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Φ-maximal Functions Measuring Smoothness

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Abstract This paper is devoted to the study of certain generalized maximal function (Φ-maximal function) measuring smoothness. In this work we essentially use the relation between maximal function measuring smoothness and oscillation of functions.

Keywords Maximal functions, Smoothness of functions, Mean oscillation, Harmonic oscillation, Φ-oscillation

1. Introduction

Let R^n be n-dimensional Euclidean space,

$$P(x) = c_n \cdot (1+|x|^2)^{-\frac{n+1}{2}}, \qquad x \in R^n,$$

$$c_n = \Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}},$$

$$P_r(x) = r^{-n} P\left(\frac{x}{r}\right), \quad r > 0, \quad f \in L_{loc}(R^n),$$

$$(P_r * f)(x) = \int_{R^n} P_r(x-t) f(t)\,dt.$$

Note that the quantity

$$\int_{R^n} P_r(x-t)|f(t) - (P_r * f)(x)|\,dt$$

is called harmonic oscillation (see, for instance, [1], [2]). In work [2] it has proven that

$$\int_{R^n} P_r(x-t)|f(t) - (\mathrm{P}_r * f)(x)|\,dt \asymp$$

$$\asymp \int_{R^n} P_r(x-t)\left|f(t) - f_{B(x,r)}\right|dt,$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(t)\,dt,$$

$|B(x,r)|$ denotes the volume of ball $B(x,r) = \{y \in R^n\colon\ |x-y| \le r\}$, and constants in the relation " $\asymp$ " depend only on dimension n. (For positive functions F and G we will use the notation $F(u) \asymp G(u),\ \ u \in U$, if there exist positive constants c_1 and c_2 such that

$$\forall u \in U\colon\quad c_1 F(u) \le G(u) \le c_2 F(u).)$$

Let

$$\int_{R^n} K(x)\,dx = 0, \qquad K_r(x) = r^{-n} K\left(\frac{x}{r}\right), \quad r > 0,$$

$$\Phi_0(x) = \frac{1}{|B(0,1)|} X_{B(0,1)}(x), \qquad x \in R^n,$$

where $X_E(x)$ is a characteristic function of the set $E \subset R^n$. It is easy to see that

$$\int_{R^n} \Phi_0(x)\,dx = 1.$$

In the papers of some other authors (see e.g. [3], [4]) the quantity

$$\|K_r * f\|_{L^p(R^n)}$$

is chosen as a characteristic to determine homogeneous classes of Besov. We can write the quantity $K_r * f(x)$ in the following form

$$\begin{aligned}(K_r * f)(x) &= ((K + \Phi_0 - \Phi_0)_r * f)(x)\\ &= ((K + \Phi_0)_r * f)(x) - (\Phi_{0,r} * f)(x)\\ &= (\Phi_r * f)(x) - \int_{R^n} \Phi_{0,r}(x-t) f(t)\,dt\\ &= (\Phi_r * f)(x) - \frac{1}{|B(x,r)|}\int_{B(x,r)} f(t)\,dt\\ &= (\Phi_r * f)(x) - f_{B(x,r)}\\ &= (\Phi_r * f)(x) - \int_{R^n} \Phi_r(x-t) f_{B(x,r)}\,dt\\ &= \int_{R^n} \Phi_r(x-t)\left[f(t) - f_{B(x,r)}\right]dt,\end{aligned}$$

where $\Phi(x) = K(x) + \Phi_0(x)$, $\Phi_{0,r}(x) = (\Phi_0)_r(x) = = r^{-n}\Phi_0\left(\frac{x}{r}\right)$.

It is obvious that

$$\int_{R^n} \Phi(x)\,dx = 1.$$

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Thus,

$$(K_r * f)(x) = \int_{R^n} \Phi_r(x-t)\left[f(t) - f_{B(x,r)}\right]dt.$$

Hence we have

$$|(K_r * f)(x)| \leq \int_{R^n} |\Phi_r(x-t)|\left|f(t) - f_{B(x,r)}\right|dt =$$

$$= \int_{R^n} |\Phi|_r(x-t) \cdot \left|f(t) - f_{B(x,r)}\right|dt.$$

In the present paper, for the principal characteristic we take the quantity

$$\Omega^{\Phi}\left(f, B(x,r)\right) = \int_{R^n} \Phi_r(x-t) \cdot \left|f(t) - f_{B(x,r)}\right|dt,$$

where $\Phi \in L^1(R^n)$, $\Phi(x) \geq 0 \quad (x \in R^n)$, $f \in L_{loc}(R^n)$. $\Omega^{\Phi}\left(f, B(x,r)\right)$ is said to be Φ-oscillation of the function f in the ball $B(x,r)$ [2].

It is known that maximal functions measuring smoothness play an important role in the study of properties of integral operators and other objects of Harmonic Analysis. The main topic of this paper is the study of certain generalized maximal function (Φ -maximal function) measuring smoothness.

The paper is organized as follows. Section 2 has auxiliary character and presents the basic definitions, some notation and well-known facts. In section 3 the relations between maximal function and metric characteristic are investigated and some useful inequalities were obtained. In section 4 estimations between Φ-maximal function and maximal function was obtained. The main results are given in Propositions 3.1, 3.3, 4.1, 4.3 and 4.4.

