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A Three-Parameter Lindley Distribution

Rama Shanker[1,*], Kamlesh Kumar Shukla[1], Ravi Shanker[2], Tekie Asehun Leonida[3]

[1]Department of Statistics, Eritrea Institute of Technology, Asmara, Eritrea
[2]Department of Mathematics, G.L.A. College, N.P University, Daltonganj, Jharkhand, India
[3]Department of Applied Mathematics, University of Twente, Enschede, The Netherlands

Abstract A three-parameter Lindley distribution, which includes some two-parameter Lindley distributions introduced by Shanker and Mishra (2013 a, 2013 b), Shanker et al (2013), Shanker and Amanuel (2013), two-parameter gamma distribution, and one parameter exponential and Lindley distributions as special cases, has been proposed for modeling lifetime data. Its statistical properties including its shape, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, order statistics, Renyi entropy measure, Bonferroni and Lorenz curves, stress-strength reliability have been discussed. For estimating its parameters, maximum likelihood estimation has been discussed. Finally, a numerical example has been presented to test the goodness of fit of the proposed distribution and the fit has been compared with the three-parameter generalized Lindley distribution.

Keywords Lindley distribution, Quasi Lindley distribution, Two-parameter Lindley distribution, Generalized Lindley distribution, Mathematical and Statistical properties, Moments, Maximum Likelihood estimation, Goodness of fit

1. Introduction

The modeling and analyzing lifetime data are crucial in many applied sciences including medicine, engineering, insurance and finance, amongst others. During a short span of time a number of one parameter, two-parameter and three-parameter lifetime distributions have been introduced in statistical literature for modeling lifetime data from biomedical science and engineering. But each of these lifetime distributions has advantages and disadvantages over one another due to the number of parameters involved, shape, hazard rate function and mean residual life function, among others.

Lindley (1958) distribution, introduced in the context of Bayesian analysis as a counter example of fiducial statistics, is defined by its probability density function (p.d.f) and cumulative distribution function (c.d.f)

$$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\quad ;x>0,\,\theta>0 \qquad (1.1)$$

$$F(x;\theta)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\quad ;x>0,\theta>0 \qquad (1.2)$$

A detailed study about its important mathematical and statistical properties, estimation of parameter and application showing the superiority of Lindley distribution over exponential distribution for the waiting times before service of the bank customers has been done by Ghitany *et al* (2008). Shanker *et al* (2015) have comparative study on modeling of lifetime data using one parameter Lindley (1958) distribution and exponential distribution and concluded that there are many lifetime data where exponential distribution gives better fit than Lindley distribution. Sankaran (1970) obtained a Poisson mixture of Lindley distribution and named it discrete Poisson-Lindley distribution (PLD) and discussed its various properties, estimation of parameter and goodness of fit. Shanker and Hagos (2015) have introduced a simple method for estimating parameter of PLD and discussed its applications for modeling count data from biological sciences.

The probability density function (p.d.f.) and cumulative distribution function (c.d.f) of quasi Lindley distribution (QLD) of Shanker and Mishra (2013a) are given by

$$f(x;\alpha,\theta)=\frac{\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}\quad ;x>0,\,\theta>0,\alpha>-1 \qquad (1.3)$$

$$F(x;\alpha,\theta)=1-\left[\frac{1+\alpha+\theta x}{\alpha+1}\right]e^{-\theta x};x>0,\theta>0,\alpha>-1 \qquad (1.4)$$

At $\alpha=\theta$, both (1.3) and (1.4) reduce to the corresponding expressions (1.1) and (1.2) of Lindley distribution. Shanker and Mishra (2016) have obtained a Poisson mixture of a quasi Lindley distribution and named it a ‘quasi Poisson-Lindley distribution (QPLD)’ and discussed its various statistical and mathematical properties, estimation of parameters, and applications. Shanker *et al* (2016 a) have discussed many interesting properties of QLD and its

* Corresponding author:
shankerrama2009@gmail.com (Rama Shanker)
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applications for modeling various lifetime data and observed that it gives better fit in most of the data sets.

The probability density function (p.d.f.) and cumulative distribution function (c.d.f) of two-parameter Lindley distribution (TPLD) of Shanker and Mishra (2013b) are given by

$$f(x;\alpha,\theta)=\frac{\theta^2}{\alpha\theta+1}(\alpha+x)e^{-\theta x}\quad ;x>0,\,\theta>0,\alpha\theta>-1 \quad (1.5)$$

$$F(x;\alpha,\theta)=1-\left[\frac{1+\alpha\theta+\theta x}{\alpha\theta+1}\right]e^{-\theta x};x>0,\theta>0,\alpha\theta>-1 \quad (1.6)$$

At $\alpha=1$, both (1.5) and (1.6) reduce to the corresponding expressions (1.1) and (1.2) of Lindley distribution. Shanker and Mishra (2014) have obtained a Poisson mixture of a two- Lindley distribution and named it a two-parameter Poisson-Lindley distribution and discussed its various statistical and mathematical properties, estimation of parameters, and applications. Shanker *et al* (2016 b) have discussed many interesting properties of TPLD and its applications for modeling various lifetime data and observed that it gives better fit in most of the data sets.

