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A Projection Method for Variational Inequalities over the Fixed Point Set

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Abstract In this paper, we introduce a new iteration method for solving a variational inequality over the fixed point set of a firmly nonexpansive mapping in $\mathbb{R}^n$, where the cost function is continuous and monotone, which is called the *projection method*. The algorithm is a variant of the subgradient method and projection methods.

Keywords Variational inequality, Subgradient algorithm, Firmly nonexpansive mapping

1. Introduction

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a firmly nonexpansive mapping, i.e., $\|T(x) - T(y)\|^2 \le \langle T(x) - T(y), x - y\rangle$ for all $x, y \in \mathbb{R}^n$, and mapping $F: \mathbb{R}^n \to R$. We consider the following variational inequalities over the fixed point set (shortly, VI(F, Fix(T))):

$$\text{Find } x^* \in Fix(T) \text{ such that } \langle F(x^*), x - x^*\rangle \ge 0 \quad \forall y \in Fix(T),$$

where Fix(T): = $\{x \in C: Tx = x\}$. Problem VI(F,Fix(T)) is a special class of equilibrium problems on the nonempty closed convex constraint set. Many iterative methods for solving such problems have been presented in [1, 2, 3, 4, 5, 8, 9].

In this paper, we investigate a new and efficient global algorithm for solving variational inequalities over the fixed point set of a firmly nonexpansive mapping. To solve the problem, most of current algorithms are based on the metric projection onto a nonempty closed convex constraint set, in general, which is not easy to compute. The fundamental difference here is that, at each main iteration in the proposed algorithm, we only require computing the simple projection. Moreover, by choosing suitable regularization parameters, we show that the iterative sequence globally converges to a solution of Problem *VI*(*F*,*Fix*(*T*)).

The paper is organized as follows. Section 2 recalls some concepts related to variational inequalities over the fixed point set of a nonexpansive mapping, that will be used in the sequel and a new iteration scheme. Section 3 investigates the convergence theorem of the iteration sequences presented in Section 2 as the main results of our paper.

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2. Preliminaries

We list some well known definitions and the projection under the Euclidean norm, which will be required in our following analysis.

Definition 2.1 Let C be a nonempty closed convex subset of in $\mathbb{R}^n$, we denote the metric projection on C by $Pr_C(\cdot)$, i.e,

$$\mathrm{Pr}_C(x) = \mathrm{argmin}\{\|y - x\|: y \in C\} \ \forall x \in R^n.$$

The mapping F: $C \to \mathbb{R}^n$ is said to be

(i) monotone on C if for each x,y ∈ C, ⟨F(x) - F(y), x - y⟩ ≥ 0;
(ii) pseudomonotone on C if for each x,y ∈ C, ⟨F(y), x - y⟩ ≥ 0 ⇒ ⟨F(y), x - y⟩ ≥ 0.

It is well-known that the gradient method in [10] solves the convex optimization problem:

$$\min\{f(x): x \in C\}, \qquad (2.1)$$

where C_i is a closed convex subset of $\mathbb{R}^n$ for all $i = 1, \ldots, m$, $C := \bigcap_{i=1}^m C_i$, and f is a differentiable convex function on C. The iteration sequence $\{x^k\}$ of the method is defined by

$$x^{k+1} := P_C\big(x^k - \lambda \nabla f(x^k)\big).$$

When C is arbitrary closed convex, in general, computation of the metric projection P_C is not necessarily easy and hence it is not effective for solving the convex optimization problem. To overcome this drawback, Yamada in [11] proposed a fixed point iteration method

$$x^{k+1} := T\big(x^k - \lambda_k \nabla f(x^k)\big).$$

where T is a nonexpansive mapping defined by

$T(x) := \sum_{i=1}^{m} \beta_i P_{C_i}(x)$ for all $x \in C$, $\beta_i \in (0, 1)$ such that $\sum_{i=1}^{m} \beta_i = 1$. Under certain parameters β_i ($i = 1, \ldots, m$), the sequence $\{x^k\}$ converges a solution to Problem (2.1). Very recently, Iiduka in [6] proposed the fixed point optimization algorithm for solving the following variational inequalities:

