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A Note on Fixed Points for Asymptotic Nonlinear Contractions

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Abstract Let (M,d) be a complete metric space and T be a self-mapping of M. W.A. Kirk proved a fixed point theorem for a continuous asymptotic contraction T in[4] .Y.Z. Chen extended Kirk's theorem in[2] by assuming weaker assumptions on T. Also Chen introduced some other conditions to replace the assumption on the boundedness of the orbit. We introduce the weaker condition $\liminf_{n\to\infty} (d(x,T^n x)) = 0$ for some x in M, and prove that this condition implies the existence of a fixed point and the convergence of the Picard iterates to this fixed point.

Keywords Fixed Point, Asymptotic Nonlinear Contraction, Picard iterates, Upper Semicontinuous

1. Introduction and Preliminaries

W.A. Kirk in[4] introduced asymptotic nonlinear contractions in metric spaces and proved the existence and uniqueness of a fixed point for such mappings by using ultrafilter methods. Y.Z. Chen in[2] showed the existence and uniqueness of a fixed point for asymptotic nonlinear contractions with weaker assumptions than Kirk and without using ultrafilter methods. The following theorem was proved by Chen in[2]:

THEOREM 1.1 Let (M, d) be a complete metric space, and let T: M → M satisfy:

$d(T^n x, T^n y) \leq h_n(d(x, y))$ for all x, y in M,

where $h_n:[0, \infty] \to [0, \infty]$ and $\lim_{n\to\infty} h_n = h$ uniformly on any bounded interval[0, b]. Assume ϕ is upper semicontinuous, $h(t) < t$ for $t > 0$, and assume there is a positive integer m such that h_m is upper semicontinuous and $h_m(0) = 0$. If there exists some x_0 in M with a bounded orbit, then T has a unique fixed point x* in M and $\lim_{n\to\infty} T^n x = x^*$ for all x in M.

In the above theorem, the assumption on the boundedness of the orbit of some x_0 is crucial to the proof of the existence of a fixed point in X.

In this note, we remove the assumption of the boundedness of the orbit and instead introduce the condition that $\liminf_{n\to\infty} (d(x,T^n x)) = 0$ for some x in M, and we are able to get the same conclusion as in Theorem 1.1.

For further details about asymptotic contractions and related topics, we refer the reader to[1, 3, 5-7] and the references therein.

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2. Main Results

We shall begin with a lemma introduced in[1] which will be used later for the proof of our theorem. We will reproduce the proof for the sake of completeness.

LEMMA 2.1 Let h: R+ → R+ be upper semicontinuous and $h(t) < t$ for $t > 0$. Suppose there exists two sequences of nonnegative real numbers $\{u_n\}$, $\{\varepsilon_n\}$, such that:

$u_{2n} \leq h(u_n) + \varepsilon_n$

with $\varepsilon_n \to 0$.

Then either $\sup_{n>0} u_n = \infty$ or $\liminf_{n\to\infty} u_n = 0$.

Proof. Assume $b := \sup_{n>0} u_n < \infty$ and $\liminf_{n\to\infty} u_n \neq 0$. Then there exists a real number m >0 and an integer $N_1 > 0$ such that $u_n > m$ for all $n > N_1$.

Since h is upper semicontinuous, then h(t)/t is upper semicontinuous on[m, b]. Therefore h(t)/t achieves its maximum in the compact interval[m,b]. Let

$L_m = \max\{h(t)/t, t \text{ in}[m,b]\} < 1$

because $h(t) < t$ for all $t > 0$. Let $\varepsilon > 0$. Then, using our assumption on $\{u_n\}$ and ε_n, there exists $N_2 > N_1$ such that

$u_{2n} \leq h(u_n) + \varepsilon \leq L_m u_n + \varepsilon$ for all $n > N_2$

Note that $L_m u_n < u_n$, since $L_m < 1$ and $u_n > 0$.

