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Weak Insertion of a $\gamma-$ Continuous Function[1]

Majid Mirmiran

Department of Mathematics, University of Isfahan Isfahan 81746-73441, Iran

Abstract A sufficient condition in terms of lower cut sets are given for the weak insertion of a $\gamma-$continuous function between two comparable real-valued functions.

Keywords Weak Insertion, Strong Binary Relation, Preopen Set, Semi-Open Set, $\gamma-$Open Set, Lower Cut Set

1. Introduction

The concept of a preopen set in a topological space was introduced by H. H. Corson and E. Michael in 1964[5]. A subset A of a topological space (X, τ) is called preopen or locally dense or nearly open if $A \subseteq Int(Cl(A))$. A set A is called preclosed if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term ,preopen, was used for the first time by A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb[13], while the concept of a , locally dense, set was introduced by H. H. Corson and E. Michael[5].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963[12]. A subset A of a topological space (X, τ) is called semi-open[12] if $A \subseteq Cl(Int(A))$. A set A is called semi-closed if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

Recall that a subset A of a topological space (X, τ) is called $\gamma-$open if $A \cap S$ is preopen, whenever S is preopen [2]. A set A is called $\gamma-$closed if its complement is $\gamma-$open or equivalently if $A \cup S$ is preclosed, whenever S is preclosed. The class $\gamma-$open sets is a topology on X[1].

A real-valued function f defined on a topological space X is called A−continuous[14] if the preimage of every open subset of R belongs to A, where A is a collection of subset of X. Most of the definitions of function used throughout this paper are consequences of the definition of A−continuity. However, for unknown concepts the reader may refer to[6,7].

Hence, a real-valued function f defined on a topological space X is called precontinuous (resp. semi-continuous or $\gamma-$continuous) if the preimage of every open subset of R is preopen (resp. semi-open or $\gamma-$open) subset of X. Precontinuity was called by V. Ptak nearly continuity[15]. Nearly continuity or precontinuity is known also as almost continuity by T. Husain[8]. Precontinuity was studied for real-valued functions on Euclidean space by Blumberg back in 1922[3].

Results of Katˇetov[9,10] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks[4], are used in order to give a sufficient condition for the insertion of a $\gamma-$continuous function between two comparable real-valued functions.

If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of conditions considered in[11].

A property P defined relative to a real-valued function on a topological space is a $\gamma-$property provided that any constant function has property P and provided that the sum of a function with property P and any $\gamma-$continuous function also has property P. If P1 and P2 are $\gamma-$property, the following terminology is used: A space X has the weak $\gamma-$insertion property for (P1, P2) if and only if for any functions g and f on X such that $g \leq f$, g has property P1 and f has property P2, then there exists a $\gamma-$continuous function h such that $g \leq h \leq f$.

In this paper, is given a sufficient condition for the weak $\gamma-$insertion property. Also several insertion theorems are obtained as corollaries of this result.

2. The Main Result

Before giving a sufficient condition for insertability of a $\gamma-$continuous function, the necessary definitions and terminology are stated.

Let (X, τ) be a topological space, the family of all $\gamma-$open, $\gamma-$closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\gamma O(X, \tau)$, $\gamma C(X, \tau)$, $sO(X, \tau)$, $sC(X, \tau)$, $pO(X, \tau)$ and $pC(X, \tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ). Respectively, we define the $\gamma-$closure, $\gamma-$interior, s-closure, s-interior, p-closure and p-interior of a set A, denoted by $\gamma Cl(A)$, $\gamma Int(A)$, $sCl(A)$, $sInt(A)$, $pCl(A)$ and $pInt(A)$ as follows:

$\gamma Cl(A)= \cap\{F : F \supseteq A, F \in \gamma C(X, \tau)\}$,

* Corresponding author:
mirmir@sci.ui.ac.ir (Majid Mirmiran)
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$\gamma Int(A)= \cup\{O : O \subseteq A, O \in \gamma O(X, \tau)\}$,
$sCl(A)= \cap\{F : F \supseteq A, F \in sC(X, \tau)\}$,
$sInt(A)= \cup\{O : O \subseteq A, O \in sO(X, \tau)\}$,
$pCl(A)= \cap\{F : F \supseteq A, F \in pC(X, \tau)\}$ and
$pInt(A)= \cup\{O : O \subseteq A, O \in pO(X, \tau)\}$.

