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Stochastic Inventory System with Reneging of Customers from the Service Facilities

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Abstract This paper analysis an ***(s, S)*** perishable inventory system with postponed demands with service facilities. We assume that customer arrive to the service according to Poisson process with parameter $\boldsymbol{\lambda}$. In this system, there is a finite buffer whose capacity varies according to the inventory level at any given time. When the maximum buffer size is reached, further demands join a pool which has finite capacity $M\ (<\infty)$ with probability γ and with probability $(1-\gamma)$ it is lost forever. When inventory level is larger than the number of customers in the buffer, an external demand can enter the buffer for service. A pooled customer is transferred to the buffer for service at a service completion epoch with probability p if the inventory exceeds $s+1$ and provided the number of customers in the buffer is less than the number items held in the inventory. It is assumed that initially the inventory level is S. When inventory level reaches s an order for replenishment is placed. The lead time is exponentially distributed with parameter β. When at least one customer is available in the buffer then reneging will be occurred with parameter ξ. We obtain the steady state analysis is made. Some performance measures are obtained and a few numerical illustrations are provided.

Keywords Finite buffer customers, Finite pool custom

1. Introduction

At most of the inventory models it is assumed that the inventory depletes at a rate equal to demand rate. However, it becomes unrealistic for the service facilities where the stocked item is delivered to the customers after some service is performed. In this paper we consider an (s, S) inventory system with service facilities. Arrival of demands form a Poisson process with parameter $\lambda\ (> 0)$ to a buffer of finite capacity equal 0 the inventory level at any given time t. When the maximum buffer size is reached, further demands join a pool of infinite capacity with probability γ or with probability $(1-\gamma)$ it is lost forever. Pooled customers are taken for service at a service completion epoch if the inventory level is at least $s+1$. The service time is assumed to be exponentially distributed with parameter μ. It is also assumed that initially the inventory level is S. When inventory level reaches s due to service, an order for replenishment is placed. The lead time is exponentially distributed with parameter. When at least one customer is available in the buffer then reneging will be occurred with parameter ξ. When $I(t>B(t)$ and $I(t) \geq s+1$ and $B(t)= 0$, then a pool customer is sent to the buffer with probability p at the previous service epoch. Even when $I(t)>B(t)$ and $I(t)<s$ and $B(t)= 0$, then there is no pool customers are sent to the buffer.

Krishnamoorthy et al [1] analyze retrial production inventory system with service time in which primary demands occur according to Markovian Arrival Process (MAP), using matrix analytic method they carry out the steady state analysis of the system and some performance measures and obtained.

Mohammad Ekramol Islam et al [2] developed the reneging concept their paper assuming pool customers can get impatience and can level the pool with probability $(1-\beta)$ or can enter the pool with probability β that indicate entering customers can have patients to get service.

Mohammad Ekramol Islam et al [3] built some inventory models related to postponed demand, reneging pool customers and rejection of customers from the system in service facilities. In that paper they introduce inventory system at service facility with ***N***- policy. Consider reneging and rejection of pool customers. Customers direct go to the pool region and get service.

Krishnamoorthy and Mohammad Islam [4] considered and ***(s, S)*** inventory system with postponed demands where

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they assumed that customers arrive to the system according to a Poisson process. When inventory level depletes to s due to demands or decay or service to pool customer, an order for replenishment is placed. The lead-time is exponentially distributed with parameter γ. When inventory level reaches zero, the incoming customers are sent to a pool of capacity M from the pool customers will be picked upfor satisfying demands.

Sivakumar and Arivarignan [5] considerd a continuous review perishable inventory system in which the demands arrive according to Morkovian Arrival process (MAP). Berman and Kim [6-7] analyzed a problem in which customers arrive at a service facility according to a Poisson process with service time exponentially distributed and each customer demands one item of the inventory, both zero lead time and positive lead time cases were discussed. Berman and Sapna [8-9] studied and inventory control problem at a service facility, which uses one item of inventory for service provided. They assumed Poisson arrivals, arbitrarily distributed service times and zero lead time and analyze the system with the restriction that the waiting space is finite. Under a specific cost structure, they derived the optimal order quantity that minimizes the long run expected cost rate.

The paper is organized as follows:

Sections:

Then we displayed all references cited in the text and the values of the all steady state probability vectors are given in appendix-A.