2. Some Definition and Auxiliary Facts

Let the function $\varphi(x,r)$ be defined on the set $R^n \times (0,+\infty)$, takes only positive values, and monotone increases with respect to the argument r on the interval $(0,+\infty)$. We denote the class of all functions $\varphi(x,r)$ with the above mentioned properties by Ψ.

Let $\Phi \in L^1(R^n)$, $\Phi(x) \geq 0 \quad (x \in R^n)$, $\varphi \in \Psi$, $f \in L_{loc}(R^n)$. Let's introduce the following Φ-maximal function

$$f_{\varphi}^{\#,\Phi}(x) = \sup_{r>0} \frac{1}{\varphi(x,r)} \int_{R^n} \Phi_r(x-t)\left|f(t) - f_{B(x,r)}\right|dt.$$

We also introduce the following metric Φ-characteristic

$$m_f^{\Phi}(x;\delta) = \sup_{0<r\leq\delta} \int_{R^n} \Phi_r(x-t)\left|f(t) - f_{B(x,r)}\right|dt,$$

$$x \in R^n, \quad \delta > 0.$$

Consider the known special cases of the introduced maximal function $f_{\varphi}^{\#,\Phi}(x)$.

1) If $\Phi(x) \equiv \Phi_0(x)$, $\varphi(x,r) \equiv 1$, then $f_{\varphi}^{\#,\Phi}(x) \equiv f^{\#}(x)$, where $f^{\#}(x)$ is the maximal function which is introduced in the paper [5].
2) If $\Phi(x) \equiv \Phi_0(x)$, $\varphi(x,r) \equiv r^{\alpha} \quad (\alpha > 0)$, then $f_{\varphi}^{\#,\Phi}(x) \equiv f_{\alpha}^{\#}(x)$. The maximal function $f_{\alpha}^{\#}(x)$ was mentioned in the papers [6], [7]. In paper [8] the function $f_{\alpha}^{\#}(x)$ was investigated.
3) If $\Phi(x) \equiv \Phi_0(x)$, $\varphi(x,r) \equiv \varphi(r)$, then the maximal function $f_{\varphi}^{\#,\Phi}(x) =: f_{\varphi}^{\#}(x)$ may be found in the papers [9], [10], [11], [12], [13] and others.

Now let's consider special cases of metric Φ-characteristic $m_f^{\Phi}(x;\delta)$.

1) If $\Phi(x) \equiv \Phi_0(x)$, then $m_f^{\Phi}(x;\delta) \equiv m_f(x;\delta)$ (see section 3), where

$$m_f(x;\delta) = \sup_{0<r\leq\delta} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(t) - f_{B(x,r)}\right|dt.$$

Note that the function $m_f(x;\delta)$ was first introduced in the paper [14] (see also, [15], [16]).

2) Let $\Phi(x) \equiv P(x)$, where $P(x)$ is the Poisson kernel, i.e. $P(x) = c_n \cdot (1+|x|^2)^{-\frac{n+1}{2}}$, where $c_n = \Gamma\left(\frac{n+1}{2}\right) \cdot \pi^{-\frac{n+1}{2}}$.

Global variant of the characteristic $m_f^{\Phi}(x;\delta)$ (more precisely, the equivalent characteristic to it which is called a modulus of harmonic oscillation) for periodic functions of one variable may be found in the paper [1].

It is known that Hardy-Littlewood's maximal function is determined by the equality

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t)|dt, \qquad x \in R^n.$$

For case of $\Phi \in L^1(R^n)$, $\Phi(x) \geq 0 \quad (x \in R^n)$, $f \in L_{loc}(R^n)$ the following maximal function is also considered [17]

$$M_{\Phi}f(x) = \sup_{r>0} \int_{R^n} \Phi_r(x-t)|f(t)|dt, \qquad x \in R^n.$$

It is easy to see that if $\Phi(x) \equiv \Phi_0(x)$, $x \in R^n$, then $M_{\Phi}f(x) \equiv Mf(x)$.

From the definition of a maximal function $f^{\#}(x)$ it follows that

$$\forall x \in R^n: \quad f^{\#}(x) \leq \sup_{r>0} \frac{2}{|B(x,r)|} \int_{B(x,r)} |f(t)|dt =$$

$$= 2Mf(x).$$

Thus,

$$f^{\#}(x) \leq 2Mf(x), \qquad x \in R^n. \tag{2.1}$$

It is known that (see e.g. [18]) if $1 < p \leq \infty$, then

$$\exists C_p > 0 \qquad \forall f \in L^p(R^n): \qquad \|Mf\|_{L^p} \leq C_p \cdot \|f\|_{L^p}.$$

Hence, from (2.1) we get

$$\exists A_p > 0 \qquad \forall f \in L^p(R^n): \qquad \|f^{\#}\|_{L^p} \leq A_p \cdot \|f\|_{L^p}.$$

The last relation means that the operator $f \longmapsto f^{\#}$ is the operator of the type (p,p) for $1 < p \leq \infty$.

