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Semi- Minimax Estimation of the Scale Parameter of Laplace Distribution under Symmetric and Asymmetric Loss Functions

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Abstract In this paper, semi- minimax estimation of the scale parameter of Laplace distribution is presented by applying the theorem of Lehmann (1950) under symmetric (quadratic) and asymmetric (entropy and MLINEX) loss functions. The results of comparison among these estimators are compared empirically using R- Code simulation study with respect to the mean square error (MSE). In general, the result has showed that the semi- minimax estimator under MLINEX loss function is the best estimator with respect to MSE for all sample sizes. It has also observed that, MSE's of the estimators is increasing with the increase of the scale parameter value. Finally, for all parameter values, an obvious reduction in MSE's has observed with the increase in sample size.

Keywords Semi- minimax estimator, Laplace distribution, Bayes estimator, Quadratic loss function, Entropy loss function, MLINEX loss function

1. Introduction

Semi- minimax estimation is an upgraded non- classical approach in the estimation area of statistical inference. The most important element in the Semi- minimax approach is the specification of the distribution function on the parameter space which is called prior distribution. In this paper we have studied the Semi- minimax estimators depend on the quadratic, entropy and MLINEX loss functions of the scale parameter of Laplace distribution. Al-kutubi and Ibrahim (2009) [1] compared Jeffrey prior and the extension of Jeffrey prior information for estimating the parameter of the exponential distribution. Asgharzadeh (2009) [2] Bayes estimators of the unknown parameter and the reliability function for the generalized exponential model has been derived under various loss functions such as the squared error, the absolute error, the squared log error, and the entropy loss functions. Dey (2008) [3] obtained Bayesian predictive intervals of the parameter of Rayleigh distribution. Amrollah (2011) [4] found the semi-minimax estimators of the scale parameter of the weibull distribution under quadratic and MLINEX loss functions. Masoud and Hassan (2010) [5] studied the minimax estimators of the shape parameter for the Burr type XII distribution under the squared log error, precautionary and weighted balanced squared error loss functions. Nassiri, Sajad and Hassan (2011) [6] studied the semi- minimax estimators of the parameter of the Rayleigh distribution for the well known quadratic and modified linear exponential loss functions and the efficiency of the estimators had also been studied. Podder (2004) [7] studied the minimax estimator of the Pareto distribution under quadratic and MLINEX loss functions. Also Shadrokh and Pazira (2010) [8] studied the minimax estimator for the minimax distribution under several loss functions. S. Ali et al [9] studied the scale parameter estimation of the Laplace model using different asymmetric loss functions for Laplace distribution.

It is our interest to study the Semi- minimax estimation of the scale parameter of Laplace distribution for different symmetric and asymmetric loss functions to see the comparative situation. Since Laplace distribution is used in hydrology to extreme events such as annual maximum one-day rainfall and river discharges. This distribution has also been used in speech recognition to model priors on discrete Furrier transform (DFT) coefficients and in joint photographic experts group (JPEG) image compression to model AC coefficients generated by a discrete cosine transform (DCT) [10].

Laplace distribution is a continuous probability distribution. It has generally two parameters. One is location parameter θ and other is scale parameter λ. Practically location parameter has limited use. Here only scale parameter is considered to estimate. A continuous random

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variable X is said to have Laplace (θ,λ) distribution if its probability density function (pdf) is given by [11]

$$f(x;\lambda,\theta)=\begin{cases}\frac{1}{2\lambda}e^{-\frac{|x-\theta|}{\lambda}} & ; -\infty<x<\infty,\ -\infty<\theta<\infty,\ \lambda>0\\ 0 & ;\ otherwise\end{cases} \quad (1)$$

where, θ is the location parameter and λ is the scale parameter.

The cumulative distribution function (cdf) of Laplace distribution is given by

$$F(x;\lambda,\theta)=\begin{cases}1-\frac{1}{2}\exp\left(-\frac{x-\theta}{\lambda}\right) & ;\text{ if } x\geq\theta\\ \frac{1}{2}\exp\left(\frac{x-\theta}{\lambda}\right) & ;\text{ if } x<\theta\end{cases}$$

2. Prior and Posterior Density Function of Parameter λ

Let us assume that λ has Inverse gamma prior distribution is [4] defined as

$$g(\lambda)=\frac{\beta^{a}}{\Gamma\alpha}e^{-\frac{\beta}{\lambda}}\lambda^{-\alpha-1};\alpha,\beta>0,\lambda>0$$

