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Amarendra Distribution and Its Applications

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Abstract In this paper a new one parameter continuous distribution named, 'Amarendra Distribution' having monotonically increasing hazard rate for modeling lifetime data, has been suggested. Its first four moments about origin and moments about mean have been obtained and expressions for coefficient of variation, skewness and kurtosis have been given. Various other characteristics such as its hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves have been discussed. The condition under which Amarendra distribution is over-dispersed, equi-dispersed, and under-dispersed has been given along with conditions under which Akash, Shanker, Sujatha, Lindley and exponential distributions are over-dispersed, equi-dispersed, and under-dispersed. Estimation of its parameter has been discussed using method of maximum likelihood and the method of moments. The applicability and the goodness of fit of the proposed distribution over one parameter Akash, Shanker, Sujatha, Lindley and exponential distributions have been illustrated with two real lifetime data- sets from medical science and engineering.

Keywords Lindley distribution, Akash distribution, Shanker distribution, Sujatha distribution, Mathematical and statistical properties, Estimation of parameter, Goodness of fit

1. Introduction

The analyzing and modeling of lifetime data are crucial in almost all applied sciences including biomedical science, engineering, insurance, finance, amongst others. A number of lifetime distributions for modeling lifetime data such as Akash, Shanker, Sujatha, Lindley, exponential, gamma, lognormal, and Weibull are available in statistical literature. The Akash, Shanker, Sujatha, Lindley, exponential, and Weibull distributions are more popular than the gamma and the lognormal distributions because the survival functions of the gamma and the lognormal distributions cannot be expressed in closed forms and both require numerical integration. Akash, Shanker, Sujatha, Lindley, and exponential distributions consists of one parameter and Akash, Shanker, Sujatha, and Lindley distributions have advantage over exponential distribution that the exponential distribution has constant hazard rate whereas Akash, Shanker, Sujatha, and Lindley distributions have monotonically increasing hazard rate. Further, it has been shown by Shanker (2015 a, 2015 b, 2015 c) that the nature of Akash, Shanker, and Sujatha distributions are more flexible than Lindley and exponential distributions for modeling lifetime data.

The probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of Lindley (1958) distribution are given by

$$f_1(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)\, e^{-\theta x} \;\; ; x > 0,\, \theta > 0 \qquad (1.1)$$

$$F_1(x;\theta) = 1 - \frac{\theta+1+\theta x}{\theta+1}\, e^{-\theta x} \;\; ; x > 0\,,\, \theta > 0 \qquad (1.2)$$

It can be easily shown that the density (1.1) is a two-component mixture of an exponential distribution with scale parameter θ and a gamma distribution having shape parameter 2 and a scale parameter θ with their mixing proportions $\frac{\theta}{\theta+1}$ and $\frac{1}{\theta+1}$ respectively. Ghitany *et al* (2008) have discussed various properties of this distribution and showed that in many ways (1.1) provides a better model for some applications than the exponential distribution. The Lindley distribution has been modified, extended, generalized suiting their applications in different areas of knowledge by many researchers including Hussain (2006), Zakerzadeh and Dolati (2009), Nadarajah *et al* (2011), Deniz and Ojeda (2011), Bakouch et al (2012), Shanker and Mishra (2013 a, 2013 b), Shanker and Amanuel (2013), Shanker *et al* (2013), Elbatal *et al* (2013), Ghitany *et al* (2013), Merovci (2013), Liyanage and Pararai (2014), Ashour and Eltehiwy (2014), Oluyede and Yang (2014), Singh *et al* (2014), Sharma *et al* (2015), Shanker *et al* (2015 a, 2015 b), Alkarni (2015), Pararai *et al* (2015), Abouammoh *et al* (2015) are some among others.

The probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of Akash distribution introduced by Shanker (2015 a) are given by

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$$f_2(x;\theta)=\frac{\theta^3}{\theta^2+2}\left(1+x^2\right)e^{-\theta x}\;;x>0,\,\theta>0 \quad (1.3)$$

$$F_2(x;\theta)=1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\;;x>0,\,\theta>0 \quad (1.4)$$

Shanker (2015 a) has shown that the density (1.3) is a two-component mixture of an exponential distribution with scale parameter θ and a gamma distribution having shape parameter 3 and a scale parameter θ with their mixing proportions $\frac{\theta^2}{\theta^2+2}$ and $\frac{2}{\theta^2+2}$ respectively. Shanker (2015 a) has discussed its various mathematical and statistical properties including its shape, moment generating function, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic orderings, mean deviations, distribution of order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, stress-strength reliability, some amongst others. Shanker (2016 a) has obtained Poisson mixture of Akash distribution named, Poisson-Akash distribution (PAD) and discussed its various mathematical and statistical properties, estimation of its parameter and applications for various count data-sets. Shanker *et al* (2015 c) has detailed and critical study about modeling and analyzing lifetime data from various fields of knowledge using one parameter Akash, Lindley and exponential distributions. Further, Shanker (2016 b, 2016 c) has also obtained the size-biased and zero-truncated versions of PAD, derived their important mathematical and statistical properties, and discussed the estimation of parameter and applications for count-data-sets.

The probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of Shanker distribution introduced by Shanker (2015 b) are given by

$$f_3(x;\theta)=\frac{\theta^2}{\theta^2+1}\left(\theta+x\right)e^{-\theta x}\;;x>0,\,\theta>0 \quad (1.5)$$

$$F_3(x;\theta)=1-\frac{\left(\theta^2+1\right)+\theta x}{\theta^2+1}e^{-\theta x}\;;x>0,\,\theta>0 \quad (1.6)$$

Shanker (2015 b) has shown that the density (1.5) is a two-component mixture of an exponential distribution with scale parameter θ and a gamma distribution having shape parameter 2 and a scale parameter θ with their mixing proportions $\frac{\theta^2}{\theta^2+1}$ and $\frac{1}{\theta^2+1}$ respectively. Shanker (2015 b) has discussed its various mathematical and statistical properties including its shape, moment generating function, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic orderings, mean deviations, distribution of order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, stress-strength reliability , some amongst others. Further, Shanker (2016 d) has obtained Poisson mixture of Shanker distribution named Poisson-Shanker distribution (PSD) and discussed its various mathematical and statistical properties, estimation of its parameter and applications for various count data-sets. Shanker and Hagos (2016 a, 2016 b) have obtained the size-biased and zero-truncated versions of Poisson-Shanker distribution (PSD), derived their interesting mathematical and statistical properties, discussed the estimation of parameter and applications for count data-sets from different fields of knowledge.

The probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of Sujatha distribution introduced by Shanker (2015 c) are given by

$$f_4(x;\theta)=\frac{\theta^3}{\theta^2+\theta+2}\left(1+x+x^2\right)e^{-\theta x}\;;x>0,\,\theta>0 \quad (1.7)$$

$$F_4(x)=1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\;;x>0,\theta>0\;; \quad (1.8)$$

Shanker (2015 c) has shown that the density (1.7) is a three-component mixture of an exponential distribution with scale parameter θ, a gamma distribution having shape parameter 2 and a scale parameter θ, and a gamma distribution having shape parameter 3 and a scale parameter θ with their mixing proportions $\frac{\theta^2}{\theta^2+\theta+2}$, $\frac{\theta}{\theta^2+\theta+2}$ and $\frac{2}{\theta^2+\theta+2}$ respectively. Shanker (2015 c) has discussed its various mathematical and statistical properties including its shape, moment generating function, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic orderings, mean deviations, distribution of order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, stress-strength reliability, some amongst others. Further, Shanker (2016 e) has obtained Poisson mixture of Sujatha distribution named, Poisson-Sujatha distribution (PSD) and discussed its various mathematical and statistical properties, estimation of its parameter and applications for various count data-sets. Shanker and Hagos (2016 c, 2016 d) have obtained the size-biased and zero-truncated versions of Poisson-Sujatha distribution (PSD), derived their interesting mathematical and statistical properties, discussed the estimation of parameter and applications for count data-sets. Shanker and Hagos (2016 e) has also done an extensive study on comparative study of zero-truncated Poisson, Poisson-Lindley and Poisson-Sujatha distribution and shown that in most of the data-sets zero-truncated Poisson-Sujatha distribution gives much closer fit.

Although Akash, Shanker, Sujatha, Lindley, and exponential distributions have been used to model various lifetime data from biomedical science and engineering, there are many situations where these distributions may not be suitable from applied and theoretical point of view. Therefore, to obtain a new distribution which is more flexible than the Akash, Shanker, Sujatha, Lindley and exponential distributions, we introduced a distribution by considering a four component mixture of exponential (θ), a gamma $(2,\theta)$, a gamma $(3,\theta)$ and a gamma $(4,\theta)$ with their mixing proportions $\dfrac{\theta^3}{\theta^3+\theta^2+2\theta+6}$, $\dfrac{\theta^2}{\theta^3+\theta^2+2\theta+6}$, $\dfrac{2\theta}{\theta^3+\theta^2+2\theta+6}$ and $\dfrac{6}{\theta^3+\theta^2+2\theta+6}$ respectively.

The probability density function (p.d.f.) of a new one parameter lifetime distribution can be introduced as

$$f_5\left(x;\theta\right)=\frac{\theta^4}{\theta^3+\theta^2+2\theta+6}\left(1+x+x^2+x^3\right)e^{-\theta x}\quad ;x>0,\ \theta>0 \tag{1.9}$$

We would name this distribution as 'Amarendra Distributiuon'. The corresponding cumulative distribution function of Amarendra distribution (1.9) can be obtained as

$$F_5\left(x;\theta\right)=1-\left[1+\frac{\theta^3x^3+\theta^2\left(\theta+3\right)x^2+\theta\left(\theta^2+2\theta+6\right)x}{\theta^3+\theta^2+2\theta+6}\right]e^{-\theta x};\ x>0,\theta>0 \tag{1.10}$$

The graphs of the p.d.f. and the c.d.f. of Amarendra distribution (1.9) for different values of θ are shown in figures 1 and 2.