The probability density function (p.d.f.) and cumulative distribution function (c.d.f) of another two-parameter Lindley distribution, introduced by Shanker *et al* (2013) are given by

$$f(x;\theta,\alpha)=\frac{\theta^2}{\theta+\beta}(1+\beta x)e^{-\theta x}\quad ;x>0,\,\theta>0,\theta+\beta>0 \quad (1.7)$$

$$F(x;\theta,\beta)=1-\left[1+\frac{\beta x}{\theta+\beta}\right]e^{-\theta x};x>0,\theta>0,\theta+\beta>0 \quad (1.8)$$

At $\beta=1$, both (1.7) and (1.8) reduce to the corresponding expressions (1.1) and (1.2) of Lindley distribution. Shanker *et al* (2012) obtained a Poisson mixture of two-parameter Lindley distribution, named it a 'discrete two-parameter Poisson-Lindley distribution' and studied its various properties, estimation of parameters and applications.

The probability density function (p.d.f.) and cumulative distribution function (c.d.f) of a new quasi Lindley distribution (NQLD), introduced by Shanker and Amanuel (2013) are given by

$$f(x;\alpha,\theta)=\frac{\theta^2}{\theta^2+\alpha}(\theta+\alpha x)e^{-\theta x}\quad ;x>0,\,\theta>0 \quad (1.9)$$

$$F(x;\alpha,\theta)=1-\left[1+\frac{\theta\alpha x}{\theta^2+\alpha}\right]e^{-\theta x};x>0,\theta>0 \quad (1.10)$$

Where $\left(\theta^2+\alpha>0 \text{ and } \theta+\alpha x<0\right)$ or $\left(\theta^2+\alpha<0 \text{ and } \theta+\alpha x<0\right)$

At $\alpha=\theta$, both (1.9) and (1.10) reduce to the corresponding expressions (1.1) and (1.2) of Lindley distribution. Shanker and Tekie (2014) obtained a new quasi Poisson-Lindley distribution by taking a Poisson mixture of NQLD, discussed its various statistical and mathematical properties, estimation of parameters and applications for count data.

The probability density function of three-parameter generalized Lindley distribution (TPGLD) introduced by Zakerzadeh and Dolati (2009) having parameters α,β, and θ is given by

$$f_1(x;\alpha,\beta,\theta)=\frac{\theta^{\alpha+1}}{(\beta+\theta)}\frac{x^{\alpha-1}}{\Gamma(\alpha+1)}(\alpha+\beta x)e^{-\theta x};\quad (1.11)$$
$$x>0,\alpha>0,\beta>0,\theta>0$$

Clearly the gamma distribution, the Lindley (1958) distribution and the exponential distribution are particular cases of (2.1) for $(\beta=0)$, $(\alpha=\beta=1)$ and $(\alpha=1,\beta=0)$ respectively. The discussion about its properties, estimation of parameters and applications are available in Zakerzadeh and Dolati (2009).

The corresponding distribution function of the TPGLD can be obtained as

$$F_1(x;\alpha,\beta,\theta)=1-\frac{\alpha(\beta+\theta)\Gamma(\alpha,\theta x)+\beta(\theta x)^{\alpha}e^{-\theta x}}{(\beta+\theta)\Gamma(\alpha+1)};$$
$$x>0,\alpha>0,\beta>0,\theta>0 \quad (1.12)$$

where $\Gamma(\alpha,z)$ is the upper incomplete gamma function defined as

$$\Gamma(\alpha,z)=\int_z^{\infty}e^{-y}\,y^{\alpha-1}\,dy;\alpha>0,\,z\geq 0$$

Recently Shanker (2016) has detailed study about TPGLD and obtained expressions for coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function and the mean residual life function. Shanker (2016) has detailed comparative study of TPGLD and three-parameter generalized gamma distribution (TPGGD) and observed that in most of the data sets from medical science and engineering TPGGD gives better fit than TPGLD.

There are many situations where these distributions are not suitable for modeling lifetime data from theoretical or applied point of view. Therefore, an attempt has been made in this paper to obtain a new distribution which is flexible than these lifetime distributions for modeling lifetime data in reliability and in terms of its hazard rate shapes. Various interesting mathematical and statistical properties of the proposed distribution have been discussed. The estimation of the parameters of the proposed distribution has been discussed using maximum likelihood estimates and the goodness of fit of the distribution has been discussed with a real lifetime data. Finally the goodness of fit of the proposed distribution has been compared with three-parameter generalized Lindley distribution, introduced by Zakerzadeh and Dolati (2009).

2. A Three-Parameter Lindley Distribution

The probability density function (p.d.f.) of a three-parameter Lindley distribution (ATPLD) can be introduced as

$$f(x;\theta,\alpha,\beta)=\frac{\theta^2}{\theta\alpha+\beta}(\alpha+\beta x)e^{-\theta x} \quad ; \tag{2.1}$$

$$x>0,\ \theta>0, \beta>0,\ \theta\alpha+\beta>0$$

It can be easily verified that the two-parameter quasi Lindley distribution of Shanker and Mishra (2013 a), two-parameter Lindley distribution of Shanker and Mishra (2013 b), two-parameter Lindley distribution of Shanker et al (2013), a new two-parameter quasi Lindley distribution of Shanker and Amanuel (2013), Lindley distribution introduced by Lindley (1958), Gamma $(2,\theta)$ distribution and exponential distribution are particular cases of a three-parameter Lindley distribution (ATPLD) for $(\beta=\theta)$, $(\beta=1)$, $(\alpha=1)$, $(\alpha=\theta,\beta=\alpha)$, $(\alpha=\beta=1)$, $(\alpha=\theta)$ and $(\beta=0)$ respectively.