Then the posterior distribution of λ for the given random sample $X=(X_1,X_2,....,X_n)$ is given by

$$h(\lambda/x)=\frac{g(\lambda)L(\lambda;x_1,x_2,...,x_n)}{\int_0^\infty g(\lambda)L(\lambda;x_1,x_2,...,x_n)d\lambda}$$

$$=\frac{\frac{\beta^\alpha}{\Gamma\alpha}e^{-\frac{\beta}{\lambda}}\lambda^{-\alpha-1}\left(\frac{1}{2\lambda}\right)^n e^{-\frac{\sum_{i=1}^n|x_i-\theta|}{\lambda}}}{\int_0^\infty\frac{\beta^\alpha}{\Gamma\alpha}e^{-\frac{\beta}{\lambda}}\lambda^{-\alpha-1}\left(\frac{1}{2\lambda}\right)^n e^{-\frac{\sum_{i=1}^n|x_i-\theta|}{\lambda}}d\lambda}$$

$$=\frac{e^{-\frac{1}{\lambda}\left(\sum_{i=1}^n|x_i-\theta|+\beta\right)}\lambda^{-(\alpha+n)-1}}{\int_0^\infty e^{-\frac{1}{\lambda}\left(\sum_{i=1}^n|x_i-\theta|+\beta\right)}\lambda^{-(\alpha+n)-1}d\lambda}$$

$$=\frac{e^{-\frac{1}{\lambda}\left(\sum_{i=1}^n|x_i-\theta|+\beta\right)}\lambda^{-(\alpha+n)-1}}{\frac{\Gamma(\alpha+n)}{\left(\sum_{i=1}^n|x_i-\theta|+\beta\right)^{\alpha+n}}}$$

$$=\frac{e^{-\frac{1}{\lambda}\left(\sum_{i=1}^n|x_i-\theta|+\beta\right)}\left(\sum_{i=1}^n|x_i-\theta|+\beta\right)^{\alpha+n}}{\Gamma(\alpha+n)\lambda^{\alpha+n+1}} \quad (2)$$

$$h(\lambda/x)=\frac{e^{-\frac{S}{\lambda}}S^{\alpha+n}}{\Gamma(\alpha+n)\lambda^{\alpha+n+1}};$$

Here, $S=\sum_{i=1}^n|x_i-\theta|+\beta$

This implies that, the posterior density is recognized as the density of the Inverse Gamma (IG) distribution: $h(\lambda/x)\sim IG[(\alpha+n),S]$.

We have applied **Lehmann's Theorem** [12] in our article. Lehmann's Theorem states:

Let $\tau=\{F_\theta:\theta\in\Omega\}$ be a family of distribution function and D be a class of estimators of θ. Suppose, that $d^*\in D$ is a Baye's estimator against a prior distribution $\xi^*(\theta)$ on the parameter space Ω, and the risk function $R(d^*,\theta)=$ constant on Ω, then d^* is a minimax estimator of θ.

3. Bayes Estimator of Parameter λ under Quadratic Loss Function

We consider the quadratic loss function [3] of the form

$$L(\lambda,\hat{\lambda}_1)=\left(\frac{\lambda-\hat{\lambda}_1}{\lambda}\right)^2.$$

Then the Bayes estimator of λ for the above loss function is given by

$$\hat{\lambda}_1=\frac{E\left(\frac{1}{\lambda}/x\right)}{E\left(\frac{1}{\lambda^2}/x\right)} \quad (3)$$

Now

$$E\left(\frac{1}{\lambda}/x\right)=\int_0^\infty\frac{1}{\lambda}h(\lambda/x)d\lambda$$

$$=\int_0^\infty\frac{1}{\lambda}\frac{e^{-\frac{1}{\lambda}\left(\sum_{i=1}^n|x_i-\theta|+\beta\right)}\left(\sum_{i=1}^n|x_i-\theta|+\beta\right)^{\alpha+n}}{\Gamma(\alpha+n)\lambda^{\alpha+n+1}}d\lambda$$

$$=\frac{\left(\sum_{i=1}^n|x_i-\theta|+\beta\right)^{\alpha+n}}{\Gamma(\alpha+n)}\int_0^\infty e^{-\frac{1}{\lambda}\left(\sum_{i=1}^n|x_i-\theta|+\beta\right)}\lambda^{-(\alpha+n+1)-1}d\lambda$$

$$=\frac{\left(\sum_{i=1}^{n}|x_i-\theta|+\beta\right)^{\alpha+n}}{\Gamma(\alpha+n)}\frac{\Gamma(\alpha+n+1)}{\left(\sum_{i=1}^{n}|x_i-\theta|+\beta\right)^{\alpha+n+1}}$$

$$=\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n)\left(\sum_{i=1}^{n}|x_i-\theta|+\beta\right)} \quad (4)$$