Finding $x^* \in C$ such that $\langle F(x^*), x - x^*\rangle \geq 0$, $\forall x \in C$,

where C is a nonempty closed convex subset of $\mathbb{R}^n$, F: $C \to \mathbb{R}^n$, over the fixed point set $Fix(T)$ of a firmly nonexpansive mapping T: $C \to \mathbb{R}^n$. In each iteration of the algorithm, in order to get the next iterate x^{k+1}, one orthogonal projection onto C included $Fix(T)$ is calculated, according to the following iterative step. Given the current iterate x^k, calculate

$$\begin{cases} y^k := T\big(x^k - \lambda_k F(x^k)\big), \\ x^{k+1} := Pr_C(\alpha_k x^k + (1-\alpha_k)y^k). \end{cases}$$

Under certain conditions over parameters λ_k, α_k ($k \geq 1$), and asymtotic optimization conditions $\bigcap_{i=1}^{m}\{u \in Fix(T): \langle F(x^k), x - x^k\rangle \leq 0\} \neq \emptyset$ is satisfied. Then, the iterative sequence x^k, converges a solution to the variational inequalities over the fixed point set of the firmly nonexpansive mapping. In fact, the asymtotic optimization condition, in some cases, is very difficult to define. In order to avoid this requirement, we propose a new iteration method without both the asymtotic optimization condition and computing the metric projection on a closed convex set. Our algorithm is described more detailed as follows.

Algorithm 2.2 *Initialization. Take a point* $x^0 \in \mathbb{R}^n$ *such that* $M \leq \|x^0\|$, $\eta_0 := \|x^0\|$, *a positive number* $\rho > 0$, *and the positive sequences* $\{\beta_k\}$, $\{\rho_k\}$, $\{\epsilon_k\}$ *verifying the following conditions:*

$$\rho < \rho_k, \lim_{k\to\infty} \epsilon_k = 0, \ \sum_{k=0}^{\infty} \frac{\beta_k}{\rho_k} < \infty. \tag{2.2}$$

Step 1. *Let* $x^k \in \mathbb{R}^n$. *Choose arbitrary* $\lambda_k \in (0, 1)$ *such that* $(1 - \lambda_k)(\|x^k\| + M) \leq \beta_k$ *for all* $k \geq 0$. *Define* $\gamma_k := max\{\rho_k, \|F(x^k)\|\}$, $\alpha_k := \frac{\beta_k}{\rho_k}$, *and* $C_k := \{x \in \mathbb{R}^n: \|x\| \leq \eta_k + 1\}$. $t^k := P_{C_k}(x^k - \alpha_k F(x^k))$.

Step 2. *Compute*

$$x^{k+1} := T(\lambda_k x^k + (1-\lambda_k)t^k), \eta_{k+1} := \max\{\eta_k, \|x^{k+1}\|\}, \ k = k+1.$$

Note that $C_k := \{x \in \mathbb{R}^n: \|x\| \leq \eta_k + 1\}$ is a closed ball. Therefore, the metric projection $P_{C_k}(x^k - \alpha_k F(x^k))$ is computed by

$$t^k = \frac{\eta_k + 1}{\|x^k - \alpha_k F(x^k)\|}(x^k - \alpha_k F(x^k)).$$

3. Convergent Results

To investigate the convergence of Algorithm 2.2, we recall the following technical lemmas, which will be used in the sequel.

Lemma 3.1 (see [7]) *Let* $\{a_k\}$, $\{b_k\}$ *and* $\{c_k\}$ *be the three nonnegative sequences satisfying the following condition:*

$$a_{k+1} \leq (1 + b_k)a_k + c_k.$$

If $\sum_{i=1}^{\infty} b_k < \infty$ and $\sum_{i=1}^{\infty} c_k < \infty$, then $\lim_{k\to\infty} a_k$ exists.

We are now in a position to prove some convergence theorems.

Theorem 3.2 *Let* C *be a nonempty closed convex subset of* $\mathbb{R}^n$, *T:* $C \to \mathbb{R}^n$ *is a firmly nonexpansive mapping such that Fix(T) is bounded by M > 0, and F:* $C \to \mathbb{R}^n$ *ismonotone. Then, the sequence* $\{c_k\}$ *generalized by Algorithm 2.2 converges to a solution of Problem VI(F, Fix(T)).*

Proof. We divide the proof into five steps.