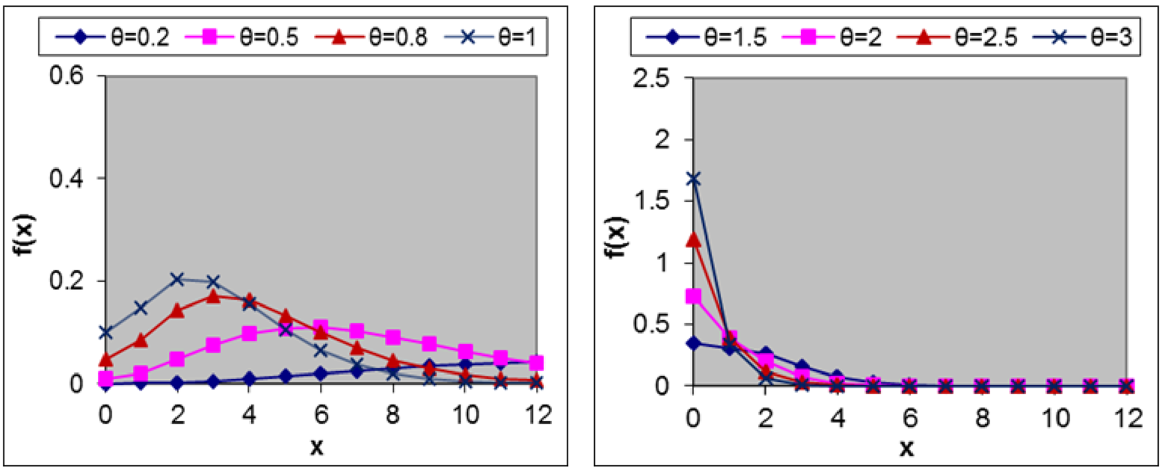

Figure 1. Graph of the pdf of Amarendra distribution for different values of parameter θ

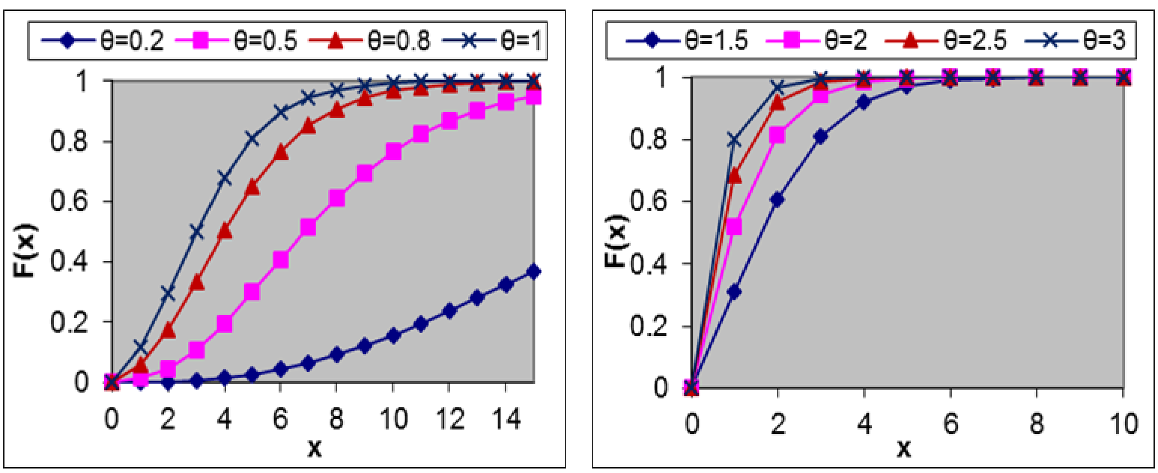

Figure 2. Graph of the cdf of Amarendra distribution for different values of parameter θ

2. Moment Generating Function, Moments and Related Measures

The moment generating function of Amarendra distribution (1.9) can be obtained as

$$M_X(t)=\frac{\theta^4}{\theta^3+\theta^2+2\theta+6}\int_0^\infty e^{-(\theta-t)x}\left(1+x+x^2+x^3\right)dx$$

$$=\frac{\theta^4}{\theta^3+\theta^2+2\theta+6}\left[\frac{1}{\theta-t}+\frac{1}{(\theta-t)^2}+\frac{2}{(\theta-t)^3}+\frac{6}{(\theta-t)^4}\right]$$

$$=\frac{\theta^4}{\theta^3+\theta^2+2\theta+6}\left[\frac{1}{\theta}\sum_{k=0}^\infty\left(\frac{t}{\theta}\right)^k+\frac{1}{\theta^2}\sum_{k=0}^\infty\binom{k+1}{k}\left(\frac{t}{\theta}\right)^k+\frac{2}{\theta^3}\sum_{k=0}^\infty\binom{k+2}{k}\left(\frac{t}{\theta}\right)^k+\frac{6}{\theta^4}\sum_{k=0}^\infty\binom{k+3}{k}\left(\frac{t}{\theta}\right)^k\right]$$

$$=\sum_{k=0}^\infty\frac{\theta^3+(k+1)\theta^2+(k+1)(k+2)\theta+(k+1)(k+2)(k+3)}{\left(\theta^3+\theta^2+2\theta+6\right)}\left(\frac{t}{\theta}\right)^k$$