If $\{Ai : i \in I\}$ be a family of preopen (resp. semi-open) sets, since $Ai \subseteq Int(Cl(Ai)) \subseteq Int(Cl(\cup\{Ai : i \in I\}))$ (resp. $Ai \subseteq Cl(Int(Ai)) \subseteq Cl(Int(\cup\{Ai : i \in I\}))$, then $\cup\{Ai : i \in I\} \subseteq Int(Cl(\cup\{Ai : i \in I\}))$ (resp. $\cup\{Ai : i \in I\} \subseteq Cl(Int(\cup\{Ai : i \in I\}))$, i. e., $\cup\{Ai : i \in I\}$ is a preopen (resp. semi-open) set. Therefore, both preopen and semi-open sets are preserved by arbitrary unions.

Hence, respectively, we have γCl(A), sCl(A), pCl(A) are γ−closed, semi-closed, preclosed and γInt(A), sInt(A), pInt(A) are γ−open, semi-open, pre-open.

The following first two definitions are modifications of conditions considered in[9,10].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set P (X) of a topological space X is called a strong binary relation in P (X) in case ρ satisfies each of the following conditions:

1) If $Ai \rho Bj$ for any $i \in\{1,...,m\}$ and for any $j \in\{1,...,n\}$, then there exists a set C in P (X) such that $Ai \rho C$ and $C \rho Bj$ for any $i \in\{1,...,m\}$ and any $j \in\{1,...,n\}$.

2) If $A \subseteq B$, then $A \bar{\rho} B$.

3) If $A \rho B$, then $\gamma Cl(A) \subseteq B$ and $A \subseteq \gamma Int(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks[4] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) <t\} \subseteq A(f, t) \subseteq \{x \in X : f(x) \leq t\}$ for a real number t, then A(f, t) is called a lower indefinite cut set in the domain of f at the level

We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on a topological space X with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets A(f, t) and A(g, t) in the domain of f and g at the level t for each rational number t such that if $t1 <t2$ then $A(f, t1) \rho A(g, t2)$, then there exists a γ−continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets A(f, t) and A(g, t) in the domain of f and g at the level t for each rational number t such that if $t1 <t2$ then $A(f, t1) \rho A(g, t2)$.

Define functions F and G mapping the rational numbers Qinto the power set of X by F (t)= A(f, t) and G(t)= A(g, t). If t1 and t2 are any elements of Q with $t1 <t2$, then $F (t1) \bar{\rho} F (t2)$, $G(t1) \rho G(t2)$, and $F (t1) \rho G(t2)$. By Lemmas 1 and 2 of[10] it follows that there exists a function H mapping Q into the power set of X such that if t1 and t2 are any rational numbers with $t1 <t2$, then $F (t1) \rho H(t2)$, $H(t1) \rho H(t2)$ and $H(t1) \rho G(t2)$.

For any x in X, let $h(x) = \inf\{t \in Q : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in H(t) then x is in G(k) for any $k >t$; since x is in G(k)= A(g, k) implies that $g(x) \leq k$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in H(t), then x is not in F (k) for any $k <t$; since x is not in F (k)= A(f, k) implies that $f(x) >k$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t1 and t2 with $t1 <t2$, we have $h^{-1}(t1,t2)= \gamma Int(H(t2)) \setminus \gamma Cl(H(t1))$. Hence $h^{-1}(t1,t2)$ is a γ−open subset of X, i. e., h is a γ−continuous function on X.

The above proof used the technique of proof of Theorem 1 of[9].

3. Applications

The abbreviations pc and sc are used for precontinuous and semicontinuous, respectively.

Before stating the consequences of Theorem 2.1, we suppose that X is a topological space that γ−open sets are semi-open and preopen.

Corollary 3.1. If for each pair of disjoint preclosed (resp. semi-closed) sets F1, F2, there exist γ−open sets G1 and G2 such that $F1 \subseteq G1$, $F2 \subseteq G2$ and $G1 \cap G2 = \emptyset$ then every precontinuous (resp. semi-continuous) function is γ−continuous.