Similarly,

$$E\left(\frac{1}{\lambda^2}/x\right)=\frac{\Gamma(\alpha+n+2)}{\Gamma(\alpha+n)\left(\sum_{i=1}^{n}|x_i-\theta|+\beta\right)^2} \quad (5)$$

Substituting (4) and (5) in (3) we get,

$$\hat{\lambda}_1=\frac{\sum_{i=1}^{n}|x_i-\theta|+\beta}{\alpha+n+1}$$

Hence, $\hat{\lambda}_1=\frac{\sum_{i=1}^{n}|x_i-\theta|+\beta}{\alpha+n+1}$ is the Bayes estimator under quadratic loss function. Now the risk function [3] of the estimator $\hat{\lambda}_1$ is

$$R\left(\hat{\lambda}_1\right)=E\left\{L\left(\lambda,\hat{\lambda}_1\right)\right\}$$

$$=\frac{1}{\lambda^2}\left\{\lambda^2-2\lambda E\left(\hat{\lambda}_1\right)+E\left(\hat{\lambda}_1^2\right)\right\}$$

$$=\frac{1}{\lambda^2}\left[\lambda^2-\frac{2\lambda}{(\alpha+n+1)}E\left(\sum_{i=1}^{n}|x_i-\theta|+\beta\right)\right]+\frac{1}{\lambda^2(\alpha+n+1)^2}E\left\{\left(\sum_{i=1}^{n}|x_i-\theta|+\beta\right)^2\right\}$$

Let, $T=\sum_{i=1}^{n}|x_i-\theta|$, such that the statistic T is distributed as $Gamma(n,\lambda^{-1})$

$$\Rightarrow R\left(\hat{\lambda}_1\right)=\frac{1}{\lambda^2}\left[\lambda^2-\frac{2\lambda}{(\alpha+n+1)}E(T+\beta)\right]+\frac{1}{\lambda^2(\alpha+n+1)^2}E(T+\beta)^2 \quad (6)$$

$$=\frac{1}{\lambda^2}\left[\begin{array}{l}\lambda^2-\frac{2\lambda}{(\alpha+n+1)}\{E(T)+\beta\}\\ +\frac{1}{(\alpha+n+1)^2}\{E(T^2)+2\beta E(T)+\beta^2\}\end{array}\right]$$

Since, $T\sim Gamma(n,\lambda^{-1})$ hence $E(T)=n\lambda$, $Var(T)=n\lambda^2$, $E(T^2)=\lambda^2 n(n+1)$

$$\Rightarrow R\left(\hat{\lambda}_1\right)=\frac{1}{\lambda^2}\left[\begin{array}{l}\lambda^2-\frac{2\lambda}{(\alpha+n+1)}(n\lambda+\beta)\\ +\frac{1}{(\alpha+n+1)^2}\{\lambda^2 n(n+1)+2\beta n\lambda+\beta^2\}\end{array}\right]$$

$$\therefore R\left(\hat{\lambda}_1\right)=1-\frac{2n}{(\alpha+n+1)}-\frac{2\beta}{\lambda(\alpha+n+1)}+\frac{n(n+1)}{(\alpha+n+1)^2}+\frac{2n\beta}{\lambda(\alpha+n+1)^2}+\frac{\beta^2}{\lambda^2(\alpha+n+1)^2} \quad (7)$$

From (7) it is clear that $R(\hat{\lambda}_1)$ is not constant. So $\hat{\lambda}_1$ is not minimax estimator exactly. Now return to (6) and let $\beta\to 0$, we get.

$$R\left(\hat{\lambda}_1\right)=\frac{1}{\lambda^2}\left[\lambda^2-\frac{2\lambda}{(\alpha+n+1)}E(T)+\frac{1}{(\alpha+n+1)^2}E(T^2)\right]$$

$$\Rightarrow R\left(\hat{\lambda}_1\right)=\frac{1}{\lambda^2}\left[\lambda^2-\frac{2\lambda}{(\alpha+n+1)}n\lambda+\frac{1}{(\alpha+n+1)^2}\lambda^2 n(n+1)\right]$$

$$\therefore R\left(\hat{\lambda}_1\right)=1-\frac{2n}{(\alpha+n+1)}+\frac{n(n+1)}{(\alpha+n+1)^2}$$

Which is independent of λ, hence according to the Lehmann's theorem it follows that $\hat{\lambda}_1$ is the semi- minimax estimator for the parameter λ.