The r the moment about origin μ_r', obtained as the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$, of Amarendra distributon (1.9) has been obtained as

$$\mu_r'=\frac{r!\left[\theta^3+(r+1)\theta^2+(r+1)(r+2)\theta+(r+1)(r+2)(r+3)\right]}{\theta^r\left(\theta^3+\theta^2+2\theta+6\right)}\quad ;r=1,2,3,...$$

and thus the first four moments about origin are given by

$$\mu_1'=\frac{\theta^3+2\theta^2+6\theta+24}{\theta\left(\theta^3+\theta^2+2\theta+6\right)}\ ,\qquad \mu_2'=\frac{2\left(\theta^3+3\theta^2+12\theta+60\right)}{\theta^2\left(\theta^3+\theta^2+2\theta+6\right)},$$

$$\mu_3'=\frac{6\left(\theta^3+4\theta^2+20\theta+120\right)}{\theta^3\left(\theta^3+\theta^2+2\theta+6\right)},\qquad \mu_4'=\frac{24\left(\theta^3+5\theta^2+30\theta+210\right)}{\theta^4\left(\theta^3+\theta^2+2\theta+6\right)}$$

Using the relationship between moments about mean and the moments about origin, the moments about mean of the Amarendra distribution (1.9) are obtained as

$$\mu_2=\frac{\theta^6+4\theta^5+18\theta^4+96\theta^3+72\theta^2+96\theta+144}{\theta^2\left(\theta^3+\theta^2+2\theta+6\right)^2}$$

$$\mu_3=\frac{2\left(\theta^9+6\theta^8+36\theta^7+242\theta^6+324\theta^5+468\theta^4+672\theta^3+720\theta^2+864\theta+864\right)}{\theta^3\left(\theta^3+\theta^2+2\theta+6\right)^3}$$

$$\mu_4=\frac{3\begin{pmatrix}3\theta^{12}+24\theta^{11}+172\theta^{10}+1312\theta^9+3064\theta^8+7008\theta^7+15984\theta^6+24576\theta^5+39312\theta^4\\+56832\theta^3+43776\theta^2+41472\theta+31104\end{pmatrix}}{\theta^4\left(\theta^3+\theta^2+2\theta+6\right)^4}$$

The coefficient of variation $(C.V)$, coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis $\left(\beta_2\right)$, and index of dispersion (γ) of Amarendra distribution (1.9) are thus obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^6 + 4\theta^5 + 18\theta^4 + 96\theta^3 + 72\theta^2 + 96\theta + 144}}{\theta^3 + 2\theta^2 + 6\theta + 24}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\left(\theta^9 + 6\theta^8 + 36\theta^7 + 242\theta^6 + 324\theta^5 + 468\theta^4 + 672\theta^3 + 720\theta^2 + 864\theta + 864\right)}{\left(\theta^6 + 4\theta^5 + 18\theta^4 + 96\theta^3 + 72\theta^2 + 96\theta + 144\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\begin{pmatrix} 3\theta^{12} + 24\theta^{11} + 172\theta^{10} + 1312\theta^9 + 3064\theta^8 + 7008\theta^7 + 15984\theta^6 + 24576\theta^5 \\ +39312\theta^4 + 56832\theta^3 + 43776\theta^2 + 41472\theta + 31104 \end{pmatrix}}{\left(\theta^6 + 4\theta^5 + 18\theta^4 + 96\theta^3 + 72\theta^2 + 96\theta + 144\right)^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^6 + 4\theta^5 + 18\theta^4 + 96\theta^3 + 72\theta^2 + 96\theta + 144}{\theta\left(\theta^3 + \theta^2 + 2\theta + 6\right)\left(\theta^3 + 2\theta^2 + 6\theta + 24\right)}$$

The condition under which Amarendra distribution is over-dispersed, equi-dispersed, and under-dispersed has been given along with conditions under which Akash, Shanker, Sujatha, Lindley and exponential distributions are over-dispersed, equi-dispersed, and under-dispersed in table 1.

Table 1. Over-dispersion, equi-dispersion and under-dispersion of Amarendra, Akash, Shanker, Sujatha, Lindley and exponential distributions for varying values of their parameter $\boldsymbol{\theta}$

Distribution	Over-dispersion $\left(\mu < \sigma^2\right)$	Equi-dispersion $\left(\mu = \sigma^2\right)$	Under-dispersion $\left(\mu > \sigma^2\right)$
Amarendra	$\theta < 1.525763580$	$\theta = 1.525763580$	$\theta > 1.525763580$
Akash	$\theta < 1.515400063$	$\theta = 1.515400063$	$\theta > 1.515400063$
Shanker	$\theta < 1.171535555$	$\theta = 1.171535555$	$\theta > 1.171535555$
Sujatha	$\theta < 1.364271174$	$\theta = 1.364271174$	$\theta > 1.364271174$
Lindley	$\theta < 1.170086487$	$\theta = 1.170086487$	$\theta > 1.170086487$
Exponential	$\theta < 1$	$\theta = 1$	$\theta > 1$

