Categories in Digital Images

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Abstract   In this paper we recall some properties in digital topology for example, digital set, digital function, digital homotopy etc. from Boxer[5] and then give the definition of digital category. Also we present digital version of important category models.

Keywords   Digital Image, Digital Continuous Maps, Digital Homotopy, Digital Path, Digital Category

1. Introduction

Digital Topology is a branch of mathematics where the image processing and digital image processing is studied. Many Mathematicians, for example Rosenfeld, Kopperman, Han, Kong, Malgouyres, Boxer, Ayala, Karaca and others have contributed this area with their research. The notion of digital image, digital continuous map and digital homotopy studied in[3, 4, 6, 7, 12, 16]. Their recognition and efficient computation became a useful material for our study.

Then we carry this notion to category theory and we construct some fundamental category models in digital topology.

In section two we recall some definitions and properties from Boxer[5] in section three we introduce the 'Digital Category' and give basic examples of digital categories in order to construct a tool for category theory researchers.

2. Preliminaries

In this paper we denote the set of integers by $\mathbb{Z}$. Then $\mathbb{Z}^n$ represents the set of lattice points in Euclidean n-dimensional spaces. A finite subset of $\mathbb{Z}^n$ is called to be digital image.

We will use a variety of adjacency relations it the digital image research. The following[7] are commonly used.

Two points p and q in $\mathbb{Z}^2$ are $8-adjacent$ if they are distinct and differ by at most 1 in each coordinate; p and q in $\mathbb{Z}^2$ are $4-adjacent$ if they are $8-adjacent$ and differ in exactly one coordinate. Two points p and q in $\mathbb{Z}^3$ are $26-adjacent$ if they are distinct and differ by at most 1 in each coordinate; they

are $18-adjacent$ if they are $26-adjacent$ and differ in at most two coordinates; they are $6-adjacent$ if they are $18-adjacent$ and differ in exactly one coordinate. For $k \in \{4, 8, 6, 18, 26\}$, a $k-neighbor$ of a lattice point p is a point that is $k-adjacent$ to p.

We generalize $4-adjacency$ in $\mathbb{Z}^2$ and $6-adjacency$ in $\mathbb{Z}^3$ by taking $p, q \in \mathbb{Z}^n$ are $2n-adjacent$ if $p \neq q$ and p and q differ by 1 in one coordinate and by 0 in all other coordinates.

More extensive adjacency relations are investigated in[5]. In the following, if κ is an adjacency relation defined for an integer κ on $\mathbb{Z}^n$ as one of the $k-adjacencies$ discussed above, that is, if
$$(n, k) \in \{(1, 2), (2, 4), (2, 8), (3, 6), (3, 18), (3, 26)\}$$
or $k = 2n$.

We assume $\kappa-adjacency$ as $k-adjacency$, $\kappa-connectedness$ as $k-connectedness$,etc.

Suppose that κ be an *adjacency* relation defined on $\mathbb{Z}^n$. A digital image $X \subset \mathbb{Z}^n$ is $\kappa-connected$ [5] if and only if for every pair of points $\{x, y\} \subset X, x \neq y$, there is a set $\{x_0, x_1, ..., x_c\} \subset X$ such that $x = x_0, x_c = y$ and x_i and x_{i+1} are $\kappa-neighbors, i \in \{0, 1, ..., c-1\}$.

Definition 2.1. Let X and Y are digital images such that $X \subset \mathbb{Z}^{n_0}$, $Y \subset \mathbb{Z}^{n_1}$. Then the digital function $f : X \rightarrow Y$ is a function which is defined between digital images.

Definition 2.2. ([3]; see also[15]) Let X and Y are digital images such that $X \subset \mathbb{Z}^{n_0}$, $Y \subset \mathbb{Z}^{n_1}$. Assume that $f : X \rightarrow Y$ be a function. Let κ_i be an *adjacency* relation defined on $\mathbb{Z}^{n_i}$, $i \in \{0, 1\}$. f is called to be $(\kappa_0, \kappa_1)-continuous$ if the image under f

of every $\kappa_0 - connected$ subset of X is $\kappa_1 - connected$.