It is also known [18] that if $f \in L^1(R^n)$, then there exists a number $A > 0$ such that for any $\lambda > 0$

$$m\{x\in R^n:\quad Mf(x)>\lambda\}\le\frac{A}{\lambda}\cdot\|f\|_{L^1(R^n)},$$

where mE denotes the Lebesgue measure of the set $E\subset R^n$. Hence, from (2.1) we get

$$m\{x\in R^n:\ f^{\#}(x)>\lambda\}\le\ m\left\{x\in R^n:\quad Mf(x)>\frac{\lambda}{2}\right\}\le$$
$$\le\frac{2A}{\lambda}\|f\|_{L^1(R^n)}.$$

Thus, if $f\in L^1(R^n)$, then there exist the number $A_1>0$ such that for any $\lambda>0$

$$m\{x\in R^n:\quad f^{\#}(x)>\lambda\}\le\frac{A_1}{\lambda}\cdot\|f\|_{L^1(R^n)}.$$

The last relation means that the operator $f\longmapsto f^{\#}$ is the operator of weak type (1,1).

In the case $\varphi(x,r)\equiv 1$, we denote the function $f_\varphi^{\#,\Phi}(x)$ by $f^{\#,\Phi}(x)$. Then for the function $f^{\#,\Phi}(x)$ we have

$$f^{\#,\Phi}(x)=\sup_{r>0}\int_{R^n}\Phi_r(x-t)\left|f(t)-f_{B(x,r)}\right|dt\le$$
$$\le\sup_{r>0}\left\{\int_{R^n}\Phi_r(x-t)|f(t)|dt+\left|f_{B(x,r)}\right|\int_{R^n}\Phi_r(x-t)dt\right\}$$
$$\le\sup_{r>0}\int_{R^n}\Phi_r(x-t)|f(t)|dt+$$
$$+C\cdot\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(t)|dt=$$
$$=M_\Phi f(x)+C\cdot Mf(x),\qquad x\in R^n,$$

where $\mathrm{C}=\int_{R^n}\Phi_r(x-t)dt=\int_{R^n}\Phi(t)dt$. Thus,

$$f^{\#,\Phi}(x)\le M_\Phi f(x)+C\cdot Mf(x),\quad x\in R^n. \tag{2.2}$$

From the inequality (2.2), the Hardy-Littlewood maximal theorem and theorem 2 of chapter 3 [18] we get the following facts.

If $\psi(x)=\sup_{|y|\ge|x|}|\Phi(y)|$, $\psi\in L^1(R^n)$, then for $1<p\le\infty$

$$\exists A_p>0\qquad\forall f\in L^p(R^n):\qquad\|f^{\#,\Phi}\|_{L^p}\le A_p\cdot\|f\|_{L^p},$$

and for $p=1$ we have

$$m\{x\in R^n:\ f^{\#,\Phi}(x)>\lambda\}\le\frac{A_1}{\lambda}\|f\|_{L^1},$$
$$f\in L^1(R^n),\qquad\lambda>0,$$

where the positive constant A_1 is independent on f and λ.

Thus, at the indicated conditions on the function Φ, the operator $f\longmapsto f^{\#,\Phi}$ is the operator of type (p,p) for $1<p\le\infty$, and is also weak type (1,1) operator.

3. Relations between Maximal Function and Metric Characteristic. Some Inequalities

In this section we'll assume that $\Phi\in$

$$\in L^1(R^n),\ \ \Phi(x)\ge 0\quad(x\in R^n),\ \ \varphi\in\Psi.$$

Proposition 3.1. If $f\in L_{loc}(R^n)$, then the following equality is satisfied

$$f_\varphi^{\#,\Phi}(x)=\sup_{r>0}\frac{m_f^\Phi(x;r)}{\varphi(x,r)},\ \ x\in R^n. \tag{3.1}$$

Proof: From the definition of the function $f_\varphi^{\#,\Phi}(x)$ we get

$$f_\varphi^{\#,\Phi}(x)=\sup_{r>0}\frac{1}{\varphi(x,r)}\int_{R^n}\Phi_r(x-t)\left|f(t)-f_{B(x,r)}\right|dt\le$$
$$\le\sup_{r>0}\frac{m_f^\Phi(x;r)}{\varphi(x,r)},\qquad x\in R^n. \tag{3.2}$$

On the other hand, for any $r>0$ and $x\in R^n$ we have

$$f_\varphi^{\#,\Phi}(x)\ge\frac{1}{\varphi(x,r)}\int_{R^n}\Phi_r(x-t)\left|f(t)-f_{B(x,r)}\right|dt.$$

Hence it follows that

$$\int_{R^n}\Phi_r(x-t)\left|f(t)-f_{B(x,r)}\right|dt\le\varphi(x,r)\cdot f_\varphi^{\#,\Phi}(x),$$

therefore

$$m_f^\Phi(x;r)\le\varphi(x,r)\cdot f_\varphi^{\#,\Phi}(x),\qquad r>0,\qquad x\in R^n\ .$$

So,

$$f_\varphi^{\#,\Phi}(x)\ge\frac{m_f^\Phi(x;r)}{\varphi(x,r)},\qquad r>0,\qquad x\in R^n.$$

From the last inequality we get

$$f_\varphi^{\#,\Phi}(x)\ge\sup_{r>0}\frac{m_f^\Phi(x;r)}{\varphi(x,r)},\qquad x\in R^n. \tag{3.3}$$

Equality (3.1) is obtained from inequalities (3.2) and (3.3).