3. Mathematical and Statistical Properties

3.1. Hazard Rate Function and Mean Residual life Function

Let X be a continuous random variable with p.d.f. $f\left(x\right)$ and c.d.f. $F\left(x\right)$. The hazard rate function (also known as the failure rate function) and the mean residual life function of X are respectively defined as

$$h\left(x\right) = \lim_{\Delta x \to 0} \frac{P\left(X < x + \Delta x \middle| X > x\right)}{\Delta x} = \frac{f\left(x\right)}{1 - F\left(x\right)} \tag{3.1.1}$$

and

$$m\left(x\right) = E\left[X - x \middle| X > x\right] = \frac{1}{1 - F\left(x\right)} \int_x^\infty \left[1 - F\left(t\right)\right] \, dt \tag{3.1.2}$$

The corresponding hazard rate function, $h\left(x\right)$ and the mean residual life function, $m\left(x\right)$ of Amarendra distribution (1.9) are thus given by

$$h(x)=\frac{\theta^4\left(1+x+x^2+x^3\right)}{\theta^3\left(1+x+x^2+x^3\right)+\theta^2\left(3x^2+2x+1\right)+2\theta\left(3x+1\right)+6} \tag{3.1.3}$$

and

$$m(x)=\frac{\theta^3+\theta^2+2\theta+6}{\left[\theta^3\left(x^3+x^2+x+1\right)+\theta^2\left(3x^2+2x+1\right)+2\theta\left(3x+1\right)+6\right]e^{-\theta x}}\times$$

$$\int_x^\infty\left[\left[\theta^3\left(t^3+t^2+t+1\right)+\theta^2\left(3t^2+2t+1\right)+2\theta\left(3t+1\right)+6\right]e^{-\theta t}\right]e^{-\theta t}dt$$

$$=\frac{\theta^3\left(x^3+x^2+x+1\right)+2\theta^2\left(3x^2+2x+1\right)+6\theta\left(3x+1\right)+24}{\theta\left[\theta^3\left(x^3+x^2+x+1\right)+\theta^2\left(3x^2+2x+1\right)+2\theta\left(3x+1\right)+6\right]} \tag{3.1.4}$$

It can be easily verified that $h(0)=\dfrac{\theta^4}{\theta^3+\theta^2+2\theta+6}=f_5(0)$ and $m(0)=\dfrac{\theta^3+2\theta^2+6\theta+24}{\theta\left(\theta^3+\theta^2+2\theta+6\right)}=\mu_1'$. The graphs of $h(x)$ and $m(x)$ of Amarendra distribution (1.9) for different values of its parameter are shown in figures 3 and 4, respectively.

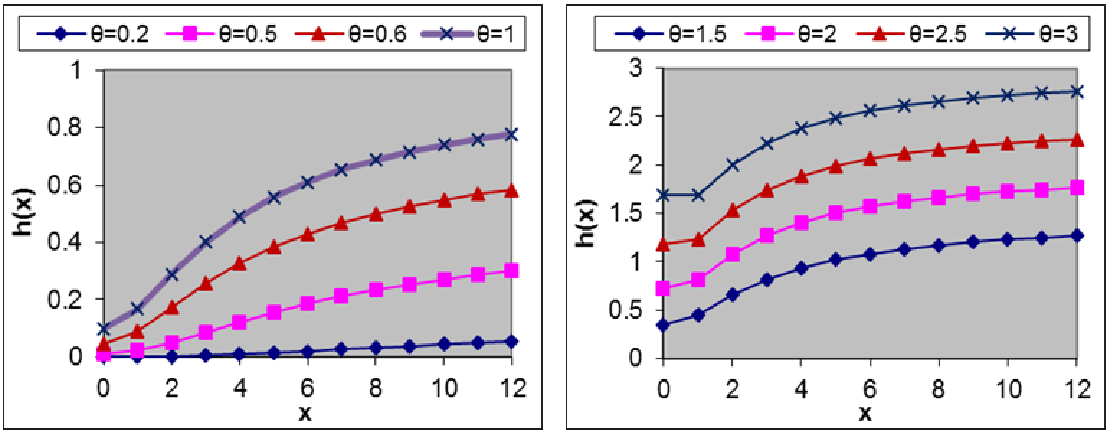

Figure 3. Graph of hazard rate function of Amarendra distribution for different values of parameter θ

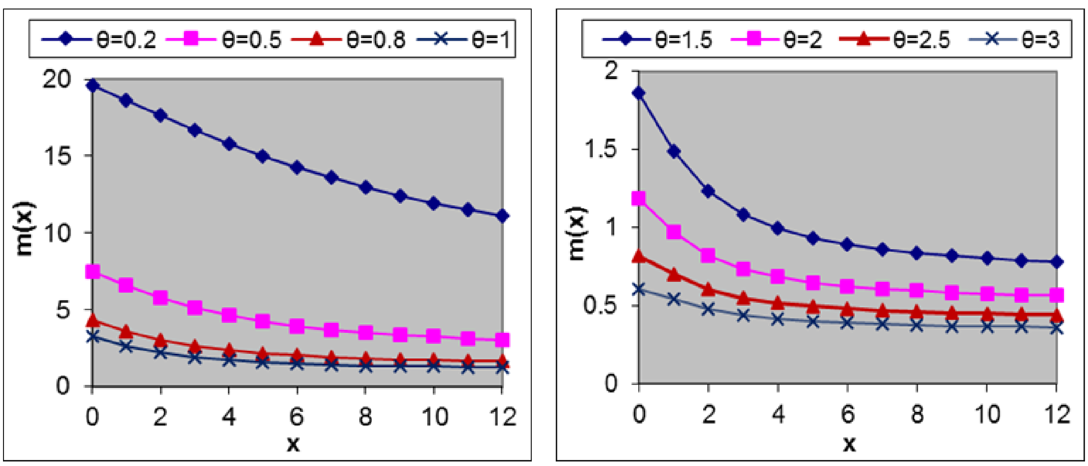

Figure 4. Graph of mean residual life function of Amarendra distribution for different values of parameter θ

It is also obvious from the graphs of $h(x)$ and $m(x)$ that $h(x)$ is monotonically increasing function of x and θ, whereas $m(x)$ is monotonically decreasing function of x and θ.