4. Bayes Estimator of Parameter λ under Entropy Loss Function

Let us consider the entropy loss function [2] of the form:

$$L(\hat{\lambda},\hat{\lambda}_2)=b\left[\left(\frac{\hat{\lambda}_2}{\lambda}\right)-\log\left(\frac{\hat{\lambda}_2}{\lambda}\right)-1\right];b>0$$

Then the Bayes estimator of λ for the above loss function is given by: $\hat{\lambda}_2=\left[E\left(\frac{1}{\lambda}/x\right)\right]^{-1}$

From (4) we get,

$$E\left(\frac{1}{\lambda}/x\right)=\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n)\left(\sum_{i=1}^{n}|x_i-\theta|+\beta\right)}=\frac{\alpha+n}{\sum_{i=1}^{n}|x_i-\theta|+\beta}$$

Hence, $\hat{\lambda}_2 = \dfrac{\sum_{i=1}^{n}|x_i-\theta|+\beta}{\alpha+n}$ is the Bayes estimator under entropy loss function. Now the risk function [2] of the estimator $\hat{\lambda}_2$ is

$$R\left(\hat{\lambda}_2\right) = E\left\{L\left(\lambda, \hat{\lambda}_2\right)\right\}$$

$$= b\left[\frac{1}{\lambda}E(\hat{\lambda}_2) - E\left\{\log(\hat{\lambda}_2)\right\} + \log(\lambda) - 1\right]$$

$$= \frac{b}{\lambda}\frac{E\left(\sum_{i=1}^{n}|x_i-\theta|+\beta\right)}{(\alpha+n)} - bE\left\{\log\left(\frac{\left(\sum_{i=1}^{n}|x_i-\theta|+\beta\right)}{(\alpha+n)}\right)\right\}$$

$$-b\left\{\log(\lambda)-1\right\}$$

After simplification we can write

$$R(\hat{\lambda}_2) = b\left[\frac{1}{\lambda}\frac{E(T+\beta)}{(\alpha+n)} - E\left\{\log(T+\beta)\right\}\right] + b\left\{\log(\alpha+n)+\log(\lambda)-1\right\} \tag{8}$$

Now $E(T+\beta) = E(T)+\beta = n\lambda+\beta$ (9)

And, $E\left\{\log(T+\beta)\right\} = \int_0^\infty \log(t+\beta) f(t)dt$

$$= \int_0^\infty \log(t+\beta)\frac{1}{\lambda^n\Gamma n}e^{-\frac{t}{\lambda}}t^{n-1}dt$$

Recall that: $\log(1+x) = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{x^m}{m}$

So, $\log(t+\beta) = \log(\beta) + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{t^m}{m\beta^m}$

Hence, $E\left\{\log(T+\beta)\right\}$

$$= \int_0^\infty \left\{\log(\beta) + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{t^m}{m\beta^m}\right\}\frac{1}{\lambda^n\Gamma n}e^{-\frac{t}{\lambda}}t^{n-1}dt$$

$$= \frac{\log(\beta)}{\lambda^n\Gamma n}\int_0^\infty e^{-\frac{t}{\lambda}}t^{n-1}dt + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{m\beta^m\lambda^n\Gamma n}\int_0^\infty e^{-\frac{t}{\lambda}}t^{(m+n)-1}dt$$

$$= \frac{\log(\beta)}{\lambda^n\Gamma n}\Gamma n\lambda^n + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{m\beta^m\lambda^n\Gamma n}\Gamma(m+n)\lambda^{m+n}$$

$$= \log(\beta) + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{\Gamma(m+n)}{m\Gamma n}\left(\frac{\lambda}{\beta}\right)^m \tag{10}$$

Substituting (9) and (10) in (8) we get,

$$R(\hat{\lambda}_2) = b\left[\frac{1}{\lambda}\frac{(n\lambda+\beta)}{(\alpha+n)} - \log(\beta) - \sum_{m=1}^{\infty}(-1)^{m+1}\frac{\Gamma(m+n)}{m\Gamma n}\left(\frac{\lambda}{\beta}\right)^m\right] + b\left\{\log(\alpha+n)+\log(\lambda)-1\right\} \tag{11}$$

From (11) it is clear that $R(\hat{\lambda}_2)$ is not constant. So $\hat{\lambda}_2$ is not minimax estimator exactly. Now return to (8) and let *β→0*, we get.