A function satisfying Definition 2.1 is referred to be digitally continuous. A consequence of this definition is given below.

Definition 2.3. ([3]; see also[15]) Let X and Y are digital images. Then the function $f : X \to Y$ is said to be $(\kappa_0, \kappa_1) - continuous$ if and only if for every $\{x, y\} \subset X$ such that x_0 and x_1 are $\kappa_0 - adjacent$, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are $\kappa_1 - adjacent$.

Definition 2.4. ([2]) Let $a, b \in \mathbb{Z}$, $a < b$. A digital interval is a set of the form

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \le z \le b\}$$

in which $2 - adjacency$ is assumed.

For example, if κ is an $adjacency$ relation on a digital image Y, then $f : [a, b]_{\mathbb{Z}} \to Y$ is $(2, \kappa) - connected$ if and only if for every $\{c, c+1\} \subset [a, b]_{\mathbb{Z}}$, either $f(c) = f(c+1)$ or $f(c)$ and $f(c+1)$ are $\kappa - adjacent$.

Definition 2.5. ([3]; see also[6]) suppose that $X \subset \mathbb{Z}^n$ and $Y \subset \mathbb{Z}^n$ be digital images. Let $f, g : X \to Y$ be $(\kappa, \lambda) - continuous$ functions. Assume there is a positive integer m and a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

i) For all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;

ii) For all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by

$$F_x(t) = F(x, t) \text{ for all } t \in [0, m]_{\mathbb{Z}}$$

is $(2, \lambda) - continuous$.

iii) For all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : [0, m]_{\mathbb{Z}} \to Y$ defined by

$$F_t(x) = F(x, t) \text{ for all } t \in [0, m]_{\mathbb{Z}}$$

is $(\kappa, \lambda) - continuous$.

Then F is called to be a digital $(\kappa, \lambda) - homotopy$ between f and g, and f and g are said to be digitally $(\kappa, \lambda) - homotopic$ in Y.

We use the notation

$$f \simeq_{\kappa, \lambda} g$$

to denote f and g are digitally $(\kappa, \lambda) - homotopic$ in Y.

Definition 2.6.[4]A digital $\kappa - path$ in a digital image is a $(2, \kappa) - continuous$ function $f : [0, m]_{\mathbb{Z}} \to X$. Also if $f(0) = f(m)$, we say that

f is a digital $\kappa - loop$, and the point $p = f(0)$ is the base point of the loop f. If f is a constant function, then it is called a trivial loop.

If f and g are digital $\kappa - paths$ in X such that g starts where f ends, the product of f and g, written $f \cdot g$, is intuitively, the $\kappa - path$ obtained by following f by g. Formally $f : [0, m]_{\mathbb{Z}} \to X$, $g : [0, m]_{\mathbb{Z}} \to X$ and $f(m_1) = g(0)$, then $(f \cdot g) : [0, m_1 + m_2]_{\mathbb{Z}} \to X$ is defined by

$$(f \cdot g)(t) = \begin{cases} f(t) & \text{if } t \in [0, m_1]_{\mathbb{Z}} \\ g(t - m_1) & \text{if } t \in [m_1, m_1 + m_2]_{\mathbb{Z}} \end{cases}.$$

Definition 2.7. Let f and f' be $(\kappa, \lambda) - continuous$ functions. Two paths f and f', mapping the digital interval $[0, m]_{\mathbb{Z}}$, are said to be digital path homotopic if they have the same initial point x_0 and the same final point x_1, and there is a $(\kappa, \lambda) - continuous$ map $F : [0, m]_{\mathbb{Z}} \times [0, m]_{\mathbb{Z}} \to X$ such that

$$F(s, 0) = f(s) \text{ and } F(s, m) = f'(s),$$
$$F(0, t) = x_0 \text{ and } F(m, t) = x_1$$

for each $s \in [0, m]_{\mathbb{Z}}$ and each $t \in [0, m]_{\mathbb{Z}}$. We call F to be a digital path homotopy between f and f', and we write $f \simeq_{p(\kappa, \lambda)} f'$.