This distribution can be easily expressed as a mixture of exponential (θ) and gamma $(2,\theta)$ distributions with mixing proportion $\frac{\theta\alpha}{\theta\alpha+\beta}$. We have

$$f(x;\theta,\alpha,\beta)=p\,g_1(x)+(1-p)\,g_2(x)$$

where $p=\frac{\theta\alpha}{\theta\alpha+\beta}$, $g_1(x)=\theta e^{-\theta x}$, and $g_2(x)=\theta^2 x e^{-\theta x}$.

The corresponding cumulative distribution function (c.d.f.) of (2.1) is given by

$$F(x;\theta,\alpha,\beta)=1-\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x} \quad ; \tag{2.2}$$

$$x>0, \theta>, \beta>0, \theta\alpha+\beta>0$$

The graph of the p.d.f. and the c.d.f. of ATPLD for different values of θ,α,β are shown in figures 1 and 2

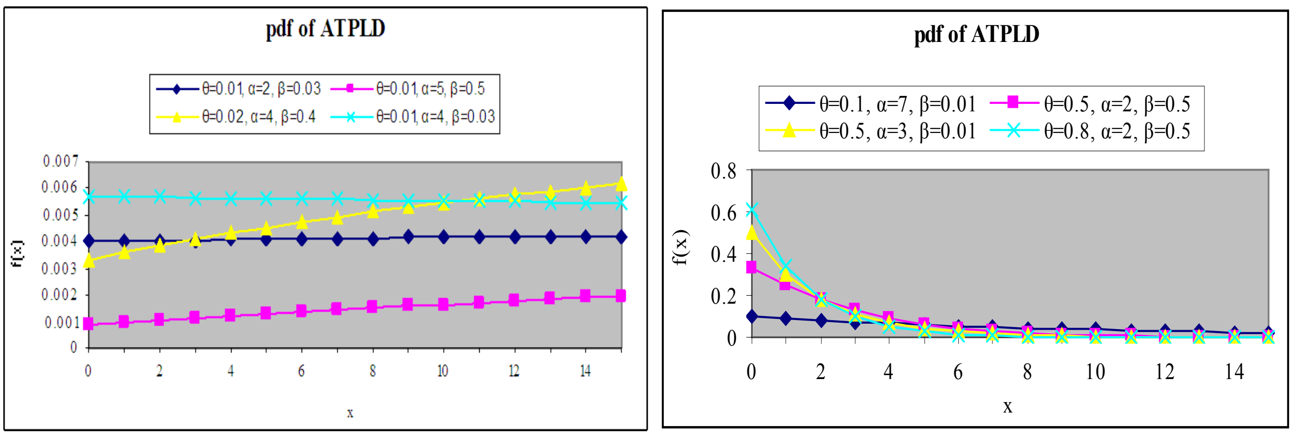

Figure 1. Graph of the pdf of ATPLD for different values of parameters θ,α,β

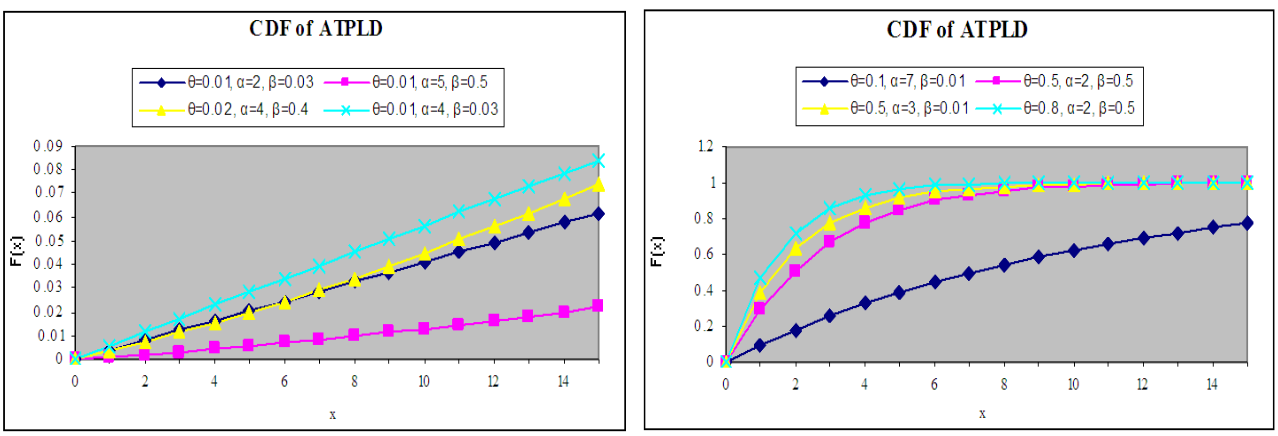

Figure 2. Graph of the cdf of ATPLD for different values of parameters θ,α,β

3. Statistical Constants

The r th moment about origin of ATPLD (2.1) can be obtained as

$$\mu_r' = \frac{r!\left[\theta\alpha+(r+1)\beta\right]}{\theta^r(\theta\alpha+\beta)} \ ; r=1,2,3,...$$

The first four moments about origin of ATPLD are as follows

$$\mu_1' = \frac{\theta\alpha+2\beta}{\theta(\theta\alpha+\beta)},\ \mu_2' = \frac{2(\theta\alpha+3\beta)}{\theta^2(\theta\alpha+\beta)},$$

$$\mu_3' = \frac{6(\theta\alpha+4\beta)}{\theta^3(\theta\alpha+\beta)},\ \mu_4' = \frac{24(\theta\alpha+5\beta)}{\theta^4(\theta\alpha+\beta)}$$

