+b_2)) \\ or \ (2,2b_1,2(b_1+b_2)) \end{cases}.$$

Case(a): $\alpha \in W(\overset{\circ}{K})$. The following relations (i), (ii),...,(vii) can be directly verified.

(i)If $\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$ with $2k_3 - k_1b_2 \le k_2 \le \frac{k_1b_1 + k_3}{2}$, then

$$< \alpha, \alpha_2^\vee >= -k_1b_1 + 2k_2 - k_3 \ and$$
$$< \alpha, \alpha_3^\vee >= -b_2k_1 - k_2 + 2k_3.$$

(ii) If $\alpha = k_1\alpha_1 + (k_1b_1 - k_2 + k_3)\alpha_2 + k_3\alpha_3$, with $2k_3 - k_1b_2 \le k_2 \le \frac{k_1b_1 + k_3}{2}$, then

$$< \alpha, \alpha_2^\vee >= k_1b_1 - 2k_2 + k_3;$$
$$< \alpha, \alpha_3^\vee >= -(b_2 + b_1)k_1 + k_2 + k_3.$$

(iii) If $\alpha = k_1\alpha_1 + k_2\alpha_2 + (k_1b_2 + k_2 - k_3)\alpha_3$, with $2k_3 - k_1b_2 \le k_2 \le \frac{k_1b_1 + k_3}{2}$, then

$$< \alpha, \alpha_2^\vee >= -(b_1 + b_2)k_1 + k_2 + k_3;$$
$$< \alpha, \alpha_3^\vee >= k_1b_2 + k_2 - 2k_3.$$

(iv)If

$$\alpha = k_1\alpha_1 + (k_1(b_1 + b_2) - k_3)\alpha_2 + (k_1b_2 + k_2 - k_3)\alpha_3$$

with $2k_3 - k_1b_2 \le k_2 \le \frac{k_1b_1 + k_3}{2}$, then

$$< \alpha, \alpha_2^\vee >= (b_1 + b_2)k_1 - k_2 - k_3;$$
$$< \alpha, \alpha_3^\vee >= -k_1b_1 + 2k_2 - k_3.$$

(v)If

$$\alpha = k_1\alpha_1 + (k_1(b_1 + b_2) - k_3)\alpha_2 + (k_1(b_1 + b_2) - k_2)\alpha_3,$$

with $2k_3 - k_1b_2 \le k_2 \le \frac{k_1b_1 + k_3}{2}$, then

$$< \alpha, \alpha_2^\vee >= b_2k_1 + k_2 - 2k_3;$$
$$< \alpha, \alpha_3^\vee >= k_1b_1 - 2k_2 + k_3.$$

(vi)If

$$\alpha = k_1\alpha_1 + (k_1b_1 + k_2 + k_3)\alpha_2 + (k_1(b_1 + b_2) - k_2)\alpha_3$$

with $2k_3 - k_1b_2 \le k_2 \le \frac{k_1b_1 + k_3}{2}$, then

$$< \alpha, \alpha_2^\vee >= -b_2k_1 - k_2 + 2k_3;$$
$$< \alpha, \alpha_3^\vee >= k_1b_1 - 2k_2 + k_3.$$

By Theorem 2.1.24., and from the above results (i), (ii), (iii), (iv), (v) and (vi), it is clear that if $-k_1b_2 - k_2 < 2k_3$ and $-k_1b_1 - k_3 < 2k_2$, that is if (in general), $\sum_{i=1, i\ne j}^{n+r} l_i a_{ji} < 2l_j$, then $\alpha \in \Delta^{sim}$ for all $\alpha \in W(\overset{\circ}{K})$.

Case (b): Let $\alpha \in W(\Pi^{im})$. The following relations (i), (ii),...,(vi) can be easily verified.

(i)If $\alpha = \alpha_1 \in \Delta_+^{im}$, we get

$$r_2(\alpha_1) = \alpha_1 + b_1\alpha_2 \ \ and \ r_3(\alpha_1) = \alpha_1 + b_2\alpha_3.$$

By Theorem 2.1.25., if $b_1, b_2 > 1$, then $r_2(\alpha_1) \ne \alpha_1$ and $r_3(\alpha_1) \ne \alpha_1$ which implies $\alpha_1 \in \Delta^{sim}$.

(ii) If $\alpha = \alpha_1 + b_1\alpha_2 \in \Delta_+^{im}$, we get

$$r_2(\alpha_1 + b_1\alpha_2) = \alpha_1 \ \ and$$
$$r_3(\alpha_1 + b_1\alpha_2) = \alpha_1 + b_1\alpha_2 + (b_1 + b_2)\alpha_3.$$

(iii) If $\alpha = \alpha_1 + b_2\alpha_3 \in \Delta_+^{im}$, we get

$$r_2(\alpha_1 + b_2\alpha_3) = \alpha_1 + (b_1 + b_2)\alpha_2 + b_2\alpha_3$$
$$and \ r_3(\alpha_1 + b_2\alpha_3) = \alpha_1.$$