3.2. Stochastic Orderings

Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behaviour of continuous distributions. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x

(ii) hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x

(iii) mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \leq m_Y(x)$ for all x

(iv) likelihood ratio order $(X \leq_{lr} Y)$ if $\dfrac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$\begin{array}{c} X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \\ \Downarrow \\ X \leq_{st} Y \end{array}$$

The Amarendra distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem: Let X ~ Amarendra distribution (θ_1) and Y ~ Amarendra distribution (θ_2). If $\theta_1 > \theta_2$, then $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^4\left(\theta_2^3+\theta_2^2+2\theta_2+6\right)}{\theta_2^4\left(\theta_1^3+\theta_1^2+2\theta_1+6\right)} e^{-(\theta_1-\theta_2)x} \;;\; x>0$$

Now

$$\log\frac{f_X(x)}{f_Y(x)} = \log\left[\frac{\theta_1^4\left(\theta_2^3+\theta_2^2+2\theta_2+6\right)}{\theta_2^4\left(\theta_1^3+\theta_1^2+2\theta_1+6\right)}\right] - (\theta_1-\theta_2)x.$$

This gives $\dfrac{d}{dx}\log\dfrac{f_X(x)}{f_Y(x)} = -(\theta_1-\theta_2)$

Thus for $\theta_1 > \theta_2$, $\dfrac{d}{dx}\log\dfrac{f_X(x)}{f_Y(x)} < 0$. This means that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

3.3. Mean Deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and the median. These are known as the mean deviation about the mean and the mean deviation about the median and are defined by

$$\delta_1(X) = \int_0^\infty |x-\mu| f(x)\,dx \text{ and } \delta_2(X) = \int_0^\infty |x-M| f(x)\,dx, \text{ respectively,}$$

where $\mu = E(X)$ and $M = \text{Median }(X)$.

The measures, $\delta_1(X)$ and $\delta_2(X)$, can be calculated using the following relationships

$$\delta_1(X)=\int_0^{\mu}(\mu-x)f(x)\,dx+\int_{\mu}^{\infty}(x-\mu)f(x)\,dx$$

$$=\mu F(\mu)-\int_0^{\mu}x\,f(x)\,dx-\mu\left[1-F(\mu)\right]+\int_{\mu}^{\infty}x\,f(x)\,dx$$

$$=2\mu F(\mu)-2\mu+2\int_{\mu}^{\infty}x\,f(x)\,dx$$

$$=2\mu F(\mu)-2\int_0^{\mu}x\,f(x)\,dx \tag{3.3.1}$$

and

$$\delta_2(X)=\int_0^{M}(M-x)f(x)\,dx+\int_{M}^{\infty}(x-M)f(x)\,dx$$

$$=M\,F(M)-\int_0^{M}x\,f(x)\,dx-M\left[1-F(M)\right]+\int_{M}^{\infty}x\,f(x)\,dx$$

$$=-\mu+2\int_{M}^{\infty}x\,f(x)\,dx$$

$$=\mu-2\int_0^{M}x\,f(x)\,dx \tag{3.3.2}$$

Using p.d.f. (1.9) and expression for the mean of Amarendra distribution, we get

$$\int_0^{\mu}x\,f_5(x)\,dx=\mu-\frac{\left\{\begin{matrix}\theta^4\left(\mu^4+\mu^3+\mu^2+\mu\right)+\theta^3\left(4\mu^3+3\mu^2+2\mu+1\right)+\\2\theta^2\left(6\mu^2+3\mu+1\right)+6\theta\left(4\mu+1\right)+24\end{matrix}\right\}e^{-\theta\mu}}{\theta\left(\theta^3+\theta^2+2\theta+6\right)} \tag{3.3.3}$$

$$\int_0^{M}x\,f_5(x)\,dx=\mu-\frac{\left\{\begin{matrix}\theta^4\left(M^4+M^3+M^2+M\right)+\theta^3\left(4M^3+3M^2+2M+1\right)+\\2\theta^2\left(6M^2+3M+1\right)+6\theta\left(4M+1\right)+24\end{matrix}\right\}e^{-\theta M}}{\theta\left(\theta^3+\theta^2+2\theta+6\right)} \tag{3.3.4}$$