Thus the moments about mean of ATPLD are obtained as

$$\mu_2 = \frac{\theta^2\alpha^2+4\theta\alpha\beta+2\beta^2}{\theta^2(\theta\alpha+\beta)^2}$$

$$\mu_3 = \frac{2\left(\theta^3\alpha^3+6\theta^2\alpha^2\beta+6\theta\alpha\beta^2+2\beta^3\right)}{\theta^3(\theta\alpha+\beta)^3}$$

$$\mu_4 = \frac{3\left(3\theta^4\alpha^4+24\theta^3\alpha^3\beta+44\theta^2\alpha^2\beta^2+32\theta\alpha\beta^3+8\beta^4\right)}{\theta^4(\theta\alpha+\beta)^4}$$

The coefficient of variation $(C.V)$, coefficient of skewness$\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis (β_2) and index of dispersion (γ) of ATPLD are thus obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^2\alpha^2+4\theta\alpha\beta+2\beta^2}}{\theta\alpha+2\beta}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\left(\theta^3\alpha^3+6\theta^2\alpha^2\beta+6\theta\alpha\beta^2+2\beta^3\right)}{\left(\theta^2\alpha^2+4\theta\alpha\beta+2\beta^2\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\left(3\theta^4\alpha^4+24\theta^3\alpha^3\beta+44\theta^2\alpha^2\beta^2+32\theta\alpha\beta^3+8\beta^4\right)}{\left(\theta^2\alpha^2+4\theta\alpha\beta+2\beta^2\right)^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^2\alpha^2+4\theta\alpha\beta+2\beta^2}{\theta(\theta\alpha+\beta)(\theta\alpha+2\beta)}$$

4. Hazard Rate Function and Mean Residual Life Function

Let $f(x)$ and $F(x)$ be the p.d.f. and c.d.f of a continuous random variable. The hazard rate function (also known as the failure rate function) and the mean residual life function of X are respectively defined as

$$h(x) = \lim_{\Delta x\to 0}\frac{P\left(X<x+\Delta x \middle| X>x\right)}{\Delta x} = \frac{f(x)}{1-F(x)} \quad (4.1)$$

and

$$m(x) = E\left[X-x \middle| X>x\right] = \frac{1}{1-F(x)}\int_x^\infty\left[1-F(t)\right]dt \quad (4.2)$$

The corresponding hazard rate function, $h(x)$ and the mean residual life function, $m(x)$ of ATPLD are obtained as

$$h(x) = \frac{\theta^2(\alpha+\beta x)}{\theta\beta x+(\theta\alpha+\beta)} \quad (4.3)$$

and

$$m(x) = \frac{1}{\left[\theta\beta x+(\theta\alpha+\beta)\right]e^{-\theta x}}\int_x^\infty\left[\theta\beta t+(\theta\alpha+\beta)\right]e^{-\theta t}dt$$

$$= \frac{\theta\beta x+(\theta\alpha+2\beta)}{\theta\left[\theta\beta x+(\theta\alpha+\beta)\right]} \quad (4.4)$$

It can be easily verified that $h(0) = \frac{\theta^2\alpha}{\theta\alpha+\beta} = f(0)$ and $m(0) = \frac{\theta\alpha+2\beta}{\theta(\theta\alpha+\beta)} = \mu_1'$. It is also obvious from the graphs of $h(x)$ and $m(x)$ that $h(x)$ is an increasing and decreasing functions of x, and θ, whereas $m(x)$ is a decreasing function of x, and θ.

The graph of the hazard rate function and mean residual life function of ATPLD are shown in figures 3 and 4.

5. Stochastic Orderings

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $(X \le_{st} Y)$ if $F_X(x) \ge F_Y(x)$ for all x

(ii) hazard rate order $(X \le_{hr} Y)$ if $h_X(x) \ge h_Y(x)$ for all x

(iii) mean residual life order $(X \le_{mrl} Y)$ if $m_X(x) \le m_Y(x)$ for all x

(iv) likelihood ratio order $(X \le_{lr} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

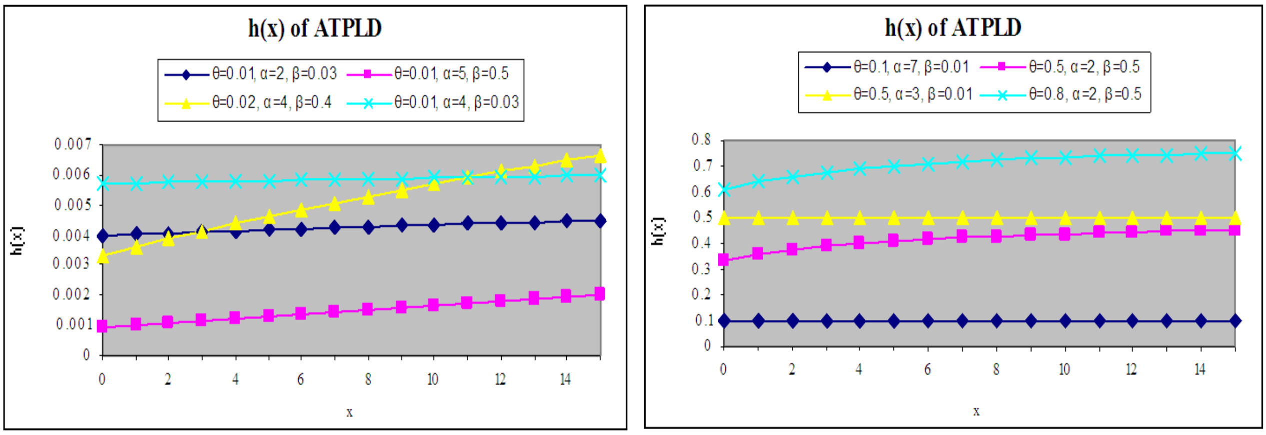

Figure 3. Graph of hazard rate function of ATPLD for different values of parameters θ, α, β