(iv) If $\alpha = \alpha_1 + (b_1 + b_2)\alpha_2 + \alpha_3 \in \Delta_+^{im}$, we get

$$r_2(\alpha_1 + (b_1 + b_2)\alpha_2 + \alpha_3) = \alpha_1 + (1 - b_2)\alpha_2 + \alpha_3 \ \ and$$
$$r_3(\alpha_1 + (b_1 + b_2)\alpha_2 + b_2\alpha_3) = \alpha_1 + (b_1 + b_2)\alpha_2 + (b_1 + 2b_2 - 1)\alpha_3.$$

(v) If $\alpha = \alpha_1 + (b_1 + b_2)\alpha_2 + (b_1 + b_2)\alpha_3 \in \Delta_+^{im}$, we get

$$r_2(\alpha_1 + (b_1 + b_2)\alpha_2 + (b_1 + b_2)\alpha_3)$$
$$= \alpha_1 + b_1\alpha_2 + (b_1 + b_2)\alpha_3 \text{ and}$$
$$r_3(\alpha_1 + (b_1 + b_2)\alpha_2 + (b_1 + b_2)\alpha_3)$$
$$= \alpha_1 + (b_1 + b_2)\alpha_2 + b_2\alpha_3.$$

(vi) If $\alpha = \alpha_1 + b_1\alpha_2 + (b_1 + b_2)\alpha_3 \in \Delta_+^{im}$, we get

$$r_2(\alpha_1 + b_1\alpha_2 + (b_1 + b_2)\alpha_3)$$
$$= \alpha_1 + (b_1 + b_2)\alpha_2 + (b_1 + b_2)\alpha_3 \ \ and$$
$$r_3(\alpha_1 + b_1\alpha_2 + (b_1 + b_2)\alpha_3) = \alpha_1 + b_1\alpha_2.$$

From the above results (ii), (iii), (iv), (v) and (vi) with $b_1, b_2 > 0$, it is clear that $r_\gamma(\alpha) \ne \alpha$ for $\alpha = \alpha_1 + b_1\alpha_2$ in (ii), $\alpha = \alpha_1 + b_2\alpha_3$ in (iii), $\alpha = \alpha_1 + (b_1 + b_2)\alpha_2 + \alpha_3$ in (iv), $\alpha = \alpha_1 + (b_1 + b_2)\alpha_2 + (b_1 + b_2)\alpha_3$ in (v), $\alpha = \alpha_1 + b_1\alpha_2 + (b_1 + b_2)\alpha_3$ in (vi). Hence, by Theorem 2.1.25., SIM property holds.

In general, if $b_1, b_2 > 1$ then $\alpha \in \Delta^{sim}$ for $\alpha \in W(\Pi^{im})$.

Case(c): Let $\alpha \in W\{2\alpha_i \mid i \in \psi^-\}$. The following relations (i),(ii),...,(vi) can be directly verified.

(i)If $\alpha = 2\alpha_1 \in \Delta_+^{im}$, we get

$$r_2(2\alpha_1) = 2\alpha_1 + 2b_1\alpha_2$$
$$and\ r_3(2\alpha_1) = 2\alpha_1 + 2b_2\alpha_3.$$

By Theorem 2.1.25., if $b_1, b_2 > 1$, then $r_2(2\alpha_1) \neq 2\alpha_1$ and $r_3(2\alpha_1) \neq 2\alpha_1$ which implies $2\alpha_1 \in \Delta^{sim}$.

(ii) If $\alpha = 2\alpha_1 + 2b_1\alpha_2 \in \Delta_+^{im}$, we get

$$r_2(2\alpha_1 + 2b_1\alpha_2) = 2\alpha_1\ and$$
$$r_3(2\alpha_1 + 2b_1\alpha_2) = 2\alpha_1 + 2b_1\alpha_2 + 2(b_1 + b_2)\alpha_3.$$

(iii) If $\alpha = 2\alpha_1 + 2b_2\alpha_3 \in \Delta_+^{im}$, we get

$$r_2(2\alpha_1 + 2b_2\alpha_3) = 2\alpha_1 + 2(b_1 + b_2)\alpha_2 + 2b_2\alpha_3$$
$$and\ r_3(2\alpha_1 + 2b_2\alpha_3) = 2\alpha_1.$$

(iv) If $\alpha = 2\alpha_1 + 2(b_1 + b_2)\alpha_2 + 2\alpha_3 \in \Delta_+^{im}$, we get

$$r_2(2\alpha_1 + 2(b_1 + b_2)\alpha_2 + 2\alpha_3)$$
$$= 2\alpha_1 + 2(1 - b_2)\alpha_2 + 2\alpha_3\ and$$
$$r_3(2\alpha_1 + 2(b_1 + b_2)\alpha_2 + 2b_2\alpha_3)$$
$$= 2\alpha_1 + 2(b_1 + b_2)\alpha_2 + 2(b_1 + 2b_2 - 1)\alpha_3.$$

(v)If $\alpha = 2\alpha_1 + 2(b_1 + b_2)\alpha_2 + 2(b_1 + b_2)\alpha_3 \in \Delta_+^{im}$, we get

$$r_2(2\alpha_1 + 2(b_1 + b_2)\alpha_2 + 2(b_1 + b_2)\alpha_3)$$
$$= 2\alpha_1 + 2b_1\alpha_2 + 2(b_1 + b_2)\alpha_3\ and$$
$$r_3(2\alpha_1 + 2(b_1 + b_2)\alpha_2 + 2(b_1 + b_2)\alpha_3)$$
$$= 2\alpha_1 + 2(b_1 + b_2)\alpha_2 + 2b_2\alpha_3.$$