3. Digital Categories

Definition 3.1. A digital category is a quintuple $C = (O, M, dom, cod, \circ)$ where

(i) O is a class whose members are called $C - object$. $C - object$ are digital images.

(ii) M is a class whose members are called $C - morphisms$. $C - morphisms$ are digital functions which is defined between digital sets.

(iii) dom and cod are digital functions from M to O ($dom(f)$ is called the domain of f and $cod(f)$ is called the codomain of f)

(iv) $\circ$ is a function from

$$D = \{(f, g) \mid f, g \in M \text{ and } dom(f) = cod(g)\}$$

into M, called the composition law of C

such that the following conditions are satisfied:

(1) Matching Condition: If $f \circ g$ is defined, then $dom(f \circ g) = dom(g)$ and $cod(f \circ g) = cod(f)$;

(2) Associativity Condition: If $f \circ g$ and $h \circ f$ are defined, then $h \circ (f \circ g) = (h \circ f) \circ g$;

(3) Identity Existence Condition: For each $C - object$ A there exist $C - morphism$ e such that $dom(e) = A = cod(e)$ and

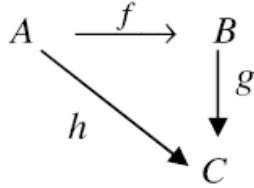**(a)** $f \circ e = f$ whenever $f \circ e$ is defined, and

(b) $e \circ g = g$ whenever $e \circ g$ is defined;

(4) Smallness of Morphism Class Condition: For any pair (A, B) of $\mathcal{C}-$ objects, the class
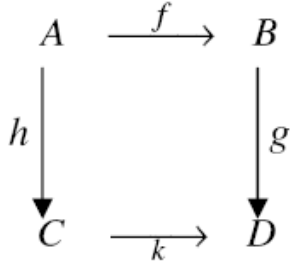
$= \{ f \mid f \in \mathcal{M}, \ dom(f) = A \ \text{and} \ cod(f) = B \}$

is a set.

Let A, B, C and D be digital images. We will use the notation $A \xrightarrow{f} B \xrightarrow{g} C$ to denote the composition $g \circ f$.

Thus the statement that the triangle

$$A \xrightarrow{\quad f \quad} B$$
$$\searrow h \qquad \downarrow g$$
$$C$$

is equivalent to the statement that

$A \xrightarrow{f} B \xrightarrow{g} C = A \xrightarrow{h} C$.   When morphisms f and g exist such that the above triangle commutes, we say that h factor through B. Similarly the statement that the square

$$A \xrightarrow{\quad f \quad} B$$
$$h \downarrow \qquad \qquad \downarrow g$$
$$C \xrightarrow{\quad k \quad} D$$

commutes means that $g \circ f = k \circ h$.

Proposition 3.1. Let $\mathcal{C}$ be a digital category and A is a $\mathcal{C}-$object such that A is a digital image. Then there exist exactly one $\mathcal{C}-$morphism $e : A \to A$ satisfiying the properties 3(a) and 3(b) of Definition 3.1; i.e. such that

(a) $f \circ e = f$ whenever $f \circ e$ is defined, and

(b) $e \circ g = g$ whenever $e \circ g$ is defined;

Proof: Suppose that each of e and $\tilde{e}$ is such a morphism. Then by (a) $\tilde{e} \circ e = \tilde{e}$ and by (b) $\tilde{e} \circ e = e$; hence, $e = \tilde{e}$.