Lemma 3.1. Let $f\in L_{loc}(R^n)$, and

$$\operatorname{essinf}\{\Phi(x):\quad x\in B(0,1)\}=\mathrm{c}_0>0. \tag{3.4}$$

Then for any constant C the following inequality is true

$$\int_{R^n}\Phi_r(x-t)\left|f(t)-f_{B(x,r)}\right|dt\le$$
$$\le\mathrm{c}_1\cdot\int_{R^n}\Phi_r(x-t)|f(t)-\mathrm{C}|dt\,,r>0,x\in R^n, \tag{3.5}$$

where the positive constant c_1 depends only on the c_0, dimension n and on the quantity $\|\Phi\|_{L^1(R^n)}$.

Proof. Let C be any constant. Then we have

$$\int_{R^n}\Phi_r(x-t)\left|f(t)-f_{B(x,r)}\right|dt\le$$
$$\le\int_{R^n}\Phi_r(x-t)|f(t)-C|dt$$
$$+\int_{R^n}\Phi_r(x-t)\left|f_{B(x,r)}-C\right|dt$$
$$\le\int_{R^n}\Phi_r(x-t)|f(t)-C|dt$$
$$+\left(\int_{R^n}\Phi_r(x-t)dt\right)\cdot\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(t)-C|dt$$

$$= \|\Phi\|_{L^1(R^n)} \cdot \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t) - C| dt$$

$$+ \int_{R^n} \Phi_r(x-t)|f(t) - C| dt.$$

Thus, for all $x \in R^n$ and $r > 0$

$$\int_{R^n} \Phi_r(x-t)\left|f(t) - f_{B(x,r)}\right| dt$$

$$\leq \int_{R^n} \Phi_r(x-t)|f(t) - C| dt +$$

$$+\|\Phi\|_{L^1(R^n)} \cdot \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t) - C| dt. \quad (3.6)$$

On the other hand, by means of condition (3.4) we get

$$\int_{R^n} \Phi_r(x-t)|f(t) - C| dt \geq$$

$$\geq \frac{1}{r^n} \int_{B(x,r)} \Phi\left(\frac{x-t}{r}\right)|f(t) - C| dt \geq$$

$$\geq c_0 \cdot \frac{|B(0,1)|}{|B(x,r)|} \int_{B(x,r)} |f(t) - C| dt, \quad x \in R^n, \quad r > 0.$$

Hence

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t) - C| dt \leq$$

$$\leq \frac{1}{c_0 \cdot |B(0,1)|} \int_{R^n} \Phi_r(x-t)|f(t) - C| dt. \quad (3.7)$$

Using this inequality, from (3.6) we get

$$\int_{R^n} \Phi_r(x-t)\left|f(t) - f_{B(x,r)}\right| dt \leq$$

$$\leq \left(1 + \frac{\|\Phi\|_{L^1(R^n)}}{c_0 \cdot |B(0,1)|}\right) \cdot \int_{R^n} \Phi_r(x-t)|f(t) - C| dt.$$

Proposition 3.2. Let $f \in L_{loc}(R^n)$, $x \in R^n$ and condition (3.4) be satisfied. Then the following inequality is true

$$m_f(x;r) \leq \frac{1}{c_0 \cdot |B(0,1)|} \cdot m_f^{\Phi}(x;r), \qquad r > 0, \quad (3.8)$$

Proof. If we take $C = f_{B(x,r)}$, then the validity of inequality (3.8) is obtained from relation (3.7).

Remark 3.1. Note that for the function $\Phi(x)$, $x \in R^n$, satisfying condition (3.4) we can take, for instance, the following functions:

1) $\Phi_0(x) = \frac{1}{|B(0,1)|} X_{B(0,1)}(x)$;

2) $\Phi^{(\alpha)}(x) = \frac{1}{1+|x|^{n+\alpha}}$, $\quad (\alpha > 0)$;

3) $P(x) = c_n \cdot (1 + |x|^2)^{-\frac{n+1}{2}}$, where $c_n = \Gamma\left(\frac{n+1}{2}\right) \pi^{-\frac{n+1}{2}}$.

Verify, that if $\Phi(x) \equiv \Phi_0(x)$, then

$$m_f^{\Phi}(x;r) \equiv m_f(x;r). \quad (3.9)$$

Indeed, if $\Phi(x) = \frac{1}{|B(0,1)|} X_{B(0,1)}(x)$, then

$$\Phi_r(x-t) = r^{-n} \Phi\left(\frac{x-t}{r}\right)$$

$$= \frac{1}{r^n \cdot |B(0,1)|} X_{B(0,1)}\left(\frac{x-t}{r}\right)$$

$$= \frac{1}{|B(x,r)|} \cdot X_{B(x,r)}(t) = \begin{cases} \dfrac{1}{|B(x,r)|} & if\ t \in B(x,r), \\ 0 & if\ t \notin B(x,r). \end{cases}$$

Therefore for this function $\Phi(x)$ we have

$$\int_{R^n} \Phi_r(x-t)\left|f(t) - f_{B(x,r)}\right| dt$$

$$= \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(t) - f_{B(x,r)}\right| dt.$$

Hence, equality (3.9) is obtained. We note that the quantity

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(t) - f_{B(x,r)}\right| dt$$

is said to be mean oscillation of the function f in the ball $B(x,r)$.