(vi) If $\alpha = 2\alpha_1 + 2b_1\alpha_2 + 2(b_1 + b_2)\alpha_3 \in \Delta_+^{im}$, we get

$$r_2(2\alpha_1 + 2b_1\alpha_2 + 2(b_1 + b_2)\alpha_3)$$
$$= 2\alpha_1 + 2(b_1 + b_2)\alpha_2 + 2(b_1 + b_2)\alpha_3\ and$$
$$r_3(2\alpha_1 + 2b_1\alpha_2 + 2(b_1 + b_2)\alpha_3) = 2\alpha_1 + 2b_1\alpha_2.$$

From the above results (ii), (iii), (iv), (v) and (vi) with $b_1, b_2 \in 2\mathbb{Z}$, it is clear that $r_\gamma(\alpha) \neq \alpha$ for $\alpha = 2\alpha_1 + 2b_1\alpha_2$ in (ii), $\alpha = 2\alpha_1 + 2b_2\alpha_3$ in (iii), $\alpha = 2\alpha_1 + 2(b_1 + b_2)\alpha_2 + 2\alpha_3$ in (iv), $\alpha = 2\alpha_1 + 2(b_1 + b_2)\alpha_2 + 2(b_1 + b_2)\alpha_3$ in (v), $\alpha = 2\alpha_1 + 2b_1\alpha_2 + 2(b_1 + b_2)\alpha_3$ in (vi). Hence, by theorem 2.1.25., SIM property holds.

In general, for $\alpha \in W\{2\alpha_i \mid i \in \psi_-\}$ SIM property holds if $b_1, b_2 > 1$.

Example: Extension of untwisted affine type

Let $A = (a_{ij})_{i,j=1}^{n+r}$ (the symmetrizable GGX) $= \begin{pmatrix} -k & -a_1 & -a_2 \\ -b_1 & 2 & -2 \\ -b_2 & -2 & 2 \end{pmatrix}$. This is a BKM supermatrix of indefinite type denoted by $SBGA_1^{(1)}$, which is an extension of untwisted affine type $A_1^{(1)}$.

If k=0, $a_1b_1 > 4$ and $a_2b_2 > 4$, then the Dynkin diagram can be drawn as follows:

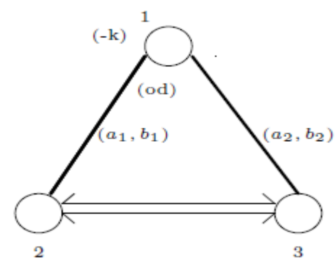


Figure 2. Dynkin diagram of $SBGA_1^{(1)}$

The Weyl group of the corresponding BKM Lie superalgebra is

$$W = \{1, r_2(r_3r_2)^j, r_3(r_2r_3)^j, (r_2r_3)^{j+1}, (r_3r_2)^{j+1} \mid j \in \mathbb{Z}_+\}.$$

Then

$$\Delta_+^{im} = \bigcup_{w \in W} W(\overset{\circ}{K}) \cup W(\Pi^{im}) \cup W\{2\alpha_i \mid i \in \psi^-\}.$$

Here

$$\bigcup_{w \in W} W(\overset{\circ}{K}) = \{\ p_1\alpha_1 + q_1\alpha_2 + r_1\alpha_3 \mid (p_1, q_1, r_1)$$
$$= (k_1, k_2, k_3) or (k_1, (k_1b_1 - k_2 + 2k_3), k_3)$$
$$or (k_1, k_2, (k_1b_2 + 2k_2 - k_3)) or (k_1, (k_1(b_1 + 2b_2)$$
$$+ 3k_2 - 2k_3), k_3) or$$
$$(k_1, (k_1b_1 - k_2 + 2k_3),$$
$$(k_1(2b_1 + b_2) - 2k_2 + 3k_3)), ...\ with$$
$$-k_1b_1 + 2k_2 - 2k_3 \le 0\ and\ -k_1b_2 - 2k_2 + 2k_3 \le 0\ \}.$$

$$W(\Pi^{im})$$
$$= \begin{cases} p_2\alpha_1 + q_2\alpha_2 + r_2\alpha_3 \mid (p_2, q_2, r_2) = \\ (1,0,0)\ or\ (1, b_1, 0)\ or\ (1, 0, b_2)\ or \end{cases}$$
$$(1, (b_1 + 2b_2), b_2)\ or\ (1, b_1, (b_2 + 2b_1))$$
$$or\ (1, (4b_1 + 2b_2), (b_2 + 2b_1))\ or$$
$$(1, (b_1 + 2b_2), 2(2b_2 + b_1))\ or$$
$$(1, (4b_1 + 2b_2), (4b_2 + 6b_1))\ or$$
$$(1, (7b_1 + 6b_2), (4b_2 + 6b_1))\ or$$
$$(1, (7b_1 + 6b_2), (7b_2 + 8b_1))\ or$$