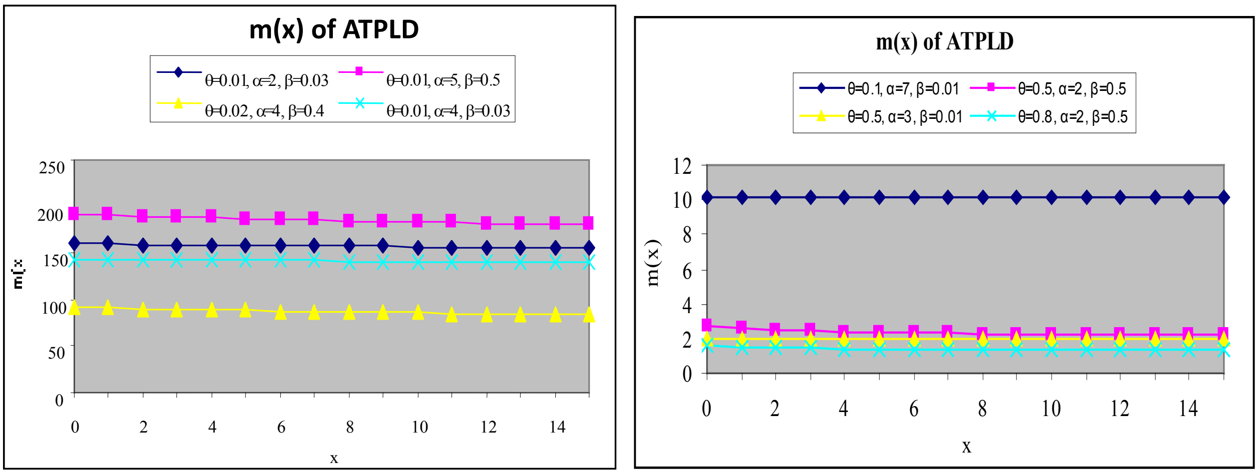

Figure 4. Graph of mean residual life function of ATPLD for different values of parameters θ, α, β

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \tag{5.1}$$

$$\Downarrow$$
$$X \le_{st} Y$$

The ATPLD is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem: Let $X \sim$ ATPLD $(\theta_1, \alpha_1, \beta_1)$ and $Y \sim$ ATPLD $(\theta_2, \alpha_2, \beta_2)$. Now under conditions (i) $\alpha_1 = \alpha_2, \beta_1 = \beta_2$, and $\theta_1 > \theta_2$, (ii) $\alpha_1 = \alpha_2, \beta_1 > \beta_2$, and $\theta_1 = \theta_2$ and (iii) $\alpha_1 > \alpha_2, \beta_1 = \beta_2$, and $\theta_1 = \theta_2$, $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \left[\frac{\theta_1^{\,2}(\theta_2\,\alpha_2 + \beta_2)}{\theta_2^{\,2}(\theta_1\,\alpha_1 + \beta_1)}\right]\left(\frac{\alpha_1 + \beta_1\,x}{\alpha_2 + \beta_2\,x}\right)e^{-(\theta_1 - \theta_2)x}\ ;\ x > 0$$

Now

$$\log\frac{f_X(x)}{f_Y(x)} = \log\left[\frac{\theta_1^{\,2}(\theta_2\,\alpha_2 + \beta_2)}{\theta_2^{\,2}(\theta_1\,\alpha_1 + \beta_1)}\right] + \log\left(\frac{\alpha_1 + \beta_1\,x}{\alpha_2 + \beta_2\,x}\right) - (\theta_1 - \theta_2)x\,.$$

This gives $\dfrac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}=\dfrac{\alpha_2-\alpha_1}{(\alpha_1+\beta_1 x)(\alpha_2+\beta_2 x)}+\dfrac{\beta_2-\beta_1}{(\alpha_1+\beta_1 x)(\alpha_2+\beta_2 x)}-(\theta_1-\theta_2)$

It can be easily verified that under conditions (1), (ii), and (iii), $\dfrac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}<0$. This means that $X\le_{lr} Y$ and hence $X\le_{hr} Y$, $X\le_{mrl} Y$ and $X\le_{st} Y$.

6. Mean Deviations

The amount of scatter in a population is measured to some extent by the totality of deviations usually from mean and median. These are known as the mean deviation about the mean and the mean deviation about the median defined by $\delta_1(X)=\int_0^\infty |x-\mu| f(x)dx$ and $\delta_2(X)=\int_0^\infty |x-M| f(x)dx$, respectively, where $\mu=E(X)$ and $M=\text{Median }(X)$.