$$R(\hat{\lambda}_2) = b\left[\frac{1}{\lambda}\frac{E(T)}{(\alpha+n)} - E\left\{\log(T)\right\} + \log(\alpha+n)+\log(\lambda)-1\right] \tag{12}$$

Now, $\mathrm{E}\left\{\log(\mathrm{T})\right\} = \int_0^\infty \log(t)g(t)dt = \int_0^\infty \log(t)\frac{1}{\lambda^n\Gamma n}e^{-\frac{t}{\lambda}}t^{n-1}dt$

Let, $y = \frac{t}{\lambda} \Rightarrow t = \lambda y$ So, $\frac{dt}{dy} = \lambda \Rightarrow dt = \lambda dy$

Then, $E\left\{\log(T)\right\} = \frac{1}{\lambda^n\Gamma n}\int_0^\infty \log(\lambda y)(\lambda y)^{n-1}e^{-y}\lambda dy$

$$= \frac{\log(\lambda)}{\Gamma n}\int_0^\infty y^{n-1}e^{-y}dy + \frac{1}{\Gamma n}\int_0^\infty \log(y)y^{n-1}e^{-y}dy$$

So, $E\left\{\log(T)\right\} = \log(\lambda) + \frac{\Gamma'(n)}{\Gamma n}$ (13)

Here, $\Gamma' n = \int_0^\infty \log(y)y^{n-1}e^{-y}dy$ is the first derivative of Γn with respect to *n*.

Substituting the value of *E(T)* and $E\left\{\log(T)\right\}$ in (12) we get

$$R(\hat{\lambda}_2) = b\left[\frac{1}{\lambda}\frac{n\lambda}{(\alpha+n)} - \log(\lambda) - \frac{\Gamma'(n)}{\Gamma n} + \log(\alpha+n)+\log(\lambda)-1\right]$$

$$\therefore R(\hat{\lambda}_2) = b\left[\frac{n}{(\alpha+n)} - \frac{\Gamma'(n)}{\Gamma n} + \log(\alpha+n) - 1\right]$$

Which is independent of λ, hence according to the Lehmann's theorem it follows that $\hat{\lambda}_2$ is the semi- minimax estimator for the parameter λ.

5. Bayes Estimator of Parameter λ under MLINEX Loss Function

Let, the MLINEX loss function [4] is defined as

$$L(\lambda,\hat{\lambda}_3) = k\left[\left(\frac{\hat{\lambda}_3}{\lambda}\right)^c - c\log\left(\frac{\hat{\lambda}_3}{\lambda}\right) - 1\right]; k>0, c\neq 0$$

Then the Bayes estimator of λ for the above loss function is given by

$$\hat{\lambda}_3 = \left[E\left(\frac{1}{\lambda^c} / x \right) \right]^{-\frac{1}{c}} \tag{14}$$

Now, $E\left(\frac{1}{\lambda^c} / x \right) = \int_0^\infty \frac{1}{\lambda^c} h\left(\lambda / x \right) d\lambda$

$$= \frac{\left(\sum_{i=1}^n |x_i - \theta| + \beta \right)^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty e^{-\frac{1}{\lambda}\left(\sum_{i=1}^n |x_i-\theta|+\beta \right)} \lambda^{-(\alpha+n+c)-1} d\lambda$$

$$= \frac{\left(\sum_{i=1}^n |x_i - \theta| + \beta \right)^{\alpha+n}}{\Gamma(\alpha+n)} \frac{\Gamma(\alpha+n+c)}{\left(\sum_{i=1}^n |x_i - \theta| + \beta \right)^{\alpha+n+c}}$$

$$= \frac{\Gamma(\alpha+n+c)}{\Gamma(\alpha+n)\left(\sum_{i=1}^n |x_i - \theta| + \beta \right)^{c}}$$

From equation (14) we get,

$$\hat{\lambda}_3 = \left[\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)} \right]^{\frac{1}{c}} \left(\sum_{i=1}^n |x_i - \theta| + \beta \right)$$

Hence, $\hat{\lambda}_3 = \left[\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)} \right]^{\frac{1}{c}} \left(\sum_{i=1}^n |x_i - \theta| + \beta \right)$ is the Bayes estimator under MLINEX loss function.