Definition 3.2. For each object A of the digital category $\mathcal{C}$, the unique $\mathcal{C}-$morphism $e : A \to A$ satishfiying (a) and (b) above is denoted by 1_A and is called the $\mathcal{C}-$identity of A.

Definition 3.3. A digital category $\mathcal{C}$ is said to be:

(1) Small provided that $\mathcal{C}$ is a digital set;

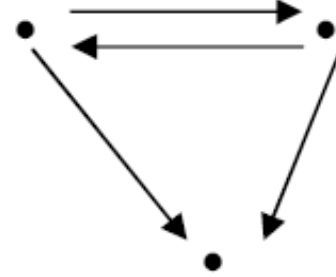(2) Discrete provided that all of its morphisms are identities;

(3) Connected that for each pair (A, B) of $\mathcal{C}-$objects,

$\hom_\mathcal{C}(A, B) \neq \varnothing$.

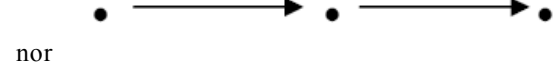**Example 3.1.** Let a diagram that shows all of the objects as dots and the non-identity morphisms as arrows. Thus

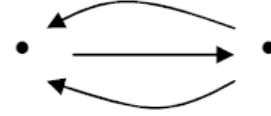
and

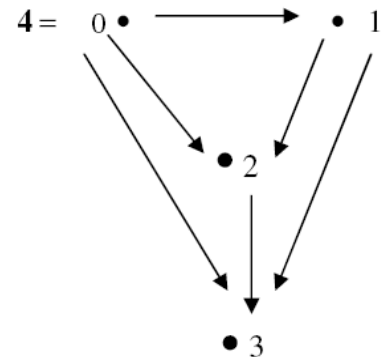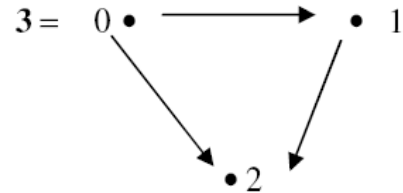can be considered to be digital categories, but neither

nor

can be digital categories.

Definition 3.4. For each natural number n, the set $\{0, 1, 2, ..., n-1\}$ supplied with the usual order can be considered to be digital category **n**, thus we have the special small digital categories:

0 = The empty category

1 = •

2 = 0 $\bullet \to \bullet$ 1

Boxer defined the path homotopy in [5]. We can consider equivalence classes of digital paths (path homotopy classes)

because digital path homotopy relation is an equivalence relation.

Example 3.2. Given a digital image X and points $x, y \in X$ a digital $\kappa-$ path from x to y is a $(2-\kappa)-$continuous mapping f from some digital set $[0, m]_{\mathbb{Z}}$ to X with $f(0) = x$ and $f(m) = y.$ If

$f : [0, m_1]_{\mathbb{Z}} \to X$ is a digital path from x to y and $g : [0, m_2]_{\mathbb{Z}} \to X$ is a digital path from y to z, there is a path $gf : [0, m_1 + m_2] \to X$ defined by

$$gf(t) = \begin{cases} f(t) & t \in [0, m_1]_{\mathbb{Z}} \\ g(t - m_1) & t \in [m_1, m_1 + m_2]_{\mathbb{Z}} \end{cases}$$

from x to z. This makes X into a category, the digital path category of X.

Now given digital paths

$f : [0, m]_{\mathbb{Z}} \to X, g : [0, m]_{\mathbb{Z}} \to X$, both from x to y.

There is a digital continuous mapping $F : [0, m]_{\mathbb{Z}} \times [0, m]_{\mathbb{Z}} \to X$ in a digital image in $\mathbb{Z}^2$ such that

$$\begin{aligned} F(t, 0) &= f(t) \\ F(t, m) &= g(t) \\ F(t, m) &= x \quad s \in [0, m]_{\mathbb{Z}} \\ F(m, s) &= y \quad s \in [0, m]_{\mathbb{Z}} \end{aligned}$$

It easy to see that this is an equivalence relation. The quotient of the digital path category by this congruence relation is a category called the digital category of digital homotopy classes of digital paths in X.