2. Mathematical Model

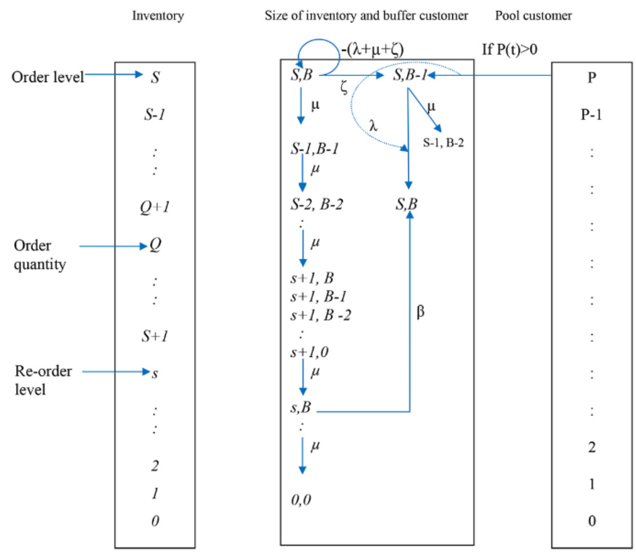


Figure 1. Diagram of the Model

2.1. Assumptions

i. Arrival of demands follows Poisson process with parameter λ.
ii. Lead time is exponentially distributed with parameter β.
iii. Service time is exponentially distributed with parameter μ.
iv. Initially the system is in S.
v. Replenishment take place when the inventory level depleted to s.
vi. If $B(t) = I(t)$, the primary arrival is directed to the pool with probability γ and with probability $(1-\gamma)$ it is lost forever.
vii. When $B(t)= I(t)$, is the highest inventory level for that situation if perishability occurs immediate one buffer customer must be pull out from buffer.

2.2. Notations

The notations used in the sequel are explained below:

i. I(t) = Inventory level at time t.
ii. B (t) = Number of customers in the buffer at time t
iii. P(t) = Number of customers in the pool at time t
iv. ξ = Reneging rate
v. e = Denote the column vector of 1*’s*

2.3. The Model Build Up and Analysis

$\{P(t), I(t), B(t) = (i,j,k) | 0 \le i \le P; 0 \le j \le S; 0 \le k \le B\}$ is formed a three dimensional Markov process with state space $E = E1 \times E2 \times E3$; $where\ E1 = \{0,1, \dots \dots . P\}$; $E2 = \{0,1 \dots \dots S\}$; $E3 = \{0,1 \dots \dots B\}$.

The infinitesimal generator of the process:

$\tilde{A} = (a(i,j,k{:}\, l,m,n); (i,j,k),(l,m,n)) \in E$ can be obtained using the following arguments:

A. The arrival of the demand makes a transition from

$$(i,j,k) \rightarrow (l = i+1, m = j;\ n = k)\ if\ 0 \le i \le P-1; 0 \le j \le S; 0 \le k \le B$$

B. The pool customer makes a transition to buffer leaves the pool size less by one as first come first serve basis.

$$(i,j,k) \rightarrow (l = i-1, m = j;\ n = k+1)\ if\ 1 \le i \le P;\ 1 \le j \le S; 0 \le k \le B-1$$

C. For reneging customer from the buffer makes a transition reducing the size of the buffer by one unit.

$$(i,j,k) \rightarrow (l=i, m=j;\ n=k-1)\ if\ 0 \le i \le P; 1 \le j \le S;\ 1 \le k \le B$$

D. If service occurred then a transition reducing the size of the buffer by one unit, as the same time reducing the size of the inventory by one unit.

$$(i,j,k) \rightarrow (l=i, m=j-1; n=k-1)\ if\ 0 \le i \le P;\ s+1 \le j \le S; k=1$$

E. Replenishment take place only when inventory less or equal to s

$$(i,j,k) \rightarrow (l=i, m=j+\beta; n=k)\ if\ 0 \le i \le P;\ 0 \le j \le s;\ 0 \le k \le B-2$$

Remark:

$$IF\ P(t) > 0\ and\ I(t) > B(t), then\ (S, B-1) \overset{\delta\gamma}{\rightarrow} (S-1, B-1)$$

The Infinitesimal generator $\tilde{A}$ of the three-dimensional Markov Process:
$[P(t), I(t), B(t); t \ge 0] can\ be\ defined\ by\ The\ Markov\ chain\ \{(P(t), I(t), B(t)), t \ge 0\}$ Has the generator $\tilde{A}$ in partitioned from given by

$$\tilde{A}=(a(i,j,k; l,m,n)) : (i,j,k), (l,m,n) \in E, (i,l) \in E_1, (j,m) \in E_2, (k,n) \in E_3$$