$(1,1,(4b_1+6b_2))\ or\ ....\ \}.$

$$W\{2\alpha_i \mid i \in \psi_-\}$$
$$= \begin{cases} p_3\alpha_1 + q_3\alpha_2 + r_3\alpha_3 \mid (p_3, q_3, r_3) \\ = (2,0,0)\ or\ (2,2b_1,0) or\ (2,0,2b_2) or \end{cases}$$
$$(2,2(b_1+2b_2),2b_2)\ or\ (2,2b_1,2(b_2+2b_1))$$
$$or\ (2,2(4b_1+2b_2),2(b_2+2b_1))\ or$$
$$(2,2(b_1+2b_2),4(2b_2+b_1))\ or$$
$$(2,2(4b_1+2b_2),2(4b_2+6b_1))\ or$$
$$(2,2(7b_1+6b_2),2(4b_2+6b_1))\ or$$
$$(2,2(7b_1+6b_2),2(7b_2+8b_1))\ or$$
$$(2,2,2(4b_1+6b_2))\ or\ ....\ \}.$$

Case(a): Let $\alpha \in \cup_{w\in W} w(\overset{\circ}{K})$. The following relations (i),(ii), ...,(vi) can be easily verified.

(i) If $\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$ with $2k_2 \le 2k_3 + k_1b_1$ and $2k_2 \le k_1b_2 - 2k_3$, then

$$<\alpha, \alpha_2^\vee> = -k_1b_1 + 2k_2 - 2k_3;$$
$$<\alpha, \alpha_3^\vee> = -k_1b_2 - 2k_2 + 2k_3.$$

(ii) If $\alpha = k_1\alpha_1 + (k_1b_1 - k_2 + 2k_3)\alpha_2 + k_3\alpha_3$ with $2k_2 \le 2k_3 + k_1b_1$ and $2k_2 \le k_1b_2 - 2k_3$, then

$$<\alpha, \alpha_2^\vee> = k_1b_1 - 2k_2 + 2k_3;$$
$$<\alpha, \alpha_3^\vee> = -k_1b_2 - 2k_1b_1 - 2k_2 + 2k_3.$$

(iii) If $\alpha = k_1\alpha_1 + k_2\alpha_2 + (k_1b_2 + 2k_2 - k_3)\alpha_3$ with $2k_2 \le 2k_3 + k_1b_1$ and $2k_2 \le k_1b_2 - 2k_3$, then

$$<\alpha, \alpha_2^\vee> = -k_1b_1 - 2k_1b_2 - 2k_2 + 2k_3;$$
$$<\alpha, \alpha_3^\vee> = k_1b_2 + 2k_2 - 2k_3.$$

(iv) If $\alpha = k_1\alpha_1 + (k_1b_1 + 2k_1b_2 + 3k_2 - 2k_3)\alpha_2 + (k_1b_2 + 2k_2 - k_3)\alpha_3$ with $2k_2 \le 2k_3 + k_1b_1$ and $2k_2 \le k_1b_2 - 2k_3$, then

$$<\alpha, \alpha_2^\vee> = k_1b_1 + 2k_1b_2 + 2k_2 - 2k_3;$$
$$<\alpha, \alpha_3^\vee> = 3k_1b_2 - 2k_1b_1 - 2k_2 + 2k_3.$$

(v) If $\alpha = k_1\alpha_1 + (k_1b_1 - k_2 + 2k_3)\alpha_2 + (k_1b_2 + 2k_1b_1 - 2k_2 + 3k_3)\alpha_3$ with $2k_2 \le 2k_3 + k_1b_1$ and $2k_2 \le k_1b_2 - 2k_3$, then

$$<\alpha, \alpha_3^\vee> = k_1b_2 + 2k_1b_1 - 2k_2 + 2k_3;$$
$$<\alpha, \alpha_2^\vee> = 3k_1b_2 - 2k_1b_1 - 2k_2 + 2k_3.$$

(vi) If $\alpha = k_1\alpha_1 + (4k_3 - 3k_2 + 2k_1b_2 + 4k_1b_1)\alpha_2 + (k_1b_2 + 2k_1b_1 - 2k_2 + 3k_3)\alpha_3$ with $2k_2 \le 2k_3 + k_1b_1$ and $2k_2 \le k_1b_2 - 2k_3$, then

$$<\alpha, \alpha_3^\vee> = k_1b_2 + 2k_1b_1 - 2k_2 + 2k_3;$$
$$<\alpha, \alpha_2^\vee> = 3k_1b_2 - 2k_1b_1 - 2k_2 + 2k_3.$$

Similarly, we can find $<\alpha, \alpha_i^\vee>$ for different $\alpha \in \Delta^{im}$ and $i = 2,3$.

Hence by Theorem 2.1.24., and by the above results (i),(ii), ...,(vi) and others, it is clear that $\alpha \in \Delta^{sim}$ for all $\alpha \in \cup_{w\in W} w(\overset{\circ}{K})$. if $-k_1b_1 + 2k_2 - 2k_3 < 0$, $-k_1b_2 - 2k_2 + 2k_3 < 0$ and $-k_1b_2 - 2k_1b_1 - 2k_2 + 2k_3 < 0$. That is if $\sum_{i=1, i\ne j}^{3} l_i a_{ji} < 2l_j$, SIM property holds.

Case(b): Let $\alpha \in W(\Pi^{im}) \cup W\{2\alpha_i \mid i \in \psi^-\}$.

The following relations (i), (ii),...,(iv) can be easily verified.