We define $f:[0, \infty) \to [0, \infty)$ by:

$f(x) = L_m x + \varepsilon$

Then $f(x)$ is a contraction, since $L_m < 1$ and

$|f(x) - f(y)| = |(L_m x + \varepsilon) - (L_m y + \varepsilon)|$

$= |L_m x - L_m y|$

$= L_m|x - y|$

By Banach's fixed point theorem, $f(x)$ has a unique fixed point $\varepsilon / (1 - L_m)$, and $\lim_{n\to\infty} f^n(x) = \varepsilon / 1 - L_m$ for every x in[0, ∞).

So for any $n > N_2$

$u_{2^2 n} \leq h(u_{2n}) + \varepsilon \leq L_m u_{2n} + \varepsilon$

$\leq L_m(L_m u_n + \varepsilon) + \varepsilon = L_m f(u_n) + \varepsilon = f^2(u_n)$

We claim that $u_{2^k n} \leq f^k(u_n)$ for all k and all $n > N_2$. We know it holds for k =2. By induction, let's assume it is true for some $h \geq 2$. Then we have

$u_{2^{h+1} n} \leq h(u_{2^h n}) + \varepsilon \leq L_m u_{2^h n} + \varepsilon$

$= f(u_{2^h n}) \leq f(f^h(u_n)) = f^{h+1}(u_n)$ (1)

Since f is increasing. Therefore (1) holds for all k.

Now let $r = \inf_{n > N_2} u_n \leq f^k(u_n)$ which converges to $\varepsilon / (1- L_m)$ as $k \to \infty$. Since ε was arbitrary we must have r = 0, which is a contradiction.

Now we proceed to the main theorem.

THEOREM 2.1 Let (M, d) be a complete metric space. Let T: M → M satisfy:

$d(T^n x, T^n y) \leq h_n(d(x, y))$ for all x, y in M

where $h_n:[0, \infty) \to [0, \infty)$ and $\lim_{n\to\infty} h_n = h$ uniformly on any bounded interval[0, b]. Assume h is upper semi-continuous, $h(t) < t$ for $t > 0$, and assume there is a positive integer n* such that h_{n^*} is upper semicontinuous and $h_{n^*}(0) = 0$. If $\liminf_{n\to\infty} (d(x,T^n x)) = 0$ then T has a unique fixed point x in M, and $\lim_{n\to\infty} T^n y = x$ for all y in M.

Proof. Let us first establish the uniqueness of the fixed point. Suppose there exist two fixed points z_1 and z_2 for T with $z_1 \neq z_2$.

Then $d(z_1, z_2) = d(T^n z_1, T^n z_2) \leq h_n(d(z_1, z_2))$.

Letting $n \to \infty$ yields $d(z_1, z_2) \leq h(d(z_1, z_2)) < d(z_1, z_2)$, which is a contradiction. Thus the fixed point is unique.

Without loss of generality we set $h_n(0) = 0$ and $h(0) = 0$.

We will first show that the sequence $a_n = d(T^{n+1}x, T^n x)$ is bounded for every fixed x.

$a_n = d(T^{n+1}x, T^n x) \leq h_n(d(Tx, x))$

hence:

$\limsup_{n\to\infty} a_n \leq \lim_{n\to\infty} h_n(d(Tx, x))$

$= h(d(Tx, x)) \leq d(Tx, x) = b$

Thus the sequence $a_n = d(T^{n+1}x, T^n x)$ is bounded.

Now we will show that $\liminf_{n\to\infty} a_n = 0$.

We have: $a_{2n} = d(T^{2n+1}x, T^{2n}x)$

$\leq h_n(d(T^{n+1}x, T^n x))$

$= h(a_n) + [h_n(a_n) - h(a_n)]$.

We choose: $\varepsilon_n = h_n(a_n) - h(a_n)$

Since $h_n \to h$ uniformly on[0, 2b], then $\varepsilon_n \to 0$. By Lemma 2.1, $\liminf_{n\to\infty} a_n = 0$.

Now we will show that $\lim_{n\to\infty} a_n = 0$. Assume by contradiction that $\lim_{n\to\infty} a_n \neq 0$. Then $\limsup_{n\to\infty} a_n > 0$.

Since $\liminf_{n\to\infty} a_n = 0$, there exists $n_0 > 0$ such that $0 < a_{n_0} < \limsup_{n\to\infty} a_n$.