Now, the infinitesimal generator $\tilde{A}$ can be conveniently express as a partition $matrix\ \tilde{A} = (a_{i,j,k})$

Or,

$$\tilde{A} = \begin{cases} (i,j,k) \rightarrow (i+1,j,k)\ is\ \lambda\ if\ i=0,1,\dots,P-1, & j=0,1,\dots,S, \quad k=0,1,\dots,B \\ (i,j,k) \rightarrow (i,j+Q,k)\ is\ \beta\ if\ i=0,1,\dots,P, & j=0,1,\dots,s, \quad k=0,\dots,1 \\ (i,j,k) \rightarrow (i,j-1,k-1)\ is\ \mu\ if\ i=0,1,\dots,P, & j=s,s+1,\dots,S, \quad k=1,2,\dots,B \\ (i,j,k) \rightarrow (i,j,k-1)\ is\ \xi\ if\ i=0,1,\dots,P, & j=s,s+1,\dots,S, \quad k=1,2,\dots,B \\ (i,j,k) \rightarrow (i-1,j,k+1)\ is\ \delta\gamma\ if\ i=1,2,\dots,P, & j=s,s+1,\dots,S, \quad k=0,1,\dots,B-1 \\ (i,j,k) \rightarrow (i,j,k)\ is -(\lambda+\beta)\ if\ i=1,2,\dots,P-1, & j=0,1,\dots,s, \quad k=0 \\ (i,j,k) \rightarrow (i,j,k)\ is -(\lambda+\xi+\mu+\beta)\ if\ i=0,1,\dots,P, & j=s, \quad k=1 \\ (i,j,k) \rightarrow (i,j,k)\ is -(\lambda+\xi+\mu)\ if\ i=0,1, & j=s+1,\dots,S, \quad k=1,2,\dots,B \\ (i,j,k) \rightarrow (i,j,k)\ is -\lambda\ if\ i=0, & j=s+1,\dots,S, \quad k=0 \\ (i,j,k) \rightarrow (i,j,k)\ is -(\lambda+\beta+\delta\gamma)\ if\ i=1,2,P-1, & j=s, \quad k=0 \\ (i,j,k) \rightarrow (i,j+Q,k)\ is -\beta\ if\ i=P, & j=0,1,\dots,s, \quad k=0 \\ (i,j,k) \rightarrow (i,j,k)\ is -(\lambda+\delta\gamma)\ if\ i=1,2,\dots,P-1, & j=s+1,\dots,S, \quad k=0 \\ (i,j,k) \rightarrow (i,j,k)\ is -(\lambda+\xi+\delta\gamma)\ if\ i=P, & j=s+1,\dots,S, \quad k=1,2,\dots,B-1 \\ (i,j,k) \rightarrow (i,j,k)\ is -(\beta+\delta\gamma)\ if\ i=P, & j=s, \quad k=0 \\ (i,j,k) \rightarrow (i,j,k)\ is -\partial\gamma\ if\ i=P, & j=s+1,\dots,S, \quad k=1 \\ (i,j,k) \rightarrow (i,j,k)\ is -(\xi+\mu)\ if\ i=P, & j=s+1,\dots,S, \quad k=2,\dots,B \\ (i,j,k) \rightarrow (i,j,k)\ is -(\beta+\xi+\mu)\ if\ i=P, & j=s, \quad k=1 \end{cases}$$

$$\tilde{A} = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & \dots & P \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ P \end{array} & \begin{pmatrix} A & B & 0 & \cdots & 0 \\ C & D & B & \cdots & 0 \\ 0 & C & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & E \end{pmatrix} \end{array}$$

Where, $A_{i,j,k}$ is a $(S+1)(B+1) \times (S+1)(B+1)$ sub-matrix, which is given by

$$A_{i,j,k} = \begin{cases} A : if\ i \rightarrow i\ ; & i=0 \\ B : if\ i \rightarrow i+1\ ; & i>0 \\ C : if\ i \rightarrow i-1, & i \ge 1 \\ D : if\ i \rightarrow i, & i=1,2,\dots,P-1 \\ E : if\ i \rightarrow i; & i=P \end{cases}$$