Now the risk function [4] of the estimator $\hat{\lambda}_3$ is

$$R\left(\hat{\lambda}_3\right) = E\left\{ L\left(\lambda, \hat{\lambda}_3 \right) \right\}$$

$$= k\left[\frac{1}{\lambda^c} E(\hat{\lambda}_3^{\,c}) - cE\left\{ \log(\hat{\lambda}_3) \right\} + c\log(\lambda) - 1 \right]$$

$$= k\left[\frac{1}{\lambda^c} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)} E\left(\sum_{i=1}^n |x_i - \theta| + \beta \right)^c \right.$$
$$-cE\left\{ \frac{1}{c}\log\left(\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)} \right) + \log\left(\sum_{i=1}^n |x_i - \theta| + \beta \right) \right\}$$
$$\left. + c\log(\lambda) - 1 \right]$$

$$R\left(\hat{\lambda}_3\right) = k\left[\frac{1}{\lambda^c} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)} E\left(T+\beta \right)^c \right]$$
$$- k\log\left\{ \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)} \right\} + \tag{15}$$
$$k\left[-cE\left\{ \log(T+\beta) \right\} + c\log(\lambda) - 1 \right]$$

Now, $E(T+\beta)^c = \int_0^\infty (t+\beta)^c f(t) dt$

$$= \int_0^\infty (t+\beta)^c \frac{1}{\lambda^n \Gamma n} e^{-\frac{t}{\lambda}} t^{n-1} dt$$

Recall that: $(x+y)^n = \sum_{m=0}^n {}^nC_m x^m y^{n-m}$

So, $(t+\beta)^c = \sum_{m=0}^c {}^cC_m t^m \beta^{c-m}$

Hence, $E(T+\beta)^c = \sum_{m=0}^c {}^cC_m \frac{\beta^{c-m}}{\lambda^n \Gamma n} \int_0^\infty e^{-\frac{t}{\lambda}} t^{(m+n)-1} dt$

$$= \sum_{m=0}^c {}^cC_m \frac{\beta^{c-m}}{\lambda^n \Gamma n} \frac{\Gamma(m+n)}{\lambda^{-(m+n)}}$$

$$\Rightarrow E(T+\beta)^c = \sum_{m=0}^c {}^cC_m \frac{\Gamma(m+n)}{\Gamma n} \beta^{c-m} \lambda^m \tag{16}$$

Substituting (10) and (16) in (15) we get,

$$R(\hat{\lambda}_3) = k\left[\frac{1}{\lambda^c} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)} \sum_{m=0}^c {}^cC_m \frac{\Gamma(m+n)}{\Gamma n} \beta^{c-m} \lambda^m \right.$$
$$- \log\left\{ \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)} \right\} \tag{17}$$
$$\left. - c\left\{ \log(\beta) + \sum_{m=1}^\infty (-1)^{m+1} \frac{\Gamma(m+n)}{m\Gamma n} \left(\frac{\lambda}{\beta} \right)^m \right\} \right]$$
$$+ c\log(\lambda) - 1$$

From (17) it is clear that $R(\hat{\lambda}_3)$ is not constant. So $\hat{\lambda}_3$ is not minimax estimator exactly. Now return to (14) and let *β→0*, we get.

$$R\left(\hat{\lambda}_3\right) = k\left[\frac{1}{\lambda^c} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)} E\left(T^c \right) - \log\left\{ \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)} \right\} \right] \tag{18}$$
$$- k\left[cE\left\{ \log(T) \right\} + c\log(\lambda) - 1 \right]$$

Here, $E(T^c) = \int_0^\infty t^c \frac{1}{\lambda^n \Gamma n} e^{-\frac{t}{\lambda}} t^{n-1} dt$

$$= \frac{1}{\lambda^n \Gamma n} \int_0^\infty e^{-\frac{t}{\lambda}} t^{(n+c)-1} dt = \frac{\Gamma(n+c)}{\Gamma n} \lambda^c \tag{19}$$

Substituting (13) and (19) in (18) we get,

$$R\left(\hat{\lambda}_3\right) = k\left[\frac{\Gamma(\alpha+n)\Gamma(n+c)}{\Gamma(\alpha+n+c)\Gamma n} - \log\left\{ \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)} \right\} - c\frac{\Gamma^{/}(n)}{\Gamma n} - 1 \right]$$

Which is independent of *λ*, hence according to the Lehmann's theorem it follows that $\hat{\lambda}_3$ is the semi- minimax estimator for the parameter *λ*.