(i) If $\alpha = \alpha_1 \in \Delta_+^{im}$, we get

$$r_2(\alpha_1) = \alpha_1 + b_1\alpha_2\ and\ r_3(\alpha_1) = \alpha_1 + b_2\alpha_3.$$

By Theorem 2.1.25., if $b_1, b_2 > 1$, then $r_2(\alpha_1) \ne \alpha_1\ and\ r_3(\alpha_1) \ne \alpha_1$ which implies $\alpha_1 \in \Delta^{sim}$.

(ii) If $\alpha = \alpha_1 + b_1\alpha_2 \in \Delta_+^{im}$, we get

$$r_2(\alpha_1 + b_1\alpha_2) = \alpha_1\ and$$
$$r_3(\alpha_1 + b_1\alpha_2) = \alpha_1 + b_1\alpha_2 + (2b_1 + b_2)\alpha_3.$$

(iii) If $\alpha = \alpha_1 + b_2\alpha_3 \in \Delta_+^{im}$, we get

$$r_2(\alpha_1 + b_2\alpha_3) = \alpha_1 + (b_1 + 2b_2)\alpha_2 + b_2\alpha_3$$
$$and\ r_3(\alpha_1 + b_2\alpha_3) = \alpha_1.$$

(iv) If $\alpha = \alpha_1 + (7b_1 + 6b_2)\alpha_2 + (8b_1 + 7b_2)\alpha_3 \in \Delta_+^{im}$, we get

$$r_2(\alpha_1 + (7b_1 + 6b_2)\alpha_2 + (8b_1 + 7b_2)\alpha_3) = \alpha_1 + (10b_1 + 8b_2)\alpha_2 + (8b_1 + 7b_2)\alpha_3$$
$$and\ r_3(\alpha_1 + (7b_1 + 6b_2)\alpha_2 + (8b_1 + 7b_2)\alpha_3) = \alpha_1 + (7b_1 + 6b_2)\alpha_2 + (6b_1 + 6b_2)\alpha_3.$$

Similarly, we can find $r_\gamma(\alpha)$ for $\alpha \in \Delta^{im}$ and $\gamma \in I^{re}$.

Hence by Theorem 2.1.25., and by the above results (ii), (iii) and others, with $b_1, b_2 > 0$, it is clear that $r_\gamma(\alpha) \ne \alpha$ for $\alpha = \alpha_1$, $\alpha = \alpha_1 + b_2\alpha_3$, $\alpha = \alpha_1 + b_1\alpha_2$, $\alpha = \alpha_1 + (7b_1 + 6b_2)\alpha_2 + (8b_1 + 7b_2)\alpha_3$,..... $\in W(\Pi^{im})$. In general, if $b_1, b_2 > 1$ we get $\alpha \in \Delta^{sim}$ for $\alpha \in W(\Pi^{im})$.

Case(C): If $\alpha \in W\{2\alpha_i \mid i \in \psi_-\}$. The following relations (i), (ii),...,(iv) can be easily verified.

(i)If $\alpha = 2\alpha_1 \in \Delta_+^{im}$, we get $r_2(2\alpha_1) = 2\alpha_1 + 2b_1\alpha_2$ and $r_3(\alpha_1) = 2\alpha_1 + 2b_2\alpha_3$.

By Theorem 2.1.25., if $b_1, b_2 > 1$, then $r_2(2\alpha_1) \neq 2\alpha_1$ *and* $r_3(2\alpha_1) \neq 2\alpha_1$ which implies $2\alpha_1 \in \Delta^{sim}$.

(ii) If $\alpha = 2\alpha_1 + 2b_1\alpha_2 \in \Delta_+^{im}$, we get

$r_2(2\alpha_1 + 2b_1\alpha_2) = 2\alpha_1$ *and*

$r_3(2\alpha_1 + 2b_1\alpha_2) = 2\alpha_1 + 2b_1\alpha_2 + 2(2b_1 + b_2)\alpha_3$.

(iii) If $\alpha = 2\alpha_1 + 2b_2\alpha_3 \in \Delta_+^{im}$, we get

$r_2(2\alpha_1 + 2b_2\alpha_3) = 2\alpha_1 + 2(b_1 + 2b_2)\alpha_2 + 2b_2\alpha_3$

and $r_3(2\alpha_1 + 2b_2\alpha_3) = 2\alpha_1$.

(iv)If

$\alpha = 2\alpha_1 + 2(7b_1 + 6b_2)\alpha_2 + 2(8b_1 + 7b_2)\alpha_3 \in \Delta_+^{im}$, we get

$$r_2(2\alpha_1 + 2(7b_1 + 6b_2)\alpha_2 + 2(8b_1 + 7b_2)\alpha_3) = 2\alpha_1 + 2(10b_1 + 8b_2)\alpha_2 + 2(8b_1 + 7b_2)\alpha_3 \text{ and}$$

$$r_3(2\alpha_1 + 2(7b_1 + 6b_2)\alpha_2 + 2(8b_1 + 7b_2)\alpha_3) = 2\alpha_1 + 2(7b_1 + 6b_2)\alpha_2 + 2(6b_1 + 6b_2)\alpha_3.$$

Similarly, we can find $r_\gamma(\alpha)$ for $\alpha \in \Delta^{im}$ and $\gamma \in I^{re}$.