Definition 3.5. Let $\mathcal{C}$ be a digital category and A, B digital digital sets. A digital morphism $f : A \to B$ in $\mathcal{C}$ said to be monic if it is left cancelable.

Theorem 3.1. Suppose that A_i, $i \in \{1,...,8\}$ be digital sets. Consider the following 'cube' of digital objects and digital morphisms in a given digital category $\mathcal{C}$:
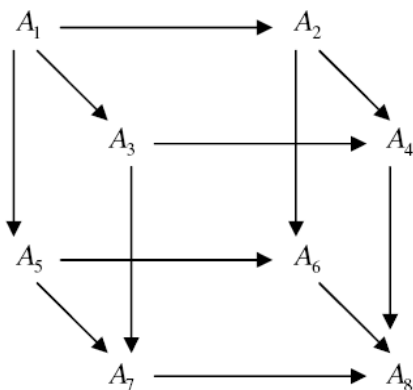
(i) Suppose that all faces except the top face are given to be commutative. If $A_4 \to A_8$ is monic, then the top face is also commutative.

(ii) Suppose that all faces except the bottom face are given to be commutative. If $A_1 \to A_5$ is epic, then the bottom face is also commutative.

Proof: Let $[i, j]$ denote the digital morphism from A_i to A_j. We have to prove that $[3, 4] \circ [1, 3] = [2, 4] \circ [1, 2]$. Now $[4, 8]$ is given to be monic and (writing $\circ$ as juxtaposition)

$$\begin{aligned} [4,8][3,4][1,3] &= [7,8][3,4][1,5] \\ &= [6,8][5,6][1,5] \\ &= [6,8][2,6][1,2] \\ &= [4,8][2,4][1,2] \end{aligned}$$

Then the result follows by left cancellation of $[4,8]$.

ii) We have to prove that $[7,8][5,7] = [6,8][5,6]. [1,5]$ is given to be epic and

$$\begin{aligned} [7,8][5,7][1,5] &= [4,8][2,4][1,2] \\ &= [7,8][3,7][1,3] \\ &= [6,8][5,6][1,5]. \end{aligned}$$

Then the result follows by right cancellation of $[1, 5]$.

Example 3.3. Given a digital image X and points $x, y \in X$ a digital κ path from x to y is a $(2-\kappa)-$continuous mapping f from some digital set $[0, m]_{\mathbb{Z}}$ to X with $f(0) = x$ and $f(m_1) = y$. If $f : [0, m_1]_{\mathbb{Z}} \to X$ is a digital path from x to y and $g : [0, m_2]_{\mathbb{Z}} \to Y$ is a digital path from y to z there is a path $gf : [0, m_1 + m_2] \to X$ defined by

$$gf(t) = \begin{cases} f(t) & t \in [0, m_1]_{\mathbb{Z}} \\ g(t - m_1) & t \in [m_1, m_1 + m_2]_{\mathbb{Z}} \end{cases}$$

from x to z. This makes X into a category the digital path category of X. Now given digital paths $f : [0, m_1]_{\mathbb{Z}} \to X, g : [0, m_2]_{\mathbb{Z}} \to Y$, both from x to y, one can define $f \sim_{(2,\kappa)} g$ if there is a continuous map $H : [0, m_1]_{\mathbb{Z}} \times [0, m_2]_{\mathbb{Z}} \to X$

in $\mathbb{Z}^2$ such that

$$\begin{aligned} H(s, 0) &= f(s) \\ H(s, m_2) &= g(s) \\ H(0, t) &= x \\ H(m_1, t) &= y. \end{aligned}$$

4. Conclusions

in this paper we construct the digital category model for digital images and gave some conditions made the diagram commutative.

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