Using expressions (3.3.1), (3.3.2), (3.3.3) and (3.3.4), the mean deviation about mean, $\delta_1(X)$ and the mean deviation about median, $\delta_2(X)$ of Amarendra distribution (1.9), after some algebraic simplifications, can be obtained as

$$\delta_1(X)=\frac{2\left[\theta^3\left(\mu^3+\mu^2+\mu+1\right)+2\theta^2\left(3\mu^2+2\mu+1\right)+6\theta\left(3\mu+1\right)+24\right]e^{-\theta\mu}}{\theta\left(\theta^3+\theta^2+2\theta+6\right)} \tag{3.3.5}$$

and

$$\delta_2(X)=\frac{2\left[\begin{matrix}\theta^4\left(M^4+M^3+M^2+M\right)+\theta^3\left(4M^3+3M^2+2M+1\right)\\+2\theta^2\left(6M^2+M+1\right)+6\theta\left(4M+1\right)+24\end{matrix}\right]e^{-\theta M}}{\theta\left(\theta^3+\theta^2+2\theta+6\right)}-\mu \tag{3.3.6}$$

3.4. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves (Bonferroni, 1930) and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p)=\frac{1}{p\mu}\int_0^q x\,f(x)\,dx=\frac{1}{p\mu}\left[\int_0^\infty x\,f(x)\,dx-\int_q^\infty x\,f(x)\,dx\right]=\frac{1}{p\mu}\left[\mu-\int_q^\infty x\,f(x)\,dx\right] \tag{3.4.1}$$

and

$$L(p)=\frac{1}{\mu}\int_0^q x\,f(x)\,dx=\frac{1}{\mu}\left[\int_0^\infty x\,f(x)\,dx-\int_q^\infty x\,f(x)\,dx\right]=\frac{1}{\mu}\left[\mu-\int_q^\infty x\,f(x)\,dx\right] \tag{3.4.2}$$

respectively or equivalently

$$B(p)=\frac{1}{p\mu}\int_0^p F^{-1}(x)\,dx \tag{3.4.3}$$

and

$$L(p)=\frac{1}{\mu}\int_0^p F^{-1}(x)\,dx \tag{3.4.4}$$

respectively, where $\mu=E(X)$ and $q=F^{-1}(p)$.

The Bonferroni and Gini indices are thus defined as

$$B=1-\int_0^1 B(p)\,dp \tag{3.4.5}$$

and

$$G=1-2\int_0^1 L(p)\,dp \tag{3.4.6}$$

respectively.

Using p.d.f. of Amarendra distribution (1.9), we get

$$\int_q^\infty x\,f_5(x)\,dx=\left[\frac{\left\{\begin{array}{l}\theta^4\left(q^4+q^3+q^2+q\right)+\theta^3\left(4q^3+3q^2+2q+1\right)+2\theta^2\left(6q^2+q+1\right)\\+6\theta(4q+1)+24\end{array}\right\}}{\theta\left(\theta^3+\theta^2+2\theta+6\right)}\right]e^{-\theta q} \tag{3.4.7}$$

Now using equation (3.4.7) in (3.4.1) and (3.4.2), we get

$$B(p)=\frac{1}{p}\left[1-\frac{\left\{\begin{array}{l}\theta^4\left(q^4+q^3+q^2+q\right)+\theta^3\left(4q^3+3q^2+2q+1\right)+2\theta^2\left(6q^2+q+1\right)\\+6\theta(4q+1)+24\end{array}\right\}e^{-\theta q}}{\theta^3+2\theta^2+6\theta+24}\right] \tag{3.4.8}$$

and

$$L(p)=1-\frac{\left\{\begin{array}{l}\theta^4\left(q^4+q^3+q^2+q\right)+\theta^3\left(4q^3+3q^2+2q+1\right)+2\theta^2\left(6q^2+q+1\right)\\+6\theta(4q+1)+24\end{array}\right\}e^{-\theta q}}{\theta^3+2\theta^2+6\theta+24} \tag{3.4.9}$$

Now using equations (3.4.8) and (3.4.9) in (3.4.5) and (3.4.6), the Bonferroni and Gini indices of Amarendra distribution (1.9) are obtained as

$$B=1-\frac{\left\{\begin{array}{l}\theta^4\left(q^4+q^3+q^2+q\right)+\theta^3\left(4q^3+3q^2+2q+1\right)+2\theta^2\left(6q^2+q+1\right)\\+6\theta\left(4q+1\right)+24\end{array}\right\}e^{-\theta q}}{\theta^3+2\theta^2+6\theta+24} \tag{3.4.10}$$

$$G=-1+\frac{2\left\{\begin{array}{l}\theta^4\left(q^4+q^3+q^2+q\right)+\theta^3\left(4q^3+3q^2+2q+1\right)+2\theta^2\left(6q^2+q+1\right)\\+6\theta\left(4q+1\right)+24\end{array}\right\}e^{-\theta q}}{\theta^3+2\theta^2+6\theta+24} \tag{3.4.11}$$

4. Estimation of the Parameter

4.1. Maximum Likelihood Estimation (MLE) of the Parameter

Let $\left(x_1, x_2, x_3, \ldots, x_n\right)$ be a random sample of size n from Amarendra distribution (1.9). The likelihood function, L of (1.9) is given by

$$L=\left(\frac{\theta^4}{\theta^3+\theta^2+2\theta+6}\right)^n\prod_{i=1}^{n}\left(1+x_i+x_i^{\,2}+x_i^{\,3}\right)e^{-n\theta\bar{x}}$$