Remark 3.2. In the case of $\Phi(x) \equiv P(x)$, $x \in R^n$, the quantity

$$\int_{R^n} \Phi_r(x-t)|f(t) - (\Phi_r * f)(x)| dt \equiv$$

$$\equiv \int_{R^n} P_r(x-t)|f(t) - (P_r * f)(x)| dt$$

is called a harmonic oscillation of the function f (see [1]). In the paper [2] it has been proven that

$$\int_{R^n} P_r(x-t)|f(t) - (P_r * f)(x)| dt \asymp$$

$$\asymp \int_{R^n} P_r(x-t)\left|f(t) - f_{B(x,r)}\right| dt,$$

where the constants in the relation " $\asymp$ " depend only on the dimension n. Hence it is obtained that

$$m_f^{P}(x;r) \asymp h_f(x;r), \quad x \in R^n, \quad r > 0,$$

where

$$h_f(x;r) = \sup_{0<\tau\leq r} \int_{R^n} P_\tau(x-t)|f(t) - (P_\tau * f)(x)| dt$$

(see [2]).

Let's show that the relation

$$\int_{R^n} \Phi_r(x-t)|f(t) - (\Phi_r * f)(x)| dt \asymp$$

$$\asymp \int_{R^n} \Phi_r(x-t)\left|f(t) - f_{B(x,r)}\right| dt \quad (3.10)$$

takes place for wider class of functions Φ.

Proposition 3.3. Let $\Phi \in L(R^n)$, $\Phi(x) \geq 0$ $(x \in R^n)$,

$\Phi_r(x) = r^{-n}\Phi\left(\frac{x}{r}\right)$, $r > 0$, $\int_{R^n}\Phi(x)dx = 1$, and condition (3.4) is satisfied. Then the relation (3.10) is true, where the constants in the relation " $\asymp$ " depend only on the constant c_0 and dimension n.

Proof. For convenience we will introduce the following notations:

$$A := \int_{R^n} \Phi_r(x-t)|f(t) - (\Phi_r * f)(x)|dt,$$

$$B := \int_{R^n} \Phi_r(x-t)\left|f(t) - f_{B(x,r)}\right|dt.$$

Then we get

$$A \le \int_{R^n} \Phi_r(x-t)\left|f(t) - f_{B(x,r)}\right|dt +$$

$$+\int_{R^n} \Phi_r(x-t)\left|f_{B(x,r)} - (\Phi_r * f)(x)\right|dt =$$

$$= B + \left|\int_{R^n} \Phi_r(x-t)f(t)dt - f_{B(x,r)}\right| \le$$

$$\le B + \int_{R^n} \Phi_r(x-t)\left|f(t) - f_{B(x,r)}\right|dt = 2B.$$

Thus

$$A \le 2B. \tag{3.11}$$

On the other hand,

$$B \le \int_{R^n} \Phi_r(x-t)|f(t) - (\Phi_r * f)(x)|dt +$$

$$+\int_{R^n} \Phi_r(x-t)\left|(\Phi_r * f)(x) - f_{B(x,r)}\right|dt =$$

$$= A + \left|(\Phi_r * f)(x) - f_{B(x,r)}\right| =$$

$$= A + \left|\int_{R^n} \Phi_r(x-t)f(t)dt - \frac{1}{|B(x,r)|}\int_{B(x,r)} f(t)dt\right| =$$

$$= A + \left|\frac{1}{|B(x,r)|}\int_{B(x,r)} \big(f(t) - (\Phi_r * f)(x)\big)dt\right| \le$$

$$\le A + \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(t) - (\Phi_r * f)(x)|dt =$$

$$\left[t \in B(x,r) \iff \frac{x-t}{r} \in B(0,1)\right]$$

$$= A + \frac{1}{|B(0,1)|}\cdot\frac{1}{r^n}\times$$

$$\times \int_{R^n} X_{B(0,1)}\left(\frac{x-t}{r}\right)|f(t) - (\Phi_r * f)(x)|dt \le$$

$$\le A + \frac{1}{c_0}\cdot\frac{1}{|B(0,1)|}\cdot\frac{1}{r^n}\times$$

$$\times \int_{R^n} \Phi\left(\frac{x-t}{r}\right)|f(t) - (\Phi_r * f)(x)|dt =$$

$$= \left(1 + \frac{1}{c_0}\cdot\frac{1}{|B(0,1)|}\right)\cdot A.$$

Thus

$$B \le \left(1 + \frac{1}{c_0}\cdot\frac{1}{|B(0,1)|}\right)\cdot A. \tag{3.12}$$

Inequalities (3.11) and (3.12) prove the required relation (3.10).

4. Estimations of Φ-Maximal Functions by Maximal Functions

Proposition 4.1. If $f \in L_{loc}(R^n)$, $\varphi \in \Psi$ and the function Φ satisfies condition (3.4), then the following inequality is true

$$f_\varphi^\#(x) \le c \cdot f_\varphi^{\#,\Phi}(x), \quad x \in R^n, \tag{4.1}$$

where $c = \frac{1}{c_0\cdot|B(0,1)|}$, and c_0 is a constant from inequality (3.4).

