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Rational Approximation on Closed Curves

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Abstract In this paper, we study a problem of approximation for the classes of functions determined only on the boundary of domain in weighted integral spaces by means of the rational functions of the form (1) where b is a point lying strictly inside the considered curve. Notice that the approximation estimations, generally speaking, coincide with the estimations of polynomial approximation for E_p classes (Smirnov's class).

Keywords Rational Approximation, Conformal Map, Smoothness Modulus

1. Introduction

Approximation problem for the classes of functions determined only on the boundary of domain is of great importance alongside with the study of approximation of functions by means of polynomials analytic in the domain G and with some conditions on the boundary Γ. Obviously, it is impossible in general to approximate such classes of functions by means of polynomials[12]. Therefore, various kinds of rational functions or so called generalized polynomials are mostly used in this case as an approximation tool[12]. J. I. Mamedkhanov, D. M. Israfilov and I. M. Botchaev investigated the approximation problems of functions determined only on the boundary of domain by means of rational functions of the form $R_n(z)=P_n(z,1/z)$ for certain classes of curves in terms of uniform metric[1-4].

In this paper, we study the approximation problems of a function from the class $L_p(\Gamma,\vartheta)$ by means of a rational function of the form

$$R_n(z)=\sum_{k=-n}^{n} a_k(z-b)^k \tag{1}$$

where b is a point lying strictly inside the considered curve Γ [1]. Without loss of generality, we will assume $b=0$ throughout this paper.

2. Basic Definitions and Notations

1. Let $\vartheta(z)$ be an almost everywhere finite, non-zero function measurable on Γ. If the function $f(z)$ determined on Γ is measurable, and the function $|\vartheta(z)f(z)|^p$ the class $L_p(\Gamma,\vartheta)$. If we define the norm in the class $L_p(\Gamma,\vartheta)$ $(p\ge 1)$ as $\|f\|_{L_p(\Gamma,\vartheta)}=\left\{\int_\Gamma |\vartheta(z)f(z)|^p|dz|\right\}^{1/p}$, then is integrable on Γ, then we will say that $f(z)$ belongs to $L(\Gamma,\vartheta)$ becomes a Banach space $L_p(\Gamma,1)=L_p(\Gamma)$.

2. By $(S_\Gamma f)(t)=\frac{1}{\pi i}\int_\Gamma \frac{f(z)}{z-t}dz,\ t\in\Gamma$ we denote the Cauchy singular integral.

3. Denote by R_p a totality of curves Γ for which $\forall f\in L_p(\Gamma)$ $(p>1)$, $\|Sf\|_{L_p(\Gamma)}\le C_p\cdot\|f\|_{L_p(\Gamma)}$, where C_p is depending on the point p.

Notice that the classes R and R_p $(p>1)$ are equivalent[5].

4. Let $u(t)$ be a measurable, almost everywhere finite function on Γ. We will say that $u(t)$ belongs to the class $W_p(u)$ if it differs from zero almost everywhere and the operator S_Γ is bounded in the space $L_p(\Gamma,u)$. Notice that this class is well studied in the theory of singular operators[5].

5. We will say that the closed curve Γ belongs to the class S_θ $(\Gamma\in S_\theta)$ if $\forall t\in\Gamma,\ \theta_t(\delta,\Gamma)\le M(\Gamma)\delta$, where $\theta_t(\delta,\Gamma)=mes\{z\in\Gamma:|z-t|<\delta\},\ 0<\delta\le diam\,\Gamma$ [2].

6. We will say that the curve Γ belongs to the class of K-curves $(\Gamma\in K)$ if the length of the greatest of the arches connecting two arbitrary points z_1 and z_2 on it has the same order as that of the chord connecting these points[10-11].

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[1] By G^+ and G^- we will denote an interior and an exterior of the curve Γ, respectively.

[2] M denotes various positive constants depending only on explicit parameters. In addition, we will use the notations $f\preccurlyeq g$, if $f\le Mg$ and $f\asymp g$, if $M_1g\le f\le M_2g$.

Obviously, $K \subset S_\theta$.

A lot of works have been devoted to the class of curves R over the past years[5]. Finally, G. David[6] proved that $R \subset S_\theta$.