The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the relationships

$$\begin{aligned}\delta_1(X)&=\int_0^\mu (\mu-x)f(x)dx+\int_\mu^\infty (x-\mu)f(x)dx\\&=\mu F(\mu)-\int_0^\mu x f(x)dx-\mu\left[1-F(\mu)\right]+\int_\mu^\infty x f(x)dx\\&=2\mu F(\mu)-2\mu+2\int_\mu^\infty x f(x)dx\\&=2\mu F(\mu)-2\int_0^\mu x f(x)dx\end{aligned}\tag{6.1}$$

and

$$\begin{aligned}\delta_2(X)&=\int_0^M (M-x)f(x)dx+\int_M^\infty (x-M)f(x)dx\\&=M F(M)-\int_0^M x f(x)dx-M\left[1-F(M)\right]+\int_M^\infty x f(x)dx\\&=-\mu+2\int_M^\infty x f(x)dx\\&=\mu-2\int_0^M x f(x)dx\end{aligned}\tag{6.2}$$

Using p.d.f. (2.1) and expression for the mean of ATPLD, we get

$$\int_0^\mu x f(x)dx=\mu-\frac{\left\{\beta\theta^2\mu^2+\left(\theta^2\alpha+2\beta\theta\right)\mu+\left(\theta\alpha+2\beta\right)\right\}e^{-\theta\mu}}{\theta(\theta\alpha+\beta)}\tag{6.3}$$

$$\int_0^M x f(x)dx=\mu-\frac{\left\{\beta\theta^2M^2+\left(\theta^2\alpha+2\beta\theta\right)M+\left(\theta\alpha+2\beta\right)\right\}e^{-\theta M}}{\theta(\theta\alpha+\beta)}\tag{6.4}$$

Using expressions from (6.1), (6.2), (6.3), and (6.4), the mean deviation about mean, $\delta_1(X)$ and the mean deviation about median, $\delta_2(X)$ of ATPLD are obtained as

$$\delta_1(X) = \frac{2\{\theta\beta\mu + (\theta\alpha + 2\beta)\}e^{-\theta\mu}}{\theta(\theta\alpha + \beta)} \tag{6.5}$$

$$\delta_2(X) = \frac{2\left\{\beta\theta^2 M^2 + \left(\theta^2\alpha + 2\beta\theta\right)M + (\theta\alpha + 2\beta)\right\}e^{-\theta M}}{\theta(\theta\alpha + \beta)} - \mu \tag{6.6}$$

7. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves (Bonferroni, 1930) and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu}\int_0^q x f(x)\,dx = \frac{1}{p\mu}\left[\int_0^\infty x f(x)\,dx - \int_q^\infty x f(x)\,dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^\infty x f(x)\,dx\right] \tag{7.1}$$

and $$L(p) = \frac{1}{\mu}\int_0^q x f(x)\,dx = \frac{1}{\mu}\left[\int_0^\infty x f(x)\,dx - \int_q^\infty x f(x)\,dx\right] = \frac{1}{\mu}\left[\mu - \int_q^\infty x f(x)\,dx\right] \tag{7.2}$$

respectively or equivalently

$$B(p) = \frac{1}{p\mu}\int_0^p F^{-1}(x)\,dx \tag{7.3}$$

and $$L(p) = \frac{1}{\mu}\int_0^p F^{-1}(x)\,dx \tag{7.4}$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$.

The Bonferroni and Gini indices are thus defined as

$$B = 1 - \int_0^1 B(p)\,dp \tag{7.5}$$

and $$G = 1 - 2\int_0^1 L(p)\,dp \tag{7.6}$$

respectively.

Using p.d.f. (2.1), we get

$$\int_q^\infty x f(x)\,dx = \frac{\left\{\beta\theta^2 q^2 + \left(\theta^2\alpha + 2\beta\theta\right)q + (\theta\alpha + 2\beta)\right\}e^{-\theta q}}{\theta(\theta\alpha + \beta)} \tag{7.7}$$

Now using equation (7.7) in (7.1) and (7.2), we get

$$B(p) = \frac{1}{p}\left[1 - \frac{\left\{\beta\theta^2 q^2 + \left(\theta^2\alpha + 2\beta\theta\right)q + (\theta\alpha + 2\beta)\right\}e^{-\theta q}}{(\theta\alpha + 2\beta)}\right] \tag{7.8}$$

and $$L(p) = 1 - \frac{\left\{\beta\theta^2 q^2 + \left(\theta^2\alpha + 2\beta\theta\right)q + (\theta\alpha + 2\beta)\right\}e^{-\theta q}}{(\theta\alpha + 2\beta)} \tag{7.9}$$

Now using equations (7.8) and (7.9) in (7.5) and (7.6), the Bonferroni and Gini indices of ATPLD are obtained as

$$B = 1 - \frac{\left\{\beta\theta^2 q^2 + \left(\theta^2\alpha + 2\beta\theta\right)q + \left(\theta\alpha + 2\beta\right)\right\}e^{-\theta q}}{\left(\theta\alpha + 2\beta\right)} \tag{7.10}$$

$$L = \frac{2\left\{\beta\theta^2 q^2 + \left(\theta^2\alpha + 2\beta\theta\right)q + \left(\theta\alpha + 2\beta\right)\right\}e^{-\theta q}}{\left(\theta\alpha + 2\beta\right)} - 1 \tag{7.11}$$

8. Order Statistics and Renyi Entropy Measure

8.1. Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n from ATPLD (2.1). Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ denote the corresponding order statistics. The p.d.f. and the c.d.f. of the k th order statistic, say $Y = X_{(k)}$ are given by

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(y)\{1 - F(y)\}^{n-k} f(y)$$

$$= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l F^{k+l-1}(y) f(y)$$

and

$$F_Y(y) = \sum_{j=k}^{n} \binom{n}{j} F^j(y)\{1 - F(y)\}^{n-j}$$

$$= \sum_{j=k}^{n} \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} (-1)^l F^{j+l}(y),$$

respectively, for $k = 1, 2, 3, ..., n$.