With

***A*= (*a*_{u, v}) _{(S+1) (B+1)} ×_{(S+1) (B+1)}**

$$= \begin{cases} (u,v) \to (u,v)\ is - (\lambda+\beta)\ if\ u=0; v=0 \\ (u,v) \to (u,v)\ is - (\lambda+\beta+\xi+\mu)\ if\ u=1; v=1 \\ (u,v) \to (u,v)\ is - \lambda\ if\ u=s+1,\dots,S; v=0 \\ (u,v) \to (u,v)\ is - (\lambda+\xi+\mu)\ if\ u=s+1,\dots,S; v=1\ ,\dots,B \\ (u,v) \to (u+\beta,v)\ is\ \beta\ if\ u=0,\dots,s; v=0,\dots,B \\ (u,v) \to (u,v-1)\ is\ \xi\ if\ u=1,\dots,S; v=1,\dots,B \\ (u,v) \to (u-1,v-1)\ is\ \mu\ if\ u=1,\dots,S; v=1,\dots,B \\ Others\ elements\ are\ zero \end{cases}$$

***B*= (*a*_{u, v}) _{(S+1) (B+1) X (S+1) (B+1)}** $= diag(\lambda,\lambda,\dots,\lambda)$

***C*= (*a*_{u, v}) _{(S+1) (B+1) X (S+1) (B+1)}** $= (u,v) \to (u,v+1)$

$if\ u=1,2,\dots,S; v=0,1,\dots,B-1$

***D*= (*a*_{u, v}) _{(S+1) (B+1)} ×_{(S+1) (B+1)}**

$$= \begin{cases} (u,v) \to (u,v)\ is - (\lambda+\beta)\ if\ u=0; v=0 \\ (u,v) \to (u,v)\ is - (\lambda+\beta+\delta\gamma)\ if\ u=1; v=0 \\ (u,v) \to (u,v)\ is - (\lambda+\beta+\xi+\mu)\ if\ u=1; v=1 \\ (u,v) \to (u,v)\ is - (\lambda+\delta\gamma)\ if\ u=s+1,\dots,S; v=0 \\ (u,v) \to (u,v)\ is - (\lambda+\mu+\xi+\delta\gamma)\ if\ u=s+1,\dots,S; v=0 \\ (u,v) \to (u,v)\ is - (\lambda+\mu+\xi)\ if\ u=s+1,\dots,S; v=2,\dots,B \\ (u,v) \to (u+\beta,v)\ is\ \beta\ if\ u=0,\dots,s; v=0,\dots,B \\ (u,v) \to (u,v-1)\ is\ \xi\ if\ u=1,\dots,S; v=1,\dots,B \\ (u,v) \to (u-1,v-1)\ is\ \mu\ if\ u=1,\dots,S; v=1 \\ (u,v) \to (u-1,v-1)\ is\ (\mu+\theta)?\ if\ u=1,\dots,S; v=1,\dots,B \\ Others\ elements\ are\ zero \end{cases}$$

***E*= (*a*_{u, v}) _{(S+1) (B+1)} ×_{(S+1) (B+1)}**

$$= \begin{cases} (u,v) \to (u,v)\ is - \beta\ if\ u=0; v=0 \\ (u,v) \to (u,v)\ is - (\beta+\delta\gamma)\ if\ u=1; v=0 \\ (u,v) \to (u,v)\ is - (\beta+\xi+\mu)\ if\ u=1; v=1 \\ (u,v) \to (u,v)\ is - \delta\gamma\ if\ u=s+1,\dots,S; v=0 \\ (u,v) \to (u,v)\ is - (\mu+\xi+\delta\gamma)\ if\ u=s+1,\dots,S; v=1 \\ (u,v) \to (u,v)\ is - (\theta+\mu+\xi)\ if\ u=s+1,\dots,S; v=2,\dots,B \\ (u,v) \to (u+\beta,v)\ is\ \beta\ if\ u=0,\dots,s; v=0,\dots,B \\ (u,v) \to (u,v-1)\ is\ \xi\ if\ u=1,\dots,S; v=1,\dots,B \\ (u,v) \to (u-1,v-1)\ is\ \mu\ if\ u=1,\dots,S; v=1,\dots,B \\ \qquad\qquad Others\ elements\ are\ zero \end{cases}$$

So, we can write the portioned matrix as follows:

$$\tilde{A} = (a_{u,v})_{(P+1)(2S+B+1)\times(P+1)(2S+B+1)}$$

$$= \begin{cases} u \to u\ is\ A\ if\ u=0 \\ u \to u+1\ is\ B\ if\ u \ge 0 \\ u \to u-1\ is\ C\ if\ u \ge 1 \\ u \to u\ is\ D\ if\ u \ge 1,\dots,P \\ u \to u\ is\ E\ if\ u = P \end{cases}$$