Hence by Theorem 2.1.25 and by the above results (ii), (iii),...,(iv) and others, with $b_1, b_2 > 0$, it is clear that $r_\gamma(\alpha) \neq \alpha$ for $\alpha = 2\alpha_1$, $\alpha = 2\alpha_1 + 2b_2\alpha_3$, $\alpha = 2\alpha_1 + 2b_1\alpha_2$, ..., $\alpha = 2\alpha_1 + 2(7b_1 + 6b_2)\alpha_2 + 2(8b_1 + 7b_2)\alpha_3$, ... $\in W(\Pi^{im})$. In general, if $b_1, b_2 > 1$ we get $\alpha \in \Delta^{sim}$ for $\alpha \in W\{2\alpha_i \mid i \in \psi_-\}$.

Class(II): BKM Lie superalgebras of infinite order and with a finite non-empty set of odd roots

We divide this class into three subclasses.

(i) All simple roots are imaginary(odd or even)

(ii) One simple real root(odd or even) and infinite number of imaginary roots(odd or even)

(iii) Finite number of simple real roots and infinite number of imaginary roots.

We discuss these cases below

(i) All simple roots are imaginary(odd or even):

For this class, all the roots are imaginary. So these algebras satisfy strictly imaginary property.

(ii) One simple real root(odd or even) and infinite number of imaginary roots(odd or even):

We prove the following theorem for this case.

Theorem 3.2.2: *Let* $A = (a_{ij})_{i,j=1}^{\infty}$ *(the symmetrizable GGX)*

$$= \begin{pmatrix} 2 & -a_1 & -a_2 & \cdots & -a_{r-1} & -a_r & \cdots & \cdots \\ -b_1 & -k_1 & -c_2 & \cdots & -c_{r-1} & -c_r & \cdots & \cdots \\ -b_2 & -d_2 & -k_2 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & & \ddots & & & & \\ -b_r & -d_r & \cdots & \cdots & -k_r & \cdots & \cdots & \cdots \\ \vdots & \vdots & & & \vdots & \ddots & & \\ \vdots & \vdots & & & \vdots & & \ddots & \end{pmatrix}.$$

Here $k_i (i \geq 2) \in \mathbb{Z}_{\geq 0}$, a_i, b_i, c_i, d_i are positive integers and GGX is the BKM supermatrix with one real simple root and infinite number of imaginary roots. If $\alpha = \sum_{i=1}^{\infty} l_i \alpha_i$ with $\sum_{i=2}^{\infty} l_i a_{1i} + 2l_1 < 0$ is true for all α, then the corresponding BKM Lie superalgebra satisfies Strictly imaginary property.

Proof:

In the usual notation $I = \{1,2,3...\}$ with $I^{re} = \{1\}$ and $I^{im} = \{i \in I \mid i \geq 2\}$. $\Pi = \{\alpha_1, \alpha_2, \alpha_3, ...\}$, with $\Pi^{re} = \{\alpha_1\}$ and $\Pi^{im} = \{\alpha_i \mid i \in I^{im}\}$. We define $\psi \in \mathbb{N}$ and

$\psi = \{1\}$ (or) $\psi = \{i \geq 2 \mid i \in I^{im}\}$ (or)

$\psi = \{1\} \cup \{i \geq 2 \mid i \in I^{im}\}$.

Let $\alpha = \sum_{i=1}^{\infty} l_i \alpha_i$ and $\beta = \alpha_1$. Then by Theorem 2.1.24.,

$$< \alpha, \alpha_1^{\vee} > = < \sum_{i=1}^{n+r} l_i \alpha_i, \alpha_1^{\vee} > = \sum_{i=1}^{n+r} < l_i \alpha_i, \alpha_1^{\vee} > = \sum_{i=1}^{n+r} l_i a_{1i}$$

If $\sum_{i=1}^{n+r} l_i a_{1i} < 0$, which is same as $\sum_{i=2}^{\infty} l_i a_{1i} + 2l_1 < 0$ then Strictly imaginary property holds.

Remark:

For BKM superalgebras which appear as extension of twisted affine type(Case 3) and extension of indefinite type(Case 4), examples were given in[17]($\Delta^{sim} \setminus \Delta^{pim} = null$ set)(Section 4, Case 3, subcase 2).

Remarks:

We have seen above that in the case of BKM Lie superalgebras of infinite order with one simple real root (odd or even) and infinite number of imaginary roots (odd or even) for $\psi \subseteq \{i \geq 2 \mid i \in I^{im}\}$, SIM property holds only when $| a_{1i} |$ and $| a_{i1} |$ are all greater than one. As a counter example, we consider Monster Lie superalgebra with one simple real root and infinite number of simple imaginary roots(odd or even). Consider $\alpha = \sum_{n=1}^{\infty} l_i \alpha_i$ with

$\sum_{i=2}^{\infty} l_i a_{1i} + 2l_1 < 0$ and $a_{1i} = 0$ for $2 \le i \le c(1)+1$ (c(1): multiplicity of the root corresponding to -2). As $a_{1i} = 0$ ($2 \le i \le c(1)+1$), the corresponding BKM Lie superalgebra does not satisfy SIM property. We prove this below. Consider Monster Lie superalgebra which has the following supermatrix as defined below:

Let $I = \{-1\} \cup \{1,2,3,.....\}$ be an index set and consider the Borcherds-Cartan super matrix $A = (-(i+j))_{i,j \in I}$ with charge $\underline{m} = (c(i) \mid i \in I)$, where $c(i)$ are the coefficients of the elliptic modular function

$$j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + ...$$
$$= \sum_{n=-1}^{\infty} c(n) q^n .$$

Here

$$A = (-(i+j))_{i,j \in I} = \begin{pmatrix} 2 & 0 & \cdots & \cdots & 0 & -1 & \cdots & \cdots \\ 0 & -2 & \cdots & \cdots & \cdots & -3 & \cdots & \cdots \\ \vdots & \vdots & \ddots & & & & & \\ 0 & \cdots & \cdots & \cdots & -2 & -3 & \cdots & \cdots \\ -1 & -3 & \cdots & \cdots & -3 & -4 & \cdots & \cdots \\ \vdots & \vdots & & & \vdots & & \ddots & \\ \vdots & \vdots & & & \vdots & & & \ddots \end{pmatrix},$$

is the BKM supermatrix and $I = \{-1\} \cup \{1,2,3.....\}$. We define $\psi \subseteq \{i \in I \mid i \ge 2\}$. α_{-1} is the real root corresponding to the diagonal element 2 and $\alpha_i (2 \le i \le c(1)+1)$ are the imaginary roots corresponding to the diagonal element -2

We consider any $\alpha_i,\ for\ 2 \le i \le c(1)+1,$ then $< \alpha_i, \alpha_{-1}^{\vee} >= a_{-1i} = 0.$

This implies α_i does not satisfy the strictly imaginary property for $2 \le i \le c(1)+1$. Hence Strictly imaginary property does not hold for a Monster Lie superalgebra.

(iii) Finite number(atleast two) of simple real roots and infinite number of imaginary roots:

We prove the following theorem for this case.

Theorem 3.2.3:

Let $A = (a_{ij})_{i,j=1}^{\infty}$ *(the symmetrizable GGX)*

$$= \begin{pmatrix} 2 & -a_1' & -a_2' & \cdots & -a_r & \cdots & \cdots & \cdots & \cdots \\ -b_{1'} & 2 & -c_1' & \cdots & -c_r & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -b_{r-1}' & -d_r' & 2 & -a_1 & -a_2 & \cdots & -a_{r-1} & -a_r & \cdots \\ -b_{r'} & -d_{r+1}' & -b_1 & -k_1 & -c_2 & \cdots & -c_{r-1} & -c_r & \cdots \\ -b_{r+1}' & -d_{r+2} & -b_2 & -d_2 & -k_2 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -d_{2r} & -d_{2r} & -b_r & -d_r & \cdots & \cdots & -k_r & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Here $k_i \in \mathbb{Z}_{\ge 0}$, $a_i, b_i, c_i, d_i, a_{i'}, b_{i'}, c_{i'}, d_{i'}$ are positive integers and GGX is the BKM supermatrix with $r (r \ge 2)$ real simple roots and infinite number of imaginary roots. If $\alpha = \sum_{i=1}^{\infty} l_i \alpha_i$ with $\sum_{i=r+1}^{\infty} l_i a_{1i} + 2l_1 < 0$ is true for all α, then the corresponding BKM Lie superalgebra satisfies Strictly imaginary property.

Proof:

In the usual notations $I = \{1,2,3...\}$ with $I^{re} = \{1,2,...,r\}$ ($r \ge 2$) and $I^{im} = \{i \mid i \ge r+1\} (i \ge 3).$ Here, $\Pi = \{\alpha_1, \alpha_2, \alpha_3, ...\}$, the set of all simple roots with $\Pi^{re} = \{\alpha_1, \alpha_2, ..., \alpha_r\}$, the set of simple real root and $\Pi^{im} = \{\alpha_i \mid i \ge r+1\}$, the set of all simple imaginary roots.

We define $\psi \in \mathbb{N}$ and $\psi = \{i \mid i \in I^{re}\}\ (or)\ \psi = \{j \mid j \in I^{im}\}\ (or)$ $\psi = \{i \mid i \in I^{re}\} \cup \{j \mid j \in I^{im}\}.$

Let $\alpha = \sum_{i=1}^{\infty} l_i \alpha_i$ and $\{\alpha_1, \alpha_2, ..., \alpha_r\} \in \Pi^{re}.$

$$< \alpha, \alpha_j^{\vee} >=< \sum_{i=1}^{\infty} l_i \alpha_i, \alpha_j^{\vee} >= \sum_{i=1}^{\infty} l_i a_{ji},\ \forall j \in I^{re}$$

As all $a_{ji}, i \ne j$ are negative integers and $a_{jj} = 2,$ we have $\sum_{i=1}^{\infty} l_i a_{ji} < 0.$ Hence $< \alpha_i, \alpha_j^{\vee} >< 0$ and SIM property holds.