7. By $w=\varphi(z)$ $\left(w=\tilde{\varphi}(z)\right)$ we will denote a function that conformally and univalently maps the exterior (interior) of the curve Γ onto the exterior (interior) of a unit circle γ_0 normalized by the conditions:

$$\varphi(\infty)=\infty, \quad \lim_{z\to\infty}\frac{\varphi(z)}{z}>0,$$

and let $z=\psi(w)$ $z=\tilde{\varphi}(w)$ denote the inverse to function to $\varphi(t)$ $\left(\tilde{\varphi}(z)\right)$. $\Gamma_{1+\sigma}\overset{df}{=}\left\{t:\left|\varphi(t)\right|=1+\sigma>1\right\}$ is a level line of the curve Γ ; $d(z,\sigma)\overset{df}{=}\inf_{t\in\Gamma_{1+\sigma}}|z-t|$ for $z\in\Gamma$; $\tilde{d}(t,\sigma)\overset{df}{=}\inf_{z\in}|z-t|$ for $t\in\Gamma_{1+\sigma}$; $d(\sigma)=\inf_{z\in} d(z,\sigma)$.

8. Let's consider the following quantities (see[7]):

$$u_{p,\vartheta}^{(2)}(\delta)=u_{p,\vartheta}^{(2)}(f,\delta)_\Gamma=\sup_{|h|\le\delta}\left\|f(z_h)-2f(z)+f(z_{-h})\right\|_{L_p(\Gamma,u)},$$

$$\tilde{u}_{p,\vartheta}^{(2)}(\delta)=\tilde{u}_{p,\vartheta}(f,\delta)_\Gamma=\sup_{|h|\le\delta}\left\|f(\tilde{z}_h)-2f(z)+f(\tilde{z}_{-h})\right\|_{L_p(\Gamma,u)},$$

where $z_{\pm h}=\psi\left(\varphi(z)e^{\pm ih}\right)$, $\tilde{z}_{\pm h}=\tilde{\psi}\left(\tilde{\varphi}(z)e^{\pm ih}\right)$.

Obviously, these quantities satisfy all the properties of smoothness modulus on good classes of curves, in particular, on smooth or piecewise-smooth curves. In case of more general classes of curves we will consider the quantities

$$\omega_{p,u}^{(2)}(f,\delta)=\delta^{(2)}\sup_{t\ge\delta} t^{-2}u_{p,\vartheta}^{(2)}(f,t)_\Gamma$$

and

$$\tilde{\omega}_{p,u}^{(2)}(f,\delta)_\Gamma=\delta^2\sup_{t\ge\delta}t^{-2}\,\tilde{u}_{p,\vartheta}^{(2)}(f,\delta)_\Gamma.$$

Proceeding in the same way as in[7], it is easy to see that these quantities satisfy all the properties of smoothness modulus (i.e. $\omega(0)=0,\ \omega(t)\in C,\ \omega(t)\uparrow,$ $\omega(\lambda t)\le(\lambda+1)^2\,\omega(t),\ \forall\lambda>0$) on any curves Γ and they are the best majorants for the functions $u_{p,\vartheta}^{(2)}(f,\delta)_\Gamma$ and $\tilde{u}_{p,\vartheta}^{(2)}(f,\delta)_\Gamma$, respectively, among all smoothness modulus type functions.

3. Main Result

Now we prove the following:

Theorem. Let $u\in W_p(\Gamma)(p>1)$ and $f\in L_p(\Gamma,u),\ p>1$. Then, for every positive integer n there exists a rational function $R_n(z)$ of the form (1) such that

$$\left\|f-R_n\right\|_{L_p(\Gamma,u)}\preccurlyeq\omega_{p,u}^{(2)}(f,1/n)+\tilde{\omega}_{p,u}^{(2)}(f,1/n).$$[3]

[3]Signs $\preccurlyeq$ and $\asymp$ define an ordinal relation. Namely, $A\preccurlyeq B$ means $A\le const\,B$. And $A\asymp B$ means $const\,A\le B\le const\,A$.

4. Proof

Obviously, $f\in L_p(\Gamma,u)$ and $u\in W_p(\Gamma)$ imply that $f\in L_1(\Gamma,u)$. Now, by virtue of $u\in W_p(\Gamma)$ we can state that

$$Sf=(S_\Gamma f)(t)=\frac{1}{\pi i}\int_\Gamma\frac{f(z)}{z-t}dz$$

exists almost everywhere on Γ. It follows that[8], the Cauchy type integral has certain angular values almost everywhere on Γ equal to

$$\Phi^{\pm}(t)=\pm\frac{1}{2}f(t)+\frac{1}{2}\int_\Gamma\frac{f(z)}{z-t}dz.$$

And this in turn implies that

$$f(t)=\Phi^+(t)-\Phi^-(t). \tag{2}$$

This relation shows that in order to approximate the function f given only on the curve Γ in terms of the metric of the space $L_p(\Gamma,u)$ it suffices to approximate the functions Φ^+ and Φ^- which are analytic inside and outside the given closed curve, respectively, and belong to $L_p(\Gamma,u)$.