2.4. Steady State Analysis

It can be seen from the structure of matrixÃ that the state space ***E*** is irreducible. Let the limiting distribution be denoted by $x^{(i,j,k)}$

$$x^{(i,j,k)} = \underset{t\to\infty}{Lt}\, pr[P(t),I(t),B(t)=(i,j,k)]; (i,j,k) \in E$$

Let $x = (x^S, x^{(S-1)}, \dots, x^1, x^0)$ with $x^j = [(x^{(i,j,o)}, x^{(i,j,1)}, \dots, x^{(i,j,B)})]$ for $i = 0,1,\dots,P$; $j = 0,1,\dots,S$

The limiting distribution exists, satisfies the following equation:

$x\tilde{A} = 0$ … **(1a)** and

$\sum_{i=0}^{P}\sum_{j=0}^{S}\sum_{k=0}^{B} x^{(i,j,k)} = 1$ … **(1b)**

The first equation of the above yields a set of equations, which can be represented as a general form in the following manner:

$$x^{(0)}A + x^{(1)}C = 0$$
$$x^{(0)}B + x^{(1)}D + x^{(2)}C = 0$$
$$x^{(1)}B + x^{(2)}D + x^{(3)}C = 0$$
$$x^{(2)}B + x^{(3)}D + x^{(4)}C = 0$$
… … … … … … … … … … … …
$$x^{(n-1)}B + x^{(n)}D + x^{(n+1)}C = 0$$
$$x^{(n-1)}B + x^{(n)}E = 0$$

The solution of the above equations (except the last one) can conveniently be expressed as:

$$\boldsymbol{x^{(i)} = x^{(0)}\beta_i};\ \text{for}\ \boldsymbol{i = 0, 1, \dots, s}$$

Where

$\boldsymbol{\beta_i = I, \forall i = 0}$

$\boldsymbol{\beta_i = -AC^{-1}, \forall i = 1}$

$\boldsymbol{\beta_i = -\beta_{i-1}DC^{-1} - \beta_{i-2}BC^{-1}, \forall i = 2, 3, \ldots, S}$

To compute $x^{(0)}$ we can use the following equation:

$x\tilde{A} = 0$ with (1b)

(see appendix-B & C)

2.5. System Performance Measures

In this section we derive some of the system performance measures of the system under consideration:

1. Expected Inventory level in the system is,

$$Q_1 = \sum_{i=0}^{p}\sum_{j=1}^{S} j \sum_{k=0}^{j} x^{(i,j,k)}$$

2. Expected number of customers in the buffer is,

$$Q_2 = \sum_{i=0}^{P}\sum_{j=1}^{S}\sum_{k=1}^{j} kx^{(i,j,k)}$$

3. Average customers lost to the system is,

$$Q_3 = \delta(1-\gamma)\sum_{i=0}^{P}\sum_{k=J-0}^{S} x^{(i,j,k)}$$

4. Expected number of customers in the pool,

$$Q_4 = \sum_{i=1}^{P}\sum_{j=0}^{S}\sum_{k=0}^{j} x^{(i,j,k)}$$

5. Expected number of reneging customer

$$Q_5 = \xi\sum_{i=0}^{P}\sum_{j=0}^{S}\sum_{k=1}^{j} kx^{(i,j,k)}$$

6. Expected re-order level of the system

$$Q_6 = \mu\sum_{i=0}^{P}\sum_{k=0}^{B} x^{(i,S+1,k)}$$

7. Expected rate that a customer will enter the pool is,

$$Q_7 = \lambda\sum_{i=0}^{P-1}\sum_{j=0}^{S}\sum_{k=0}^{j} x^{(i,j,k)}$$

8. The average rate at which the pooling customers will enter the buffer is given by

$$Q_8 = \delta\gamma\sum_{i=0}^{P}\sum_{j=s+1}^{S}\sum_{k=1}^{j} x^{(i,j,k)}$$

2.6. Cost Function

Define

c_1 Inventory holding cost per unit,

c_2=Cost of buffer customers in the system,

c_3= Cost of lost customers in the system,

c_4= Cost of pool customer in the system,

c_5=Cost of customer reneging to the system,

c_6= Cost of re-order of the system

So the total expected cost of the system is

E(TC)= $C_1Q_1 + C_2Q_2 + C_3Q_3 + C_4Q_4 + C_5Q_5 + C_6Q_6$

By using above cost function, we can exploit a lot of interesting feature and can make a sensitivity analysis.