6. Empirical Analysis

The estimated value and MSE of the different estimators of the scale parameter λ are computed by R- Code from the Laplace distribution, where

$$MSE(\hat{\lambda}) = E[\hat{\lambda} - \lambda]^2 = Var(\hat{\lambda}) + [Bias(\hat{\lambda})]^2$$

In the simulation study, we have chosen n= 5, 15, 25, 35 and 50 for several values of c= -1, 1, 2, also for λ= 1, 3; θ= 0.5, 1, 1.5, 2, -0.5, -1, -1.5, -2; α= 0.2, 1.2, 1.5 and β= 0.3, 0.5, all results are based on simulations (S= 3000). The results have summarized and tabulated in the following tables for each estimated value (EV) of the estimators and for all sample sizes (n) where C is for Criteria.

Table 1. Estimated Value and MSE of Different Estimators of Parameter λ of Laplace Distribution for Different Values of n When $\lambda = 1, \theta = 0.5, \alpha = 0.2$ and $\beta = 0.3$

n	C	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$		
				c=-1	c=1	c=2
5	EV	0.490	0.326	0.640	0.608	0.847
	MSE	0.233	0.185	0.195	0.182	0.198
15	EV	0.686	0.409	1.021	0.676	0.594
	MSE	0.149	0.131	0.103	0.124	0.135
25	EV	0.700	0.713	0.714	0.773	0.638
	MSE	0.130	0.116	0.099	0.114	0.117
35	EV	1.128	0.498	0.537	0.511	0.691
	MSE	0.122	0.108	0.097	0.108	0.113
50	EV	0.726	0.795	0.740	0.669	0.729
	MSE	0.113	0.105	0.095	0.104	0.106

Table 2. Estimated Value and MSE of Different Estimators of Parameter λ of Laplace Distribution for Different Values of n When $\lambda = 1, \theta = 1, \alpha = 0.2$ and $\beta = 0.5$

n	C	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$		
				c= -1	c=1	c=2
5	EV	0.523	0.623	0.810	0.648	0.592
	MSE	0.209	0.168	0.187	0.164	0.177
15	EV	0.638	0.680	0.739	0.689	0.667
	MSE	0.143	0.121	0.097	0.117	0.127
25	EV	0.745	0.775	0.813	0.781	0.766
	MSE	0.127	0.112	0.095	0.109	0.116
35	EV	0.499	0.513	0.531	0.516	0.509
	MSE	0.118	0.107	0.094	0.105	0.111
50	EV	0.658	0.671	0.687	0.673	0.667
	MSE	0.111	0.103	0.093	0.101	0.105

Table 3. Estimated Value and MSE of Different Estimators of Parameter λ of Laplace Distribution for Different Values of n When $\lambda = 1, \theta = 1.5, \alpha = 1.2$ and $\beta = 0.3$

n	C	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$		
				c=-1	c=1	c=2
5	EV	0.422	0.273	0.640	0.608	0.847
	MSE	0.287	0.230	0.195	0.182	0.198
15	EV	0.646	0.383	1.021	0.676	0.594
	MSE	0.173	0.154	0.103	0.124	0.135
25	EV	0.675	0.686	0.421	0.773	0.947
	MSE	0.146	0.131	0.097	0.114	0.121
35	EV	1.097	0.484	0.537	0.510	0.691
	MSE	0.133	0.119	0.097	0.109	0.113
50	EV	0.712	0.779	0.741	0.669	0.729
	MSE	0.121	0.114	0.095	0.104	0.106

Table 4. Estimated Value and MSE of Different Estimators of Parameter λ of Laplace Distribution for Different Values of n When $\lambda = 1, \theta = 2, \alpha = 1.5$ and $\beta = 0.5$

n	C	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$		
				c=-1	c=1	c=2
5	EV	0.432	0.498	0.810	0.648	0.592
	MSE	0.277	0.224	0.187	0.164	0.177
15	EV	0.591	0.627	0.739	0.689	0.667
	MSE	0.173	0.150	0.097	0.118	0.127
25	EV	0.710	0.737	0.813	0.781	0.766
	MSE	0.146	0.131	0.095	0.109	0.116
35	EV	0.482	0.495	0.531	0.516	0.509
	MSE	0.133	0.122	0.094	0.105	0.110
50	EV	0.641	0.654	0.687	0.673	0.667
	MSE	0.121	0.113	0.093	0.101	0.105