Proof. By means of Proposition 3.1 and inequality (3.8) we get

$$f_\varphi^\#(x) = \sup_{r>0}\frac{m_f(x;r)}{\varphi(x,r)} \le \frac{1}{c_0\cdot|B(0,1)|}\cdot\sup_{r>0}\frac{m_f^\Phi(x;r)}{\varphi(x,r)} =$$

$$= \frac{1}{c_0\cdot|B(0,1)|}\cdot f_\varphi^{\#,\Phi}(x),\ \ x \in R^n.$$

Proposition 4.2. [2]. Let $\Phi_\alpha(x) = \frac{1}{1+|x|^{n+\alpha}}$, $\alpha > 0$, $x \in R^n$, $f \in L_{loc}(R^n)$. Then the following inequality is true

$$m_f^{\Phi_\alpha}(x;r) \le c\cdot r^\alpha \int_r^\infty \frac{m_f(x;t)}{t^{\alpha+1}}dt, \quad r > 0, \tag{4.2}$$

where the constant $c > 0$ is independent on f, x and r.

Lemma 4.1. If $\omega(t)$ is a non-negative, monotone increasing function on the interval $(0, +\infty)$, $\alpha > 0$, and

$$\int_1^\infty \frac{\omega(t)}{t^{1+\alpha}}dt < +\infty,$$

then the function

$$\mu(r) := r^\alpha \int_r^\infty \frac{\omega(t)}{t^{1+\alpha}}dt$$

also monotone increases on interval $(0, +\infty)$.

Proof. Let $r_1, r_2 \in (0, +\infty)$ and $r_1 < r_2$. Then we have

$$\mu(r_2) - \mu(r_1) = r_2^\alpha \int_{r_2}^\infty \frac{\omega(t)}{t^{1+\alpha}}dt - r_1^\alpha \int_{r_1}^\infty \frac{\omega(t)}{t^{1+\alpha}}dt =$$

$$= (r_2^\alpha - r_1^\alpha)\int_{r_2}^\infty \frac{\omega(t)}{t^{1+\alpha}}dt - r_1^\alpha \int_{r_1}^{r_2} \frac{\omega(t)}{t^{1+\alpha}}dt \ge$$

$$\omega(r_2)(r_2^\alpha - r_1^\alpha)\int_{r_2}^{\infty} t^{-1-\alpha}dt - \omega(r_2)r_1^\alpha \int_{r_1}^{r_2} t^{-1-\alpha}dt =$$

$$= \omega(r_2)\cdot(r_2^\alpha - r_1^\alpha)\cdot\left(-\frac{1}{\alpha}\cdot t^{-\alpha}\Big|_{r_2}^{\infty}\right) -$$

$$-\omega(r_2)\cdot r_1^\alpha\cdot\left(-\frac{1}{\alpha}\cdot t^{-\alpha}\Big|_{r_1}^{r_2}\right) =$$

$$= \frac{1}{\alpha}\omega(r_2)(r_2^\alpha - r_1^\alpha)\cdot\frac{1}{r_2^\alpha} + \frac{1}{\alpha}\cdot\omega(r_2)r_1^\alpha\left(\frac{1}{r_2^\alpha} - \frac{1}{r_1^\alpha}\right) =$$

$$= \frac{1}{\alpha}\cdot\omega(r_2)\cdot\left(1 - \frac{r_1^\alpha}{r_2^\alpha} + \frac{r_1^\alpha}{r_2^\alpha} - 1\right) = 0,$$

i.e. $\mu(r_2) \ge \mu(r_1)$.

Proposition 4.3. Let $f \in L_{loc}(R^n)$, $\varphi \in \Psi$, $\Phi_\alpha(x) = = \frac{1}{1+|x|^{n+\alpha}}$, $x \in R^n$, $\alpha > 0$ and

$$\int_1^\infty \frac{\varphi(x,t)}{t^{\alpha+1}}dt < +\infty. \qquad (4.3)$$

Then the following inequality holds

$$f_\psi^{\#,\Phi_\alpha}(x) \le c\cdot f_\varphi^{\#}(x), \qquad (4.4)$$

where $\psi(x,r) \coloneqq r^\alpha \int_r^\infty \frac{\varphi(x,t)}{t^{\alpha+1}}dt$, and the positive constant c does not depend on f and x.

Proof. By means of relations (3.1), (4.2) and (4.3), we have

$$f_\psi^{\#,\Phi_\alpha}(x) = \sup_{r>0}\frac{m_f^{\Phi_\alpha}(x;r)}{\psi(x;r)} \le$$

$$\le c\cdot\sup_{r>0}\frac{1}{\psi(x;r)}\cdot r^\alpha\int_r^\infty \frac{m_f(x;t)}{t^{\alpha+1}}dt =$$

$$= c\cdot\sup_{r>0}\frac{1}{\psi(x;r)}\cdot r^\alpha\int_r^\infty \frac{m_f(x;t)}{\varphi(x,t)}\cdot\frac{\varphi(x,t)}{t^{\alpha+1}}dt \le$$

$$\le c\cdot\sup_{t>0}\frac{m_f(x;t)}{\varphi(x,t)}\cdot\sup_{r>0}\frac{1}{\psi(x,r)}\cdot r^\alpha\int_r^\infty\frac{\varphi(x,t)}{t^{\alpha+1}}dt$$

$$= c\cdot\sup_{t>0}\frac{m_f(x;t)}{\varphi(x,t)} = c\cdot f_\varphi^{\#}(x).$$

Corollary 4.1. Let $f \in L_{loc}(R^n)$, $\varphi \in \Psi$, $\Phi_\alpha(x) = \frac{1}{1+|x|^{n+\alpha}}$, $x \in R^n$, $\alpha > 0$ and

$$r^\alpha\int_r^\infty \frac{\varphi(x,t)}{t^{\alpha+1}}dt = O\big(\varphi(x,r)\big),\ r > 0, x \in R^n. \quad (4.5)$$