2.7. Numerical Illustration

By giving values to the underlying parameters we provide some numerical illustrations. Take

$S = 3, s = 1, Q = 2, \lambda = 0.4, \beta = 0.6, \delta = 0.7,$

$\mu = 0.5, \zeta = 0.3, \gamma = 0.8, c1 = 1, c2 = 2, c3 = 3,$

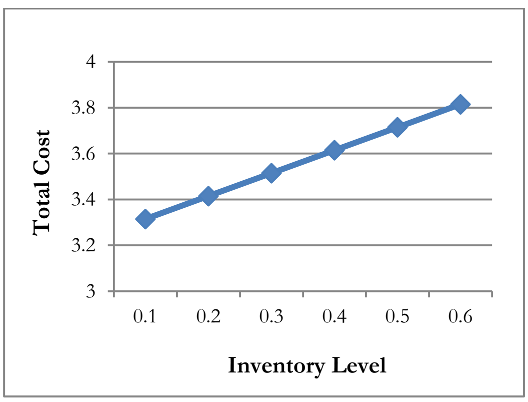
$c4 = 1, c5 = 2, c6 = 3,$

Then we get the measures as described in the table below:

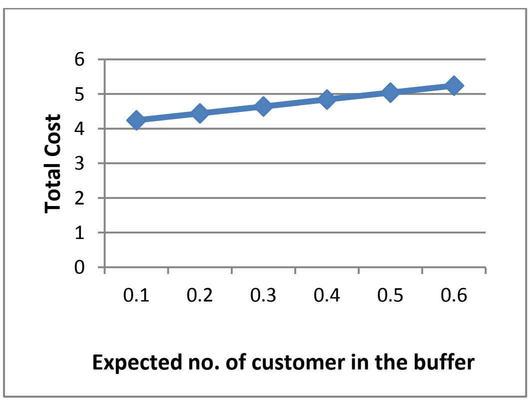
Table 1. Numerical values of different system characteristics

Q_1	Q_1	Q_1	Q_1	Q_1	Q_1	Q_1	Q_1	Q_1
1.883407100	0.528281000	0.023503314	0.703294510	0.331579506	0.240419735	0.320739264	0.083339060	5.098191769

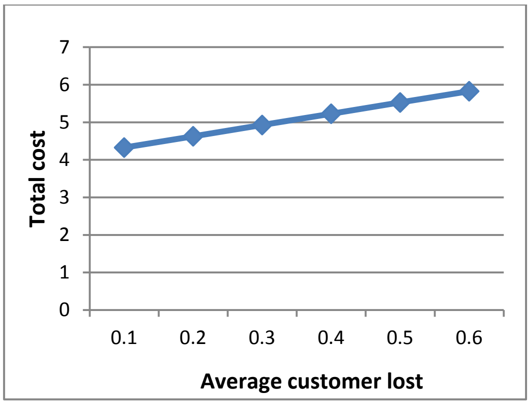
3. Graphical Presentation



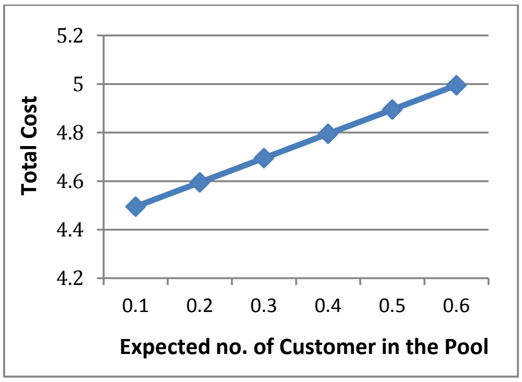
Graph 1. Inventory Level vs Total Cost



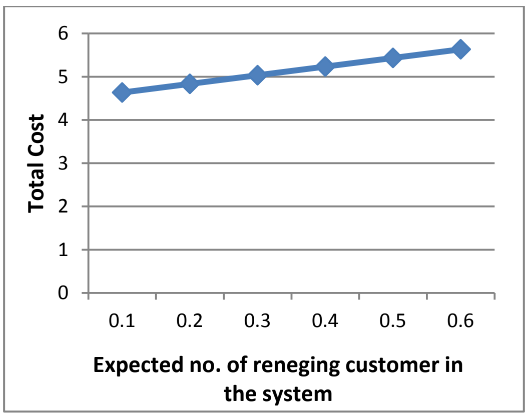
Graph 2. Expected no. of customer in the buffer vs Total Cost