Step 1. For each $x^* \in Sol(F, Fix(T))$, we have

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2(2-\lambda_k)\beta_k^2. \tag{3.1}$$

Indeed, from $t^k := P_{C_k}(x^k - \alpha_k F(x^k))$ it follows that

$$\langle \alpha_k F(x^k) + t^k - x^k, x - t^k\rangle \geq 0, \quad \forall x \in C_k. \tag{3.2}$$

Using the assumption $\|x\| \leq \|x^0\|$ for all x $\in$ $Fix(T)$ and $C_k \subseteq C_{k+1}$ for all $k \geq 0$, we have $Fix(T) \subseteq C$. Then, substituting $x = x^*$ into (3.2), we get

$$\langle \alpha_k F(x^k) + t^k - x^k, x^* - t^k\rangle \geq 0$$

Combinating this and the inequality

$$\|t^k - x^*\|^2 = \|x^k - x^*\|^2 - \|t^k - x^k\|^2 + 2\langle x^k - t^k, x^* - t^k\rangle,$$

we have

$$\|t^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|t^k - x^k\|^2 + 2\alpha_k\langle F(x^k), x^* - t^k\rangle,$$

Since (3.3), $x^{k+1} := T(\lambda_k x^k + (1-\lambda_k)t^k)$ and the equaltity

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \forall \lambda \in [0,1], x, y \in \mathbb{R}^n, \tag{3.4}$$

we get

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|T(\lambda_k x^k + (1-\lambda_k)t^k) - T(x^*)\|^2 \\ &\leq \|\lambda_k x^k + (1-\lambda_k)t^k - x^*\|^2 \\ &= \lambda_k\|x^k - x^*\|^2 + (1-\lambda_k)\|t^k - x^*\|^2 - \lambda_k(1-\lambda_k)\|t^k - x^k\|^2 \end{aligned}$$

$$\begin{aligned}&\leq \lambda_k\|x^k - x^*\|^2 + (1-\lambda_k)[\|x^k - x^*\|^2 - \|t^k - x^k\|^2 + 2\alpha_k\langle F(x^k), x^* - t^k\rangle]\\ &\quad - \lambda_k(1-\lambda_k)\|t^k - x^k\|^2\\ &= \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2\alpha_k(1-\lambda_k)\langle F(x^k), x^* - t^k\rangle.\end{aligned}$$

Thus

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2\alpha_k(1-\lambda_k)\langle F(x^k), x^* - t^k\rangle. \tag{3.5}$$

From $\gamma_k := \max\{\rho_k, \|F(x^k)\|\}$ and $\alpha_k := \frac{\beta_k}{\rho_k}$ it follows that

$$\alpha_k\|F(x^k)\| = \frac{\beta_k}{\rho_k}\|F(x^k)\| = \frac{\beta_k\|F(x^k)\|}{\max\{\rho_k, \|F(x^k)\|\}} \leq \beta_k. \tag{3.6}$$

By the definition of the metric projection Pr_{C_k} and (3.6), we have

$$\|t^k - x^k\|^2 \leq \langle \alpha_k F(x^k), x^k - t^k\rangle \|F(x^k)\|\|x^* - t^k\| \leq \beta_k\|t^k - x^k\|. \tag{3.7}$$

Combinating (3.5), (3.6) and (3.7), we get

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2\alpha_k(1-\lambda_k)\langle F(x^k), x^* - t^k\rangle\\ &\quad + 2\alpha_k(1-\lambda_k)\|F(x^k)\|\|x^k - t^k\|\\ &\leq \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2\alpha_k(1-\lambda_k)\|F(x^k)\|\,\|x^* - x^k\|\\ &\quad + 2(1-\lambda_k)\beta_k^2\\ &\leq \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2\beta_k(1-\lambda_k)(\|x^k\| + M) + 2(1-\lambda_k)\beta_k^2\\ &\leq \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2(2-\lambda_k)\beta_k^2.\end{aligned}$$

This implies (3.1).

Step 2. Claim that $\lim_{k\to\infty}\|x^{k+1} - x^k\| = 0$ and $\lim_{k\to\infty}\|x^k - T(x^k)\| = 0$.