Thus, the p.d.f. and the c.d.f of k th order statistics of ATPLD are obtained as

$$f_Y(y) = \frac{n!\theta^2(\alpha + \beta x)e^{-\theta x}}{(\theta\alpha + \beta)(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \times \left[1 - \frac{\theta\beta x + (\theta\alpha + \beta)}{\theta\alpha + \beta} e^{-\theta x}\right]^{k+l-1}$$

and

$$F_Y(y) = \sum_{j=k}^{n} \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} (-1)^l \left[1 - \frac{\theta\beta x + (\theta\alpha + \beta)}{\theta\alpha + \beta} e^{-\theta x}\right]^{j+l}$$

8.2. Renyi Entropy Measure

An entropy of a random variable X is a measure of variation of uncertainty. A popular entropy measure is Renyi entropy (1961). If X is a continuous random variable having probability density function $f(.)$, then Renyi entropy is defined as

$$T_R(\gamma) = \frac{1}{1-\gamma} \log\left\{\int f^{\gamma}(x)\,dx\right\}$$

where $\gamma > 0$ and $\gamma \neq 1$.

Thus, the Renyi entropy for ATPLD (2.1) can be obtained as

$$T_R(\gamma) = \frac{1}{1-\gamma} \log\left[\int_0^{\infty} \frac{\theta^{2\gamma}}{(\theta\alpha + \beta)^{\gamma}} (\alpha + \beta x)^{\gamma} e^{-\theta\gamma x}\,dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\int_0^\infty \frac{\theta^{2\gamma}\alpha^\gamma}{(\theta\alpha+\beta)^\gamma}\left(1+\frac{\beta}{\alpha}x\right)^\gamma e^{-\theta\gamma x}dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\int_0^\infty \frac{\theta^{2\gamma}\alpha^\gamma}{(\theta\alpha+\beta)^\gamma}\sum_{j=0}^{\infty}\binom{\gamma}{j}\left(\frac{\beta}{\alpha}x\right)^j e^{-\theta\gamma x}dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\theta^{2\gamma}\alpha^{\gamma-j}\beta^j}{(\theta\alpha+\beta)^\gamma}\int_0^\infty e^{-\theta\gamma x}x^{j+1-1}dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\theta^{2\gamma}\alpha^{\gamma-j}\beta^j}{(\theta\alpha+\beta)^\gamma}\frac{\Gamma(j+1)}{(\theta\gamma)^{j+1}}\right]$$

$$=\frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\theta^{2\gamma-j-1}\alpha^{\gamma-j}\beta^j}{(\theta\alpha+\beta)^\gamma}\frac{\Gamma(j+1)}{(\gamma)^{j+1}}\right]$$

9. Stress-Strength Reliability

The stress- strength reliability describes the life of a component which has random strength X that is subjected to a random stress Y. When the stress applied to it exceeds the strength, the component fails instantly and the component will function satisfactorily till $X > Y$. Therefore, $R = P(Y < X)$ is a measure of component reliability and in statistical literature it is known as stress-strength parameter. It has wide applications in almost all areas of knowledge especially in engineering such as structures, deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels etc.

Let X and Y be independent strength and stress random variables having ATPLD (2.1) with parameter $(\theta_1,\alpha_1,\beta_1)$ and $(\theta_2,\alpha_2,\beta_2)$ respectively. Then the stress-strength reliability R of ATPLD can be obtained as

$$R = P(Y < X) = \int_0^\infty P(Y < X \mid X = x) f_X(x)dx$$

$$= \int_0^\infty f(x;\theta_1,\alpha_1,\beta_1)\, F(x;\theta_2,\alpha_2,\beta_2)dx$$

$$= 1-\frac{\theta_1^{\,2}\left[2\beta_1\beta_2\theta_2+(\theta_1+\theta_2)(\alpha_1\theta_2\beta_2+\beta_1\theta_2\alpha_2+\beta_1\beta_2)+\alpha_1(\theta_1+\theta_2)^2(\theta_2\alpha_2+\beta_2)\right]}{(\theta_1\alpha_1+\beta_1)(\theta_2\alpha_2+\beta_2)(\theta_1+\theta_2)^3}.$$

10. Maximum Likelihood Estimate (MLE)

Let $(x_1,x_2,x_3,\ldots,x_n)$ be a random sample of size n from ATPLD (2.1). The likelihood function, L of (2.1) is given by

$$L = \left(\frac{\theta^2}{\theta\alpha+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}$$

The natural log likelihood function is thus obtained as

$$\ln L = n\ln\left(\frac{\theta^2}{\theta\alpha+\beta}\right)+\sum_{i=1}^{n}\ln(\alpha+\beta x_i)-n\theta\bar{x}$$

The maximum likelihood estimates (MLE) $\hat{\theta}$, $\hat{\alpha}$ and $\hat{\beta}$ of θ, α and β are then the solutions of the following non-linear equations

$$\frac{\partial \ln L}{\partial \theta}=\frac{2n}{\theta}-\frac{n\alpha}{\theta\alpha+\beta}-n\overline{x}=0 \tag{10.1.1}$$

$$\frac{\partial \ln L}{\partial \alpha}=\frac{-n\theta}{\theta\alpha+\beta}+\sum_{i=1}^{n}\frac{1}{\alpha+\beta x_i}=0 \tag{10.1.2}$$

$$\frac{\partial \ln L}{\partial \beta}=\frac{-n}{\theta\alpha+\beta}+\sum_{i=1}^{n}\frac{x_i}{\alpha+\beta x_i}=0 \tag{10.1.3}$$

where $\overline{x}$ is the sample mean. The equation (10.1.1) gives $\overline{X}=\dfrac{\theta\alpha+2\beta}{\theta(\theta\alpha+\beta)}$, which is the mean of ATPLD (2.1).