Then the following inequality holds

$$f_\varphi^{\#,\Phi_\alpha}(x) \le c\cdot f_\varphi^{\#}(x), \qquad x \in R^n, \qquad (4.6)$$

where the positive constant c is independent on f and x.

Proof. If condition (4.5) is satisfied, then by virtue of proposition 4.3 the inequality (4.4) holds. Furthermore, from a condition (4.5) follows that

$$\exists A > 0 \qquad \forall x \in R^n,\ \ \forall r > 0: \quad \psi(x,r) \le A\cdot\varphi(x,r).$$

Taking this into account, we have

$$f_\varphi^{\#,\Phi_\alpha}(x) = \sup_{r>0}\frac{m_f^{\Phi_\alpha}(x;r)}{\varphi(x;r)} \le A\cdot\sup_{r>0}\frac{m_f^{\Phi_\alpha}(x;r)}{\psi(x;r)}$$

$$= A\cdot f_\psi^{\#,\Phi_\alpha}(x) \le A\cdot c\cdot f_\varphi^{\#}(x), \quad x \in R^n,$$

where $c > 0$ is a constant from inequality (4.4).

Proposition 4.4. Let $\varphi \in \Psi$, $\varphi(x,t) \equiv \varphi(t)$, $\Phi_\alpha(x) = \frac{1}{1+|x|^{n+\alpha}}$, $x \in R^n$, $\alpha > 0$, and

$$\int_1^\infty \frac{\varphi(t)}{t^{\alpha+1}}dt = +\infty. \qquad (4.7)$$

Then there exists a function $f \in L_{loc}(R^n)$ such that $f_\varphi^{\#}(0) < +\infty$, $f_\psi^{\#,\Phi_\alpha}(0) = +\infty$ for any function $\psi \in \Psi$, $\psi(x,t) \equiv \psi(t)$.

Proof. Consider the function

$$f(x) = \int_{|x|}^{1}\frac{\varphi(t)}{t}dt, \quad x \in R^n.$$

In the paper [14] it is shown that

$$f_{B(0,r)} = \frac{1}{r^n}\int_0^r \varphi(x)x^{n-1}dx + \int_r^1\frac{\varphi(x)}{x}dx,$$

$$m_f(0;r) \le \frac{2}{n}\cdot\varphi(r), \quad r > 0.$$

From the last inequality it follows that

$$f_\varphi^{\#}(0) = \sup_{r>0}\frac{m_f(0;r)}{\varphi(r)} \le \frac{2}{n} < +\infty.$$

Further, for $0 < r \le 1$ we have

$$\Omega^{\Phi_\alpha}\big(f,B(0,r)\big) = \frac{1}{r^n}\int_{R^n}\Phi\left(\frac{0-t}{r}\right)\left|f(t) - f_{B(0,r)}\right|dt$$

$$= \frac{1}{r^n}\int_{R^n}\frac{1}{1+\left|\frac{t}{r}\right|^{n+\alpha}}\left|f(t) - f_{B(0,r)}\right|dt$$

$$= r^\alpha\int_{R^n}\frac{1}{r^{n+\alpha}+|t|^{n+\alpha}}\left|f(t) - f_{B(0,r)}\right|dt$$

$$\ge \frac{1}{2}r^\alpha\int_{|t|\ge 1}\frac{1}{|t|^{n+\alpha}}\left|f(t) - f_{B(0,r)}\right|dt$$

$$= \frac{1}{2}r^\alpha\int_{|t|\ge 1}\frac{1}{|t|^{n+\alpha}}\left|\int_{|t|}^{1}\frac{\varphi(x)}{x}dx - \frac{1}{r^n}\int_0^r\varphi(x)x^{n-1}dx - \int_r^1\frac{\varphi(x)}{x}dx\right|dt$$

$$= \frac{1}{2}r^\alpha\int_{|t|\ge 1}\frac{1}{|t|^{n+\alpha}}\left|-\int_r^{|t|}\frac{\varphi(x)}{x}dx - \frac{1}{r^n}\int_0^r\varphi(x)x^{n-1}dx\right|dt$$

$$= \frac{1}{2}r^\alpha\int_{|t|\ge 1}\frac{1}{|t|^{n+\alpha}}\left(\int_r^{|t|}\frac{\varphi(x)}{x}dx + \frac{1}{r^n}\int_0^r\varphi(x)x^{n-1}dx\right)dt$$