Proof. First verify that X has the weak γ−insertion property for (pc, pc) (resp. (sc, sc)): Let g and f be real-valued functions defined on the X, such that f and g are pc (resp. sc), and $g \leq f$.If a binary relation ρ is defined by AρB in case $pCl(A) \subseteq pInt(B)$ (resp. $sCl(A) \subseteq sInt(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t1 and t2 are any elements of Q with $t1 <t2$, then

$A(f, t1) \subseteq\{x \in X : f(x) \leq t1\} \subseteq \{x \in X : g(x) <t2\} \subseteq A(g, t2)$;

since $\{x \in X : f(x) \leq t1\}$ is a preclosed (resp. semi-closed) set and since $\{x \in X : g(x) <t2\}$ is a preopen (resp. semi-open) set, it follows that $pCl(A(f, t1)) \subseteq pInt(A(g, t2))$ (resp. $sCl(A(f, t1)) \subseteq sInt(A(g, t2))$). Hence $t1 <t2$ implies that $A(f, t1) \rho A(g, t2)$. The proof follows from Theorem 2.1.

Also, if f be a real-valued precontinuous (resp. semi-continuous) function defined on the X, by setting g = f, then there exists a γ−continuous function h such that g = h = f.

Corollary 3.2. If for each pair of disjoint subsets F1,F2 of X , such that F1 is preclosed and F2 is semi-closed, there exist γ−open subsets G1 and G2 of X such that $F1 \subseteq G1$, $F2 \subseteq G2$ and $G1 \cap G2 = \emptyset$ then X have the weak γ−insertion property for (pc, sc) and (sc, pc).

Proof. Let g and f be real-valued functions defined on the X, such that g is pc (resp. sc) and f is sc (resp. pc), with $g \leq f$.If a binary relation ρ is defined by AρB in case $sCl(A) \subseteq pInt(B)$ (resp. $pCl(A) \subseteq sInt(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t1 and t2 are any elements of Q with $t1 <t2$, then

$A(f, t1) \subseteq \{x \in X : f(x) \leq t1\} \subseteq \{x \in X : g(x) <t2\} \subseteq A(g,$

t2);

since $\{x \in X : f(x) \leq t1\}$ is a semi-closed (resp. preclosed) set and since $\{x \in X : g(x) < t2\}$ is a preopen (resp. semi-open) set, it follows that $sCl(A(f, t1)) \subseteq pInt(A(g, t2))$ (resp. $pCl(A(f, t1)) \subseteq sInt(A(g, t2))$). Hence $t1 < t2$ implies that $A(f, t1)\ \rho A(g, t2)$. The proof follows from Theorem 2.1.

Remark 3.1. See[1,2], for examples of topological spaces are said in corollaries 3.1 and 3.2.

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REFERENCES

[1] D. Andrijevic, On SPO-equivalent topologies, V International Meeting on Topology in Italy (Italian) (Lecce, 1990/Otranto, 1990), Rend. Circ. Mat. Palermo (2) Suppl., 29 (1992), 317-328

[2] D. Andrijevic and M. Ganster, On PO-equivalent topologies, IV International Meeting on Topology in Italy (Sorrento, 1988), Rend. Circ. Mat. Palermo (2) Suppl., 24 (1990), 251-256

[3] H. Blumberg, New properties of all functions, Trans. Amer. Math. soc., 24 (1922), 113-128

[4] F. Brooks, Indefinite cut sets for real functions, Amer. Math. Monthly, 78 (1971), 1007-1010

[5] H. H. Corson and E. Michael, Metrizability of certain countable unions, Illinois J. Math., 8 (1964), 351-360

[6] J. Dontchev, The characterization of some peculiar topological space via $\alpha-$ and $\beta-$sets, Acta Math. Hungar., 69 (1-2) (1995), 67-71

[7] M. Ganster and I. Reilly, A decomposition of continuity, Acta Math. Hungar., 56 (3-4) (1990), 299-301

[8] T. Husain, Almost continuous mappings, Prace Mat., 10 (1966), 1-7

[9] M. Katˇetov, on real-valued functions in topological spaces, Fund. Math., 38 (1951), 85-91

[10] M. Katˇetov, Correction to, ”On real-valued functions in topological spaces”, Fund. Math., 40 (1953), 203-205

[11] E. Lane, Insertion of a continuous function, Pacific J. Math., 66 (1976), 181-190

[12] N. Levine, Semi-open sets and semi-continuity in topological space, Amer. Math. Monthly, 70 (1963), 36-41

[13] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On pre-continuous and weak pre-continuous mappings, Proc. Math. Phys. Soc. Egypt, 53(1982), 47-53

[14] M. Przemski, A decompositionof continuity and $\alpha-$continuity, Acta Math. Hungar., 61 (1-2) (1993), 93-98

[15] V. Ptak, Completeness and open mapping theorem, Bull. Soc. Math. France, 86 (1958), 41-74

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