Table 5. Estimated value and MSE of different estimators of parameter λ of Laplace distribution for different values of n when $\lambda = 3, \theta = -0.5, \alpha = 0.2$ and $\beta = 0.3$

n	C	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$		
				c=-1	c=1	c=2
5	EV	1.375	1.638	2.130	1.704	1.556
	MSE	2.340	1.897	1.796	1.830	2.007
15	EV	1.841	1.962	2.130	1.988	1.925
	MSE	1.436	1.227	0.994	1.186	1.287
25	EV	2.189	2.276	2.390	2.295	2.250
	MSE	1.229	1.093	0.935	1.066	1.133
35	EV	1.464	1.506	1.559	1.514	1.493
	MSE	1.130	1.028	0.908	1.008	1.058
50	EV	1.949	1.988	2.037	1.996	1.977
	MSE	1.042	0.969	0.883	0.955	0.991

Table 6. Estimated Value and MSE of Different Estimators of Parameter λ of Laplace Distribution for Different Values of *n* When $\lambda=3, \theta=-1, \alpha=0.2$ and $\beta=0.5$

n	C	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$		
				c=-1	c=1	c=2
5	EV	1.407	1.677	2.180	1.744	1.592
	MSE	2.258	1.826	1.765	1.762	1.931
15	EV	1.853	1.975	2.144	2.001	1.938
	MSE	1.411	1.202	0.973	1.162	1.262
25	EV	2.197	2.284	2.398	2.303	2.258
	MSE	1.214	1.079	0.921	1.052	1.118
35	EV	1.470	1.511	1.565	1.520	1.499
	MSE	1.119	1.018	0.898	0.997	1.048
50	EV	1.953	1.992	2.041	2.000	1.981
	MSE	1.034	0.962	0.876	0.947	0.983

Table 7. Estimated Value and MSE of Different Estimators of Parameter λ of Laplace Distribution for Different Values of *n* When $\lambda=3, \theta=-1.5, \alpha=1.5$ and $\beta=0.3$

n	C	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$		
				c=-1	c=1	c=2
5	EV	1.136	1.311	2.130	1.704	1.556
	MSE	2.969	2.485	1.796	1.830	2.007
15	EV	1.704	1.807	2.130	1.988	1.925
	MSE	1.716	1.501	0.994	1.186	1.287
25	EV	2.086	2.165	2.390	2.295	2.250
	MSE	1.408	1.270	0.935	1.066	1.133
35	EV	1.413	1.452	1.559	1.514	1.493
	MSE	1.263	1.161	0.908	1.008	1.058
50	EV	1.901	1.938	2.037	1.996	1.977
	MSE	1.136	1.063	0.883	0.955	0.991

Table 8. Estimated Value and MSE of Different Estimators of Parameter λ of Laplace Distribution for Different Values of *n* When $\lambda=3,\ \theta=-2, \alpha=1.5$ and $\beta=0.5$

n	C	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$		
				c=-1	c=1	c=2
5	EV	1.163	1.342	2.180	1.744	1.592
	MSE	2.886	2.403	1.766	1.762	1.931
15	EV	1.715	1.819	2.144	2.001	1.938
	MSE	1.689	1.474	0.973	1.162	1.262
25	EV	2.093	2.172	2.398	2.303	2.258
	MSE	1.392	1.255	0.921	1.052	1.118
35	EV	1.419	1.458	1.565	1.520	1.499
	MSE	1.252	1.150	0.898	0.997	1.048
50	EV	1.905	1.942	2.041	2.000	1.981
	MSE	1.129	1.056	0.876	0.947	0.983

7. Conclusions

The results of the simulation study for estimating the semi- minimax estimator of the scale parameter λ of Laplace distribution when the location parameter θ is known are summarized and tabulated in above tables which contain the estimated value and MSE's we have observed. In the most cases, the results in table 1 to 8 have showed that the semi- minimax estimator under MLINEX loss function has the smallest MSE. That means this estimator is better than other estimators with different sample sizes.

Table (1, 2, 3, 4, 6 and 8) shows that for n= 5, the semi-minimax estimator under MLINEX loss function has smallest MSE for c= 1. But for n≥ 15, this estimator has smallest MSE for c= -1 and this is also smallest among all estimators. Also table (5 and 7) shows that for different sample sizes, the semi- minimax estimator under MLINEX loss function has smallest MSE for c= -1.

It has observed that, MSE's of estimators of scale parameter is increasing with the increase of the scale parameter value. It has also observed that the MSE's of different estimators of scale parameter λ decreases with increasing the sample sizes which matches with the result of S. Ali et al [9]. Finally, for all parameter values, an obvious reduction in MSE's has observed with the increase in sample size. We can apply our study in calculating annual maximum one- day rainfall and river discharges, to model priors on discrete Furrier transform (DFT) coefficients and AC coefficients generated by a discrete cosine transform.

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