Remarks:

As in the case of Monster Lie superalgebra with one simple real root and infinite number of imaginary simple roots with the condition $a_{1i} = a_{i1} = 0,$ we can consider BKM Lie superalgebras with two simple real roots and infinite number of imaginary simple roots with the condition,

$a_{ji} = a_{ij} = 0\ for\ some\ i\ and\ for\ j = 1,2.$

In this case, as $< \alpha_i, \alpha_j^{\vee} >= a_{ji} = 0\ \ for\ all\ j \in I^{re}$ and for some $i \in I^{im}$, (similar to Monster Lie superalgebra as in theorem 3.2.3.) SIM property does not hold . Hence we understand that for the infinite order case, the SIM property depends on the non-diagonal non-zero entries of the corresponding BKM supermatrix.

4. Conclusions

In this paper, a complete classification of Borcherds Kac-Moody Lie superalgebras possessing strictly imaginary property is given. From this classification, one can understand that strictly imaginary property depends mainly on the coefficients of the corresponding BKM supermatrix. With these findings, different complete classifications of Borcherds Kac-Moody Lie superalgebras possessing special imaginary roots, purely imaginary roots and strictly imaginary roots were separately found out in different research papers. In fact, these classifications will be very much helpful to the researchers to extend these classes of root systems to other types of finite and infinite dimensional Lie (super)algebras. Moreover, other characteristics of these classes of Borcherds Kac-Moody Lie superalgebras possessing these root systems can also be studied. These findings may also lead to many other applications.

ACKNOWLEDGEMENTS

The research has been financially supported by the University Grants Commission (UGC), Govt. of India, through the Major research Project, F.No.36-272/2008(SR). The authors, N.Sthanumoorthy (Principal Investigator of the Project) and K.Priyadharsini(Project Fellow for the Project) are thankful to the UGC for the same.

REFERENCES

[1] Bennett, C. Imaginary Roots of a Kac-Moody algebra whose reflections preserve root multiplicities, J. Algebra, vol. 158, pp. 244-267, 1993.

[2] Borcherds, R.E. Generalized Kac-Moody algebras, J. Algebra vol.115, pp.501-512, 1988.

[3] Casperson, D. Strictly imaginary roots of Kac-Moody algebrs, J. Algebra, vol 168,pp 90-122, 1994.

[4] Kac, V.G., Lie superalgebras .Adv.Math., vol.26, pp . 8-96, 1977.

[5] Kac, V.G., Infinite dimensional algebras, Dedekind's η -function, classical Mobius function and the very strange formula, Adv. Math, vol 30, pp 85-136, 1978.

[6] Kac, V.G., Infinite dimensional Lie algebra. 3rd ed., Cambridge University Press, Cambridge, 1990.

[7] Li,Z.-H., Jin,Y.-D., Bai,R.-P., Relationship between reflections determined by imaginary roots and the Weyl group for a special GKM algebra. J. Math. Res. Exp. Vol.19(3), pp. 501- 507, 1999.

[8] Marotta, V., Sciarrino, A., Realization of Borcherds algebras. Int. J. Mod. Phys. A vol.10(27), pp. 3921-3936, 1995.

[9] Satoshi, N., Kazhdan - Lusztig conjecture for generalized Kac- Moody algebras II. Proof of the conjecture. Trans. Amer. Math. Soc., vol. 347(10), pp 3891-3919, 1995.

[10] Sthanumoorthy, N., Uma Maheswari, A Purely imaginary roots of Kac-Moody algebras. Comm. Algebra, vol. 24(2), pp. 677- 693, 1996.

[11] Sthanumoorthy, N., Lilly, P.L., On the root system of generalized Kac-Moody algebras, J.Madras University (WMY- 2000 Special Issue) Section B : Sciences, vol. 52, pp. 81-103, 2000.

[12] Sthanumoorthy, N., Lilly, P.L., Special imaginary roots of generalized Kac-Moody algebras, Comm. Algebra, vol 30(10), pp. 4771-4787, 2002.

[13] Sthanumoorthy, N., Lilly, P.L., A note on purely imaginary roots of generalized Kac-Moody algebras, Comm. Algebra, vol. 31(11), pp 5467-5479, 2003.

[14] Sthanumoorthy, N., Lilly, P.L.,On some classess of root systems of generalized Kac-Moody algebras. Contemporary Mathematics,AMS, vol.343, pp. 289-313, 2004.

[15] Sthanumoorthy,N. and Lilly,P.L., Complete classifications of generalized Kac-Moody algebras possessing special imaginary roots and strictly imaginary property, Comm. Algebra, vol. 35, pp .2450-2471, 2007.

[16] Sthanumoorthy,N., Lilly,P.L. and Nazeer Basha,A., Special imaginary roots of BKM Lie superalgebras Inter. Jour. Pure and applied Maths, vol 38(4), pp.513-542, 2007.

[17] Sthanumoorthy,N., Lilly,P.L. and Nazeer Basha,A.Strictly imaginary roots and purely imaginary roots of BKM Lie superalgebras,Comm. Algebra, vol. 37(7), pp. 2357-2390, 2007.

[18] Sthanumoorthy,N.,Priyadharsini,K., Complete classification of BKM Lie superalgebras possessing special imaginary roots,accepted for publication in Comm. Algebra(USA)(36 pages)- to appear 2012.

[19] M.Wakimoto, Infinite dimensional Lie algebras, Originally published in Japanese by Iwanami Publishers, Tokyo, 1999. Translated from the Japanese by Kenji Iohara.