These three natural log likelihood equations do not seem to be solved directly. However, the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \ln L}{\partial \theta^2}=-\frac{2n}{\theta^2}+\frac{n\alpha^2}{(\theta\alpha+\beta)^2}$$

$$\frac{\partial^2 \ln L}{\partial \theta\partial \alpha}=\frac{-n\beta}{(\theta\alpha+\beta)^2}=\frac{\partial^2 \ln L}{\partial \alpha\partial \theta}$$

$$\frac{\partial^2 \ln L}{\partial \theta\partial \beta}=-\frac{n\alpha}{(\theta\alpha+\beta)^2}=\frac{\partial^2 \ln L}{\partial \beta\partial \theta}$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2}=\frac{-n\theta^2}{(\theta\alpha+\beta)^2}-\sum_{i=1}^{n}\frac{1}{\left(\alpha+\theta x_i^{\,2}\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial \alpha\partial \beta}=-\frac{n\theta}{(\theta\alpha+\beta)^2}-\sum_{i=1}^{n}\frac{x_i}{(\alpha+\beta x_i)^2}=\frac{\partial^2 \ln L}{\partial \beta\partial \alpha}$$

The following equations can be solved for MLEs $\hat{\theta}$, $\hat{\alpha}$ and $\hat{\beta}$ of θ, α and β of ATPLD

$$\begin{bmatrix} \dfrac{\partial^2 \ln L}{\partial \theta^2} & \dfrac{\partial^2 \ln L}{\partial \theta\partial \alpha} & \dfrac{\partial^2 \ln L}{\partial \theta\partial \beta} \\ \dfrac{\partial^2 \ln L}{\partial \alpha\partial \theta} & \dfrac{\partial^2 \ln L}{\partial \alpha^2} & \dfrac{\partial^2 \ln L}{\partial \alpha\partial \beta} \\ \dfrac{\partial^2 \ln L}{\partial \beta\partial \theta} & \dfrac{\partial^2 \ln L}{\partial \beta\partial \alpha} & \dfrac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0\\ \hat{\beta}=\beta_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \\ \hat{\beta}-\beta_0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \ln L}{\partial \theta} \\ \dfrac{\partial \ln L}{\partial \alpha} \\ \dfrac{\partial \ln L}{\partial \beta} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0\\ \hat{\beta}=\beta_0}}$$

where θ_0, α_0 and β_0 are the initial values of θ, α and β respectively. These equations are solved iteratively till sufficiently close values of $\hat{\theta}$, $\hat{\alpha}$ and $\hat{\beta}$ are obtained.

11. Goodness of Fit

A three-parameter Lindley distribution (ATPLD) has been fitted to a number of lifetime data to test its goodness of fit. In this section, we present the goodness of fit of ATPLD for a real lifetime data and its fit has been compared with the three-parameter generalized Lindley distribution (TPGLD), introduced by Zakerzadeh and Dolati (2009). The following lifetime data has been considered for testing the goodness of fit of ATPLD and TPGLD.

Data Set: This data represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960)

10	33	44	56	59	72	74	77	92	93	96	100
100	102	105	107	107	108	108	108	109	112	113	115
116	120	121	122	122	124	130	134	136	139	144	146
153	159	160	163	163	168	171	172	176	183	195	196
197	202	213	215	216	222	230	231	240	245	251	253
254	254	278	293	327	342	347	361	402	432	458	555

In order to compare ATPLD and TPGLD, values of $-2\ln L$ and K-S Statistics (Kolmogorov-Smirnov Statistics) for real life time data has been computed. The formulae for computing K-S Statistics is as follows:

$K-S=\underset{x}{\operatorname{Sup}}\left|F_n\left(x\right)-F_0\left(x\right)\right|$, where k = the number of parameters, n = the sample size and $F_n\left(x\right)$ is the empirical distribution function.

The best distribution is the distribution which corresponds to lower value of $-2\ln L$ and K-S statistics and higher p-value.

Table 1. MLE's, - 2ln L, AIC, AICC, BIC, and K-S Statistics of the fitted distributions of data

Distributions	ML Estimate $\hat{\theta}$ $\hat{\alpha}$ $\hat{\beta}$	$-2\ln L$	K-S statistic	P-value
ATPLD	0.0232 -4.4223 0.4253	783.145	0.152	0.070
TPGLD	0.0209 1.0932 5.0688	788.575	0.439	0.000

It can be easily seen from above table that ATPLD gives better fit than the TPGLD, introduced by Zakerzadeh and Dolati (2009).

12. Concluding Remarks

A three-parameter Lindley distribution (ATPLD) which includes two-parameter Lindley distributions introduced by Shanker and Mishra (2013 a, 2013 b) , Shanker et al (2013), Shanker and Amanuel (2013), two-parameter gamma distribution, and one parameter exponential and Lindley distributions, has been introduced for modeling lifetime data. Its mathematical and statistical properties including its shape, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, order statistics, Renyi entropy measure, Bonferroni and Lorenz curves, stress-strength reliability have been discussed. For estimating its parameters, maximum likelihood estimation has been discussed. The goodness of fit of ATPLD has been found better than the goodness of fit given by TPGLD, and hence ATPLD can be considered an important lifetime distribution for modeling lifetime data over TPGLD.

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