$$\geq \frac{1}{2} r^{\alpha} \int\limits_{|t|\geq 1} \frac{1}{|t|^{n+\alpha}} \left(\int\limits_{r}^{|t|} \frac{\varphi(x)}{x} dx \right) dt$$

$$= \frac{1}{2} r^{\alpha} \int\limits_{1}^{\infty} \tau^{n-1} \cdot \frac{1}{\tau^{n+\alpha}} \cdot \left(\int\limits_{S^{n-1}} \left(\int\limits_{r}^{|\tau\xi|} \frac{\varphi(x)}{x} dx \right) ds_{\xi} \right) d\tau$$

$$= |S^{n-1}| \cdot \frac{1}{2} r^{\alpha} \int\limits_{1}^{\infty} \frac{1}{\tau^{1+\alpha}} \left(\int\limits_{r}^{\tau} \frac{\varphi(x)}{x} dx \right) d\tau$$

$$= |S^{n-1}| \cdot \frac{1}{2} r^{\alpha} \int\limits_{r}^{1} \frac{\varphi(x)}{x} \left(\int\limits_{1}^{\infty} \frac{1}{\tau^{1+\alpha}} d\tau \right) dx$$

$$+ |S^{n-1}| \cdot \frac{1}{2} r^{\alpha} \int\limits_{1}^{\infty} \frac{\varphi(x)}{x} \left(\int\limits_{x}^{\infty} \frac{1}{\tau^{1+\alpha}} d\tau \right) dx$$

$$= |S^{n-1}| \cdot \frac{1}{2} \cdot \frac{1}{\alpha} \cdot r^{\alpha} \int\limits_{r}^{1} \frac{\varphi(x)}{x} dx$$

$$+ |S^{n-1}| \cdot \frac{1}{2} \cdot \frac{1}{\alpha} \cdot r^{\alpha} \int\limits_{1}^{\infty} \frac{\varphi(x)}{x^{1+\alpha}} dx = +\infty,$$

where $|S^{n-1}|$ denotes the area of the surface of a unit sphere $S^{n-1} \subset R^n$. Thus, for $0 < r \leq 1$ the equality $\Omega^{\Phi_\alpha}(f, B(0,r)) = +\infty$ is true. Therefore, for any function $\psi(x,t) \equiv \psi(t) \in \Psi$

$$f_{\psi}^{\#,\Phi_\alpha}(0) = \sup_{r>0} \frac{m_f^{\Phi_\alpha}(0;r)}{\psi(r)} \geq \sup_{0<r\leq 1} \frac{m_f^{\Phi_\alpha}(0;r)}{\psi(r)} = +\infty.$$

Corollary 4.2. Let $f \in L_{loc}(R^n)$, $\varphi \in \Psi$, $\Phi_\alpha(x) = = \frac{1}{1+|x|^{n+\alpha}}$, $x \in R^n$, $\alpha > 0$, and condition (4.5) satisfied. Then there exist the numbers $c_1 > 0$, $c_2 > 0$ such that

$$c_1 \cdot f_{\varphi}^{\#}(x) \leq f_{\varphi}^{\#,\Phi_\alpha}(x) \leq c_2 \cdot f_{\varphi}^{\#}(x), \quad x \in R^n,$$

where the constants c_1 and c_2 are independent on f and x.

Now consider the case of the function $\Phi(x) \equiv P(x)$, where $P(x)$ is a Poisson kernel. It is easy to see that there exist the numbers $c_1 > 0$, $c_2 > 0$ such that for all $x \in R^n$ the relation.

$$c_1 \cdot \frac{1}{1+|x|^{n+1}} \leq P(x) \leq c_2 \cdot \frac{1}{1+|x|^{n+1}},$$

holds. That is $P(x) \asymp \Phi_1(x)$, $x \in R^n$, where $\Phi_1(x) = \frac{1}{1+|x|^{n+1}}$. Hence it follows that if $f \in L_{loc}(R^n)$ and $\varphi \in \Psi$, then the following relations are true

$$f_{\varphi}^{\#,P}(x) \asymp f_{\varphi}^{\#,\Phi_1}(x), \qquad x \in R^n,$$

$$m_f^{P}(x;r) \asymp m_f^{\Phi_1}(x;r), \qquad x \in R^n, \quad r > 0.$$

By means of these considerations, from corollary 4.2 we get

Corollary 4.3. Let $f \in L_{loc}(R^n)$, $P = P(x)$ be a Poisson kernel, $\varphi \in \Psi$ and

$$r \int\limits_{r}^{\infty} \frac{\varphi(x,t)}{t^2} dt = O(\varphi(x,r)), \quad r > 0, \quad x \in R^n.$$

Then the following relation is true

$$f_{\varphi}^{\#,P}(x) \asymp f_{\varphi}^{\#}(x), \quad x \in R^n.$$

5. Conclusions

Maximal functions play an important role in the study of differentiation of functions, almost everywhere convergence of singular integrals, mapping properties of singular integral operators and potential type integral operators.

Maximal functions measuring smoothness are useful in the study of smoothness of functions and the mapping properties of various operators of Harmonic Analysis on smoothness spaces.

The main theme of this paper is to study certain maximal functions and Φ-maximal functions measuring smoothness. Relations between maximal and Φ-maximal functions measuring smoothness are studied. These relations allow to unite and compare the results received in terms of various characteristics.

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