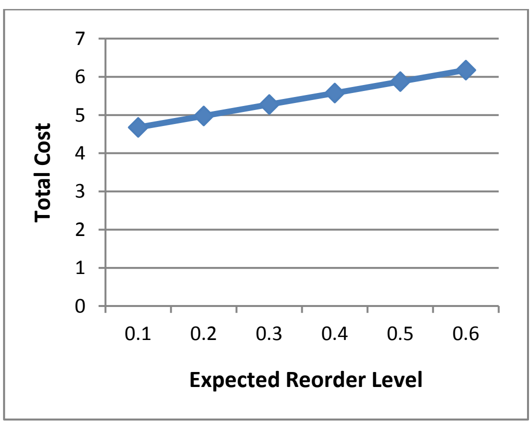
Graph 3. Average customer Lost vs Total Cost



Graph 4. Expected No. of Customers in the Pool vs Total Cost



Graph 5. Expected no. reneging customer in the system vs Total Cost



Graph 6. Expected Reorder Level vs Total Cost

4. Sensitivity Analysis

From the graph-1, it is observed that if the inventory level increased then the total cost is increased in significant amount. The result is obvious as the inventory level is increase it has impact on higher re-ordering, lost sales and also increase the cost of carrying pool and buffer customers. Hence the inventory level is vital to this system. From the graph-2, it is observed that expected number of customers in the buffer has a little impact as it increases total cost is increase. From the graph-3, it is observed that average customer lost has a little impact as it increases, total cost is increase. From the graph-4, it is observed that if expected number of customers in the full is increased than the total is measured in a significant amount. From the graph-5, it is observed that if expected of reneging customer has a little impact, as it increases total cost is increased. From the graph-6, it is observed that if expected the re-order level has a little impact, as it increase total cost is increased .Hence it must be scrutinized properly for selection per unit cost for reneging customers.

5. Conclusions

Generally, inventory models with perishability concept is more complex to analysis than the models where non-perishable items are tackled. The model become more complex, if the perishable rate is depending upon the number of items kept to the stock but sound more realistic. In this present paper, we tackled the same and explored some valuable properties for the model which have a great importance for the management to minimize the total cost.

Appendix-A

Table 2. State probability vectors of the system

x000	0.00603186	x200	0.01048370
x010	0.02447430	x210	0.01968480
x011	0.01206370	x211	0.01261830
x020	0.07989980	x220	0.06014400
x021	0.04171040	x221	0.03692030
x022	0.01973390	x222	0.01635130
x030	0.06045300	x230	0.04071170
x031	0.03165540	x231	0.02808250
x032	0.01453880	x232	0.00726022
x033	0.00614424	x233	0.00296096
x100	0.01043640	x300	0.01270210
x110	0.02115670	x310	0.01449690
x111	0.01604740	x311	0.00685556
x120	0.06582670	x320	0.07450700
x121	0.03680100	x321	0.01377160
x122	0.02607250	x322	0.00910097
x130	0.04711890	x330	0.05094810
x131	0.02786300	x331	0.01182710
x132	0.01316620	x332	0.00246194
x133	0.00543618	x333	0.00148048

Appendix-B

Theorem: If $x = \{x_i, i \geq 0\}$ is stationary distribution, then $\boldsymbol{x\tilde{A}} = \mathbf{0}$

Proof: We have from Kolmogorov forward differential equation

$$x_t' = x_t \tilde{A} \qquad \text{(a)}$$

$$x_{ij}'(t) = -P_{ij}(t)a(j,j) + \sum_{k \neq j} x_{ik}(t)a(k,j) \qquad \text{(b)}$$

Since **x** is stationary then $t \to \infty$, if limit exists, it is independent of time parameter and hence $x_{ij}'(t) \to 0$.

From equation (a)

we get, $-x_{ij}(t)a(j,j) + \sum_{k \neq j} P_{ik}(t)a(k,j) = 0$

In matrix notation which can be written as $\mathbf{x\tilde{A} = 0}$ and Normalizing condition hold; then

$\sum\sum\sum x^{(i,j,k)}$ and $\sum\sum\sum x^{(i,j,k)}$ can be completely evaluated.