The natural log likelihood function thus obtained as

$$\ln L=n\ln\left(\frac{\theta^4}{\theta^3+\theta^2+2\theta+6}\right)+\sum_{i=1}^{n}\ln\left(1+x_i+x_i^{\,2}+x_i^{\,3}\right)-n\theta\bar{x}$$

where $\bar{x}$ is the sample mean. Now

$$\frac{d\ln L}{d\theta}=\frac{4n}{\theta}-\frac{n\left(3\theta^2+2\theta+2\right)}{\theta^3+\theta^2+2\theta+6}-n\bar{x}$$

The maximum likelihood estimate, $\hat{\theta}$ of θ is the solution of the equation $\frac{d\ln L}{d\theta}=0$ and is given by the solution of the following non linear equation

$$\bar{x}\,\theta^4+\left(\bar{x}-1\right)\theta^3+2\left(\bar{x}-1\right)\theta^2+6\left(\bar{x}-1\right)\theta-24=0 \tag{4.1.1}$$

4.2. Method of Moment Estimation (MOME) of the Parameter

Let $\left(x_1, x_2, x_3, \ldots, x_n\right)$ be a random sample of size n from Amarendra distribution (1.9). Equating the first population moment about origin to the corresponding sample mean $\bar{x}$, the method of moment (MOM) estimate $\tilde{\theta}$ of θ of Amarendra distribution is found as the solution of the same non-linear equation (4.1.1), confirming that the ML estimate and MOM estimate of θ are identical.

5. Applications and Goodness of Fit

A number of data-sets have been fitted using Amarendra distribution to test its goodness of fit over one parameter Akash, Shanker, Sujatha, Lindley and exponential distributions. In this section, we present the fitting of Amarendra distribution to two real data -sets using maximum likelihood estimate and the goodness of fit is compared with the one parameter Akash, Shanker, Sujatha, Lindley and exponential distributions and it is clear from the fitting of these distributions that Amarendra distribution provides better fit for modeling lifetime data.

In order to compare the goodness of fit of these distributions, $-2\ln L$, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), BIC (Bayesian Information Criterion), and K-S Statistics (Kolmogorov-Smirnov Statistics) for two real data sets have been computed and presented in table 2. The formulae for computing AIC, AICC, BIC, and K-S Statistics are as follows:

$$AIC = -2\ln L + 2k\,,\ \ AICC = AIC + \frac{2k\left(k+1\right)}{\left(n-k-1\right)}\,,\ \ BIC = -2\ln L + k\ln n \ \text{ and}$$

$D = \underset{x}{\mathrm{Sup}}\left|F_n\left(x\right) - F_0\left(x\right)\right|$, where k = the number of parameters, n = the sample size, and $F_n\left(x\right)$ = the empirical distribution function. The best distribution is the distribution which corresponds to the lower values of $-2\ln L$, AIC, AICC, BIC, and K-S statistics

Data set 1: The first data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark (1975, P. 105). The data are as follows:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8,

1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0

Data set 2: The second data set is the strength data of glass of the aircraft window reported by Fuller *et al* (1994):

18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80,

26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76,

33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381

Table 2. MLE's, $-2\ln L$, AIC, AICC, BIC, and K-S Statistics of the fitted distributions of data -sets 1 and 2

	Model	Parameter estimate	$-2\ln L$	AIC	AICC	BIC	K-S statistic
Data 1	Amarendra	1.4807	55.64	57.64	57.86	58.63	0.286
	Sujatha	1.1367	57.50	59.50	59.72	60.49	0.309
	Shanker	0.8039	59.70	61.80	62.00	62.80	0.315
	Akash	1.1569	59.50	61.70	61.72	61.72	0.320
	Lindley	0.8161	60.50	62.50	62.72	63.49	0.341
	Exponential	0.5263	65.67	67.67	67.90	68.67	0.389
Data 2	Amarendra	0.1283	233.41	235.41	235.55	236.84	0.225
	Akash	0.0971	240.70	242.70	242.80	244.10	0.266
	Sujatha	0.0956	241.50	243.50	243.64	244.94	0.270
	Shanker	0.0647	252.30	254.30	254.50	255.80	0.326
	Lindley	0.0629	253.99	255.99	256.13	257.42	0.333
	Exponential	0.0324	274.53	276.53	276.67	277.96	0.426

It is obvious from above table that Amarendra distribution gives much closer fit than Akash, Shanker, Sujatha, Lindley and exponential distributions and hence it may be preferred over Akash, Shanker, Sujatha, Lindley and exponential distributions for modeling various lifetime data from medical science and engineering.

6. Concluding Remarks

A new lifetime distribution named, 'Amarendra distribution' has been introduced to model lifetime data. Its moment generating function, moments about origin and moments about mean and expressions for skewness and kurtosis have been given. Various mathematical and statistical properties of the distribution such as its hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, have been discussed. The method of maximum likelihood and the method of moments for estimating its parameter have also been discussed. Two examples of real lifetime data- sets have been presented to show the applications and goodness of fit of Amarendra distribution over one parameter Akash, Shanker, Sujatha, Lindley and exponential distributions.

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