So let us prove that for every positive integer n there exist the polynomials $P_n(z)$ and $\tilde{P}_n(1/n)$ of degree n such that

$$\left\|\Phi^+-P_n\right\|_{L_p(\Gamma,u)}\preccurlyeq\omega_{p,u}^{(2)}(f,1/n)_\Gamma \tag{3}$$

and

$$\left\|\Phi^--\tilde{P}_n\right\|_{L_p(\Gamma,u)}\preccurlyeq\tilde{\omega}_{p,u}^{(2)}(f,1/n)_\Gamma \tag{4}$$

First we will prove the validity of relation (3). From $f\in L_p(\Gamma,u)$ and $u\in W_p(\Gamma)$ it follows that if

$$\left\|f(z_h)-2f(z)+f(z_{-h})\right\|_{L_p(\Gamma,u)}<+\infty,$$

then $\mathcal{F}(z,h)\in L_1(\Gamma)$ $\left(\mathcal{F}(z,h)=f(z_h)+f(z_{-h})\right)$. This allows to state that the singular integral $\int_\Gamma\frac{f(z_h)+f(z_{-h})}{z-t}dz$ exists almost everywhere on Γ in the sense of principal value. The last statement enables us to approximate the function Φ^+ by the Jackson-Dzyadyk polynomials[9] represented in the following form:

$$P_n(t)=\frac{1}{4\pi^2 i}\int_{-\pi}^{\pi}K_n(h)\,dh\int_\Gamma\frac{f(z_h)+f(z_{-h})}{z-t}dz+\frac{1}{4\pi}\int_0^{\pi}K_n(h)\left[f(t_h)+f(t_{-h})\right]d\,h, \tag{5}$$

where $t\in\Gamma$ and $K_n(h)$ is a kernel that represents a trigonometric polynomial of at most n-th degree and satisfies the conditions

$$\frac{1}{\pi}\int_{-\pi}^{\pi}K_n(h)\,dh=1,\qquad n=0,1,2,\ldots \tag{6}$$

$$\int_{-\pi}^{\pi}\left|K_n(h)\right|dh\le const, \tag{7}$$

$$\int_{-\pi}^{\pi} |h|^k |K_n(h)| dh \le C(k)(n+1)^k \quad (k = 0,1,2,...) \tag{8}$$

Furthermore, from fulfillment of conditions (6) - (8) we directly get

$$\int_{-\pi}^{\pi} \left(|t| + \frac{1}{n}\right)^k |K_n(t)| dt \preccurlyeq n^{-k} \tag{9}$$

Now, taking into account relation (6), we represent the function Φ^+ as follows:

$$\Phi^+(t) = \frac{1}{4\pi^2 i} \int_0^{\pi} K_n(h) dh \int_\Gamma \frac{2f(z)}{z-t} dz + \\ + \frac{1}{4\pi} \int_0^{\pi} K_n(h) \cdot 2f(t) dh \tag{10}$$

Further, by virtue of relations (5) and (10), we estimate the difference $\Phi^+(t) - P_n(t)$ in the sense of $L_p(\Gamma,u)$ metric.

Obviously, we have

$$\|\Phi^+ - P_n\|_{L_p(\Gamma,u)} = \\ = \Big\| \frac{1}{4\pi^2 i} \int_0^{\pi} K_n(h) dh \int_\Gamma \frac{f(z_h) - 2f(z) + f(z_{-h})}{z-t} dz + \\ + \frac{1}{4\pi} \int_0^{\pi} K_n(h) d h(t_h) + 2f(t) + f(t_{-h}) \Big\|_{L_p(\Gamma,u)}.$$

We apply Minkovski inequality, and then Minkovski's generalized inequality to find that

$$\|\Phi^+ - P_n\|_{L_p(\Gamma,u)} \preccurlyeq \\ \preccurlyeq \left\| \int_0^{\pi} K_n(h) dh \int_\Gamma \frac{f(z_h) + 2f(z) + f(z_{-h})}{z-t} dz \right\|_{L_p(\Gamma,u)} + \\ + \int_0^{\pi} K_n(h) d h \|f(t_h) - 2f(t_h) + f(t_{-h})\|_{L_p(\Gamma,u)} \preccurlyeq \\ \preccurlyeq \int_0^{\pi} K_n(h) d h \|f(t_h) - 2f(t) + f(t_{-h})\|_{L_p(\Gamma,u)} + \\ + \int_0^{\pi} |K_n(h)| dh \left\| \int_\Gamma \frac{f(t_h) - 2f(t) + f(t_{-h})}{z-t} dz \right\|_{L_p(\Gamma,u)} \preccurlyeq \\ \preccurlyeq \int_0^{\pi} |K_n(h)| \cdot A(h) dh + \int_0^{\pi} |K_n(h)| \omega_{p,u}^{(2)}(f,h) dh, \tag{11}$$

where

$$A(h) = \left(\int_\Gamma \left| \int_\Gamma \frac{f(z_h) - 2f(z) + f(z_{-h})}{z-t} dz \right|^p |u(t)|^p |dt| \right)^{1/p}.$$