Indeed, using (3.3), (3.4) and the definition of the firmly nonexpansive mapping, we have

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\leq \|T(\lambda_k x^k + (1-\lambda_k)t^k) - T(x^*)\|^2\\ &\leq \langle \lambda_k x^k + (1-\lambda_k)t^k - x^*, x^{k+1} - x^*\rangle\\ &= \frac{1}{2}[\|\lambda_k x^k + (1-\lambda_k)t^k - x^*\|^2 + \|x^{k+1} - x^*\|^2 - \|\lambda_k x^k + (1-\lambda_k)t^k - x^{k+1}\|^2]\\ &\leq \frac{1}{2}[\lambda_k\|x^k - x^*\|^2 + (1-\lambda_k)\|t^k - x^*\|^2 - \lambda_k(1-\lambda_k)\|x^k - t^k\|^2 + \|x^{k+1} - x^*\|^2]\\ &\quad - \frac{1}{2}[\lambda_k\|x^k - x^{k+1}\|^2 + (1-\lambda_k)\|t^k - x^{k+1}\|^2 - \lambda_k(1-\lambda_k)\|x^k - t^k\|^2]\\ &= \frac{1}{2}[\lambda_k\|x^k - x^*\|^2 + \|x^{k+1} - x^*\|^2 - \lambda_k\|x^k - x^{k+1}\|^2 - (1-\lambda_k)\|t^k - x^{k+1}\|^2\\ &\quad + (1-\lambda_k)\|t^k - x^*\|^2]\\ &\leq \frac{1}{2}[\lambda_k\|x^k - x^*\|^2 + \|x^{k+1} - x^*\|^2 - \lambda_k\|x^k - x^{k+1}\|^2 - (1-\lambda_k)\|t^k - x^{k+1}\|^2\\ &\quad + (1-\lambda_k)(\|x^k - x^*\|^2 - \|t^k - x^k\|^2 + 2\alpha_k\langle F(x^k), x^* - t^k\rangle)].\end{aligned}$$

Hence, we have

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - \lambda_k\|x^k - x^{k+1}\|^2 - (1-\lambda_k)\|t^k - x^{k+1}\|^2\\ &\quad - (1-\lambda_k)\|t^k - x^k\|^2 + 2(1-\lambda_k)\langle F(x^k), x^* - t^k\rangle.\end{aligned} \tag{3.8}$$

Applying Lemma 3.1 for the sequences in the inequality (3.1), there exists

$$A := \lim_{k\to\infty}\|x^k - x^*\|. \tag{3.9}$$

From Initialization of Algorithm 2.2 that $\lambda_k \in (0,1), (1-\lambda_k)(\|x^k\| + \|x^0\|) \leq \beta_k$ and $\sum_{k=1}^{\infty}\beta_k^2 < \infty$, it follows that

$$\lim_{k\to\infty}\lambda_k = 1.$$

Combinating this, (3.8) and (3.9), we get

$$\lim_{k\to\infty}\|x^{k+1} - x^k\| = 0$$

Using this, the nonexpansive property of T and $\lim_{k\to\infty}\lambda_k = 1$, have

$$\begin{aligned}&\|x^{k+1} - T(x^{k+1})\|^2\\ &= \|T(\lambda_k x^k + (1-\lambda_k)t^k) - T(x^{k+1})\|\\ &\leq \|\lambda_k x^k + (1-\lambda_k)t^k - x^{k+1}\|\\ &\leq \lambda_k\|x^k - x^{k+1}\| + (1-\lambda_k)\|t^k - x^{k+1}\|\\ &\leq \lambda_k\|x^k - x^{k+1}\| + (1-\lambda_k)\|t^k - x^{k+1}\|\\ &\to 0 \ as\ k \to \infty\end{aligned}$$

Since $\{x^k\}$ is bounded, there exists $\eta := \sup\{\lambda\eta_k : k \geq 0\}$

$< \infty$ and a subsequence $\{x^{k_i}\}$ which converges to $\bar{x}$ as $i \rightarrow \infty$.

Step 3. Claim that $\bar{x} \in Fix(T) \cap B(0, \eta + 1 - \delta)$, where $\delta \in (0,1)$ and the open ball is defined by

$$B(0, \eta + 1 - \delta) := \{x \in \mathbb{R}^n : \|x\| < \eta + 1 - \delta\}.$$

Indeed, from $\eta := \sup\{\eta_k : k \geq 0\} < \infty \; and \; \delta \in (0,1,)$, it follows that the existence of k_0 such that for $\eta_k \geq \eta - \delta$ for all $k \geq k_0$. It means that $B(0, \eta + 1 - \delta) \subseteq C_k$ for all $k \geq k_0$. Then, we have

$$\|\bar{x}\| = \lim_{i \to \infty} \|x^{k_i}\| \leq \eta < \eta + 1 - \delta.$$

Thus, $\bar{x} \in B(0, \eta + 1 - \delta)$.