By using the equation (1) with normalizing condition, we calculate all the steady state probability vector by using Mathematic software can be measured.

Appendix-C

To solve the equation $\boldsymbol{x\tilde{A}} = \mathbf{0}$, we use the Mathematica Code:

Solve[{-1.0x000+0x010+0.5x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000-1.0x010+0.3x011+0x020+0.5x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0x010-1.8x011+0x020+0x021+0.5x022+0x030+0x031+0x032+0x033+0x100+0.56x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0.6x000+0x010+0x011-0.4x020+0.3x021+0x022+0x030+0.5x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0x010+0x011+0x020-1.2x021+0.3x022+0x030+0x031+0.5x032+0x033+0x100+0x110+0x111+0.56x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0x010+0x011+0x020+0x021-1.2x022+0x030+0x031+0x032+0.5x033+0x100+0x110+0x111+0x120+0.56x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0.6x010+0x011+0x020+0x021+0x022-0.4x030+0.3x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0x010+0.6x011+0x020+0x021+0x022+0x030-1.2x031+0.3x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0.56x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031-1.2x032+0.3x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0.56x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032-1.2x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0.56x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0.4x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033-1.0x100+0x110+0.5x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0.4x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100-1.56x110+0.3x111+0x120+0.5x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0x010+0.4x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110-1.8x111+0x120+0x121+0.5x122+0x130+0x131+0x132+0x133+0x200+0.56x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0x010+0x011+0.4x020+0x021+0x022+0x030+0x031+0x032+0x033+0.6x100+0x110+0x111-0.96x120+0.3x121+0x122+0x130+0.5x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0x010+0x011+0x020+0.4x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120-1.76x121+0.3x122+0x130+0x131+0.5x132+0x133+0x200+0x210+0x211+0.56x220+0x221+0x222+0x230+0x231+0x232+0x

233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

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0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0.4x130+0x131+0x132+0x133+0x200+0.6x210+0x211+0x220+0x221+0x222-0.96x230+0.3x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0.4x131+0x132+0x133+0x200+0x210+0.6x211+0x220+0x221+0x222+0x230-1.76x231+0.3x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0.56x330+0x331+0x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0.4x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231-1.76x232+0.3x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0.56x331+0x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0.4x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232-1.2x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0.56x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0.4x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233-0.6x300+0x310+0.5x311+0x320+0x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0.4x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300-1.16x310+0.3x311+0x320+0.5x321+0x322+0x330+0x331+0x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0.4x211+0x220+0x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310-1.4x311+0x320+0x321+0.5x322+0x330+0x331+0x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0.4x220+0x221+0x222+0x230+0x231+0x232+0x233+0.6x3

00+0x310+0x311-0.56x320+0.3x321+0x322+0x330+0.5x331+0x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0.4x221+0x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320-1.36x321+0.3x322+0x330+0x331+0.5x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0.4x222+0x230+0x231+0x232+0x233+0x300+0x310+0x311+0x320+0x321-0.8x322+0x330+0x331+0x332+0.5x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0.4x230+0x231+0x232+0x233+0x300+0.6x310+0x311+0x320+0x321+0x322-0.56x330+0.3x331+0x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0.4x231+0x232+0x233+0x300+0x310+0.6x311+0x320+0x321+0x322+0x330-1.36x331+0.3x332+0x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0.4x232+0x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331-1.36x332+0.3x333==0,

0x000+0x010+0x011+0x020+0x021+0x022+0x030+0x031+0x032+0x033+0x100+0x110+0x111+0x120+0x121+0x122+0x130+0x131+0x132+0x133+0x200+0x210+0x211+0x220+0x221+0x222+0x230+0x231+0x232+0.4x233+0x300+0x310+0x311+0x320+0x321+0x322+0x330+0x331+0x332-0.8x333==0,

x000+x010+x011+x020+x021+x022+x030+x031+x032+x033+x100+x110+x111+x120+x121+x122+x130+x131+x132+x133+x200+x210+x211+x220+x221+x222+x230+x231+x232+x233+x300+x310+x311+x320+x321+x322+x330+x331+x332+x333==1},

{x000,x010,x011,x020,x021,x022,x030,x031,x032,x033,x100,x110,x111,x120,x121,x122,x130,x131,x132,x133,x200,x210,x211,x220,x221,x222,x230,x231,x232,x233,x300,x310,x311,x320,x321,x322,x330,x331,x332,x333}]

Output:

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