Let $\{a_{n_i}\}$ be a subsequence of $\{a_n\}$ such that $a_{n_i} > a_{n_0}$ for all i and $\lim_{i\to\infty} a_{n_i} = \limsup_{n\to\infty} a_n$. Then:

$$a_{n_0} < a_{n_i} = d(T^{n_i+1}x, T^{n_i}x)$$
$$\leq h_{n_i - n_0}(d(T^{n_0+1}x; T^{n_0}x))$$
$$= h(a_{n_0}) + h_{n_i - n_0}(a_{n_0}) - h(a_{n_0})$$

Letting $i \to \infty$ we get: $a_{n_0} \leq h(a_{n_0}) < a_{n_0}$, which is a contradiction. Thus $\lim_{n\to\infty} a_n = 0$.

Now we show that x is a fixed point for T.

Since $\liminf_{n\to\infty} (d(T^n x, x)) = 0$, there exists a subsequence, $\{m_k\}$, such that $\lim_{k\to\infty} d(T^{m_k}x, x) = 0$

We will first show that $T^{n^*}x = x$. We have:

$\limsup_{k\to\infty} (d(T^{m_k+n^*}x, T^{n^*}x)$

$\leq \limsup_{k\to\infty} h_{n^*}(d(T^{m_k}x, x)) \leq h_{n^*}(0) = 0$

and

$d(T^{m_k+n^*}x, T^{m_k}x)$

$\leq d(T^{m_k+n^*}x, T^{m_k+n^*-1}x) + d(T^{m_k+1}x, T^{m_k}x)$

by the triangle inequality.

$= a_{m_k+n^*-1} + a_{m_k}$.

Letting $k \to \infty$, we get:

$d(T^{m_k+n^*}x, T^{m_k}x) \to 0$, since $\lim_{n\to\infty} a_n = 0$

Thus $T^{m_k}x \to T^{n^*}x$. Hence by the uniqueness of the limit of $T^{m_k}x$, $T^{n^*}x = x$.

We claim that T^{n^*} has a unique fixed point. Using the same argument as above, let z_1, z_2 be two fixed points of T^{n^*} with $z_1 \neq z_2$.

Note that $z_1 = T^{n^*}z_1$, and $T^{n^*}z_1 = T^{2n^*}z_1$.

therefore $z_1 = T^{kn^*}z_1$ for any positive integer k.

Then $d(T^{n^*}z_1, T^{n^*}z_2) = d(T^{kn^*}z_1, T^{kn^*}z_2) \leq h_{kn^*}d(z_1, z_2)$. Letting $k \to \infty$, we get:

$d(z_1, z_2) \leq \lim_{k\to\infty} h_{kn^*}(d(z_1, z_2))$

$= h(d(z_1, z_2)) < d(z_1, z_2)$

which is a contradiction. So T^{n^*} has a unique fixed point.

Note that $T^{n^*+1}x = T(T^{n^*}x) = Tx$. Hence Tx is also a fixed point of T^{n^*}. By the uniqueness of the fixed point of T^{n^*}, we have:

$Tx = x$.

To show that $\lim_{n\to\infty} d(T^n y, x) = 0$, for any y in M, by Theorem 1.1 we need only show that $d(T^n y, x)$ is bounded. We have:

$d(T^n y, x) = d(T^n y, T^n x) \leq h_n(d(y, x))$.

Letting $n \to \infty$, we get:

$\limsup_{n\to\infty} d(T^n y, x) \leq \limsup_{n\to\infty} h_n(d(y, x))$

$= h(d(y, x)) \leq d(y, x)$

Thus the orbit of y is bounded, and hence $T^n y \to x$ for every y in M.

3. Conclusions and Future Directions for Research

We proved the existence and uniqueness of the fixed point as well as the convergence of the Picard iterates for asymptotic nonlinear contractions without assuming the boundedness of the orbits but with a weaker assumption, namely that $\liminf_{n\to\infty} d(T^n x, x) = 0$ for some x in M. It still remains an open problem whether the same conclusions hold by assuming only that $\liminf_{n\to\infty} d(T^n x, x) < +\infty$.

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