As $u \in W_p(\Gamma)$, the latter relation yields

$$A(h) \preccurlyeq \left(\int_\Gamma |f(z_h) - 2f(z) + f(z_{-h})|^p |u(z)|^p |dz| \right)^{1/p}.$$

It follows from (11) that

$$\|\Phi^+ - P_n\|_{L_p(\Gamma,u)} \preccurlyeq \int_0^{\pi} |K_n(h)| \omega_{p,u}^{(2)}(f,h) dh \tag{12}$$

Therefore, using classical technique we get (3).

To complete the proof, we only have to show the validity of relation (4). To this end, we map the plane z onto the plane z' by means of the function

$$z = \frac{1}{z'}. \tag{13}$$

Obviously, the contour Γ is mapped into some contour Γ' and the functions f and u are transformed into the functions $f_1(z')$ $\left(f(z) = f\left(\frac{1}{z'}\right) = f_1(z') \right)$ and $u_1(z')$ $\left(u(z) = u\left(\frac{1}{z'}\right) = u_1(z') \right)$, respectively.

Let us prove the validity of the following statements:

$$f(z) \in L_p(\Gamma,u) \Leftrightarrow f_1(z') \in L_p(\Gamma',u_1) \tag{14}$$

and

$$u(z) \in W_p(\Gamma) \Leftrightarrow u_1(z') \in W_p(\Gamma). \tag{15}$$

Validity of (14) is obvious.

Further, combining relation

$$(S_\Gamma f)(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(z)}{z-t} dz = \frac{1}{\pi i} \int_{\Gamma'} \frac{f_1(z')}{z'-t'} dz' - \\ - \frac{1}{\pi i} \int_\Gamma \frac{f_1(z')}{z'} dz' = (S_{\Gamma'} f_1)(t') - \frac{1}{\pi i} \int_{\Gamma'} \frac{f_1(z')}{z'} dz' \tag{16}$$

with relation (14) and the fact that $z' = 0$ lies strictly outside Γ', we get relation (15).

At the same time, we notice that for $z \in \overline{G}^+$ ($G^+(G)$ is the interior of the curve Γ) the function $\Phi(t)$ on the plane z' takes the form

$$\Phi(t) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-t} dz = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f_1(z')}{z'-t'} dz' - \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(z')}{z'} dz'.$$

If we take into account

$$\frac{1}{2\pi i} \int_{\Gamma'} \frac{f_1(z')}{z'} dz' = 0,$$

then we get:

$$\Phi(t) = \Phi_1(t').$$

Obviously, the function $\Phi^-(t)$ in the plane z corresponds to the function $\Phi_1^+(t')$ in the plane z'. Hence, by virtue of relation (3) we get

$$\left\|\Phi_1^+(t') - \tilde{P}_n(t')\right\|_{L_p(\Gamma',u_1)} \preccurlyeq \omega_{p,u_1}^{(2)}(f_1, 1/n)_\Gamma. \tag{17}$$

Further, taking into account that $\Phi_1^+(t_1') = \Phi^-(t)$, $u_1(t') = u(t)$ and the point $z = 0$ lies inside Γ, we get

$$\left\|\Phi_1^+(t') - \tilde{P}_n(t')\right\|_{L_p(\Gamma',u_1)} \asymp \left\|\Phi^-(t) - \tilde{P}_n(1/t)\right\|_{L_p(\Gamma,u)} \tag{18}$$

To complete the proof of relation (4) we use the obvious relation

$$\omega_{p,u_1}^{(2)}(f,\delta)_{\Gamma'} \asymp \tilde{\omega}_{p,u}^{(2)}(f,\delta)_\Gamma. \tag{19}$$

Thus, from relation (17), by virtue of (18) and (19), we get the required relation (4).

Now, to complete the proof of the theorem it suffices to make use of relations (3) and (4) and for $R_n(z) = P_n(z, 1/z)$,

$$R_n(z) = P_n(z) + \tilde{P}_n(1/z),$$

from whence, by virtue of relation (2), the statement of theorem follows.

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