Now we suppose that $\bar{x} \neq T(\bar{x})$. By Step 2 and Opial's condition, we get

$$\begin{aligned}
&\lim_{i\to\infty} \|x^{k_i} - x^k\| < \lim_{i\to\infty} \|x^{k_i} - T(\bar{x})\| \\
&\leq \lim_{i\to\infty} (\|x^{k_i} - T(x^{k_i})\| + \|T(x^{k_i}) - T(\bar{x})\|) \\
&= \lim_{i\to\infty} \|T(x^{k_i}) - T(\bar{x})\| \\
&\leq \lim_{i\to\infty} \|x^{k_i} - \bar{x}\|
\end{aligned}$$

This is a contradiction. So, $\bar{x} = T(\bar{x})$.

Step 4. Claim that $\bar{x} \in Sol(F, Fix(T))$ and the sequence $\{x^k\}$ converges to $\bar{x}$.

Indeed, from (3.6), it follows that

$$0 \leq \rho \|F(x^k)\| \leq \|F(x^k)\| \max \{\rho_{k,} \|F(x^k)\|\} = \alpha_k \|F(x^k)\| \leq \beta_k.$$

Using $\sum_{k=0}^{\infty} \beta_k < \infty$ and $\rho > 0$, we have $\lim_{i\to\infty} \|F(x^k\| = 0$. Combinating this and Step 3, we have

$$\langle F(\bar{x}), y - \bar{x} \rangle \geq 0, \;\; \forall y \in Fix(T) \cap B(0, \eta + 1 - \delta).$$

Denote g(z):= $\langle F(\bar{x}), y - \bar{x} \rangle$. Then, g is convex and

$$g(z) \geq g(\bar{x}) = 0, \;\; \forall z \in Fix(T) \cap B(0, \eta + 1 - \delta).$$

Thus, $\bar{x}$ is a local minimizer of g. Since Fix(T) is nonempty convex, $\bar{x}$ is also a global minimizer of g, i.e., g(z) $\geq$ g($\bar{x}$) for all $z \in Fix(T)$. This means that

$$\langle F(\bar{x}), y - \bar{x} \rangle \geq 0, \;\; \forall z \in Fix(T).$$

So, $\bar{x} \in$ Sol(F, Fix(T)).

To prove $\{x^k\}$ converges to $\bar{x}$, we suppose that the subsequence $\{x^{k_i}\}$ also converges to $\hat{x}$ as j $\rightarrow \infty$. By a same way, we also have $\hat{x} \in$ VI(F, Fix(T)). Suppose that $\bar{x} \neq \hat{x}$. Then, using Opial's condition, we have

$$\begin{aligned}
\lim_{i\to\infty} \|x^k - \bar{x}\| &= \lim_{i\to\infty} \|x^{k_i} - \bar{x}\| \\
&\leq \lim_{i\to\infty} \|x^{k_i} - \hat{x}\| \\
&= \lim_{k\to\infty} \|x^k - \hat{x}\| \\
&= \lim_{k\to\infty} \|x^{k_i} - \hat{x}\| \\
&< \lim_{j\to\infty} \|x^{k_i} - \bar{x}\| \\
&= \lim_{k\to\infty} \|x^k - \bar{x}\|
\end{aligned}$$

This is a contradiction. Thus, the sequence $\{x^k\}$ converges to $\bar{x} \in$ Sol(F, Fix(T)).

4. Conclusions

This paper presented an iterative algorithm for solving variational inequalities over the fixed point set of a nonexpansive mapping *T*. By choosing the suitable regular parameters, we show that the sequences generated by the algorithm globally converge to a solution of Problem *VI*(*F*, *fix*(*T*)). Comparing with the current methods, the fundamental difference here is that, the algorithm only requires the continuity of the mapping *F* and convergence of the proposed algorithms only require *F* to satisfy monotonicity. Moreover, in general, computing the exact subgradient of a subdifferentiable function is too expensive, our algorithm only requires to compute approximate.

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