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Applications of different Types of Lorenz Transformations

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Abstract The Lorentz transformation is well known. In this paper, we have presented various types of applications of different Lorenz Transformations according to the nature of movement of one inertial frame relative to the other inertial frame such as relativistic aberration, relativistic Doppler's effect and reflection of light ray by a moving mirror. When one frame moves along X- axis with respect to the rest frame then we can find these applications for special Lorentz transformation. When the motion of the moving frame is not along X-axis relative to the rest frame but the motion is along any arbitrary direction then we can find these formulae for most general Lorentz transformation. We can generate these formulae for different types of most general Lorentz transformations using mixed number, quaternion and geometric product.

Keywords Special Lorentz Transformation, Most General, Mixed Number, Quaternion, Geometric product, Relativistic Aberration, Relativistic Doppler's Effect, Reflection of Light Ray.

1. Introduction

1.1. Special Lorentz Transformation

Let us consider two inertial frames of reference S and S′, where the frame S is at rest and the frame S′ is moving along the X-axis with velocity v with respect to the S frame. The space and time co-ordinates of S and S′ are (x, y, z, t) and (x′, y′, z′, t′) respectively. The relation between the co-ordinates of S and S′ is called the special Lorentz transformation [1] which can be written as

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}},\ y' = y,\ z' = z \qquad (1)$$

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

and the inverse special Lorentz transformation[1] can be written as

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}, y = y', z = z' \qquad (2)$$

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

1.2. Most General Lorentz Transformation

When the motion of the moving frame is along any arbitrary direction instead of the X-axis, i.e., the velocity $\vec{V}$ has three components V_x, V_y and V_z, then the relation between the space and time co-ordinates of S and S' is called the most general Lorentz transformation[2] which can be written as

$$\begin{aligned} \vec{r}' &= \vec{r} + \vec{V}\left\{\frac{(\vec{r}.\vec{V})}{V^2}(\gamma - 1) - t\gamma\right\} \\ t' &= \gamma\{t - \vec{r}.\vec{V}\} \end{aligned} \qquad (3)$$

and the inverse most general Lorentz transformation[2] which can be written as

$$\begin{aligned} \vec{r} &= \vec{r}' + \vec{V}\left\{\frac{(\vec{r}'\vec{V})}{V'^2}(\gamma - 1) + t'\gamma\right\} \\ t &= \gamma\left(t' + \vec{r}'\vec{V}\right) \end{aligned} \qquad (4)$$

Where, $\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}}$ and $c = 1$

1.3. Mixed Number Lorentz Transformation

In the case of the most general Lorentz transformation, the velocity $\vec{V}$ of S' with respect to S is not along the X-axis; i.e., the velocity $\vec{V}$ has three components, V_x, V_y,

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and V_z. Let in this case $\vec{r}$ and $\vec{r}'$ be the space parts in S and S' frames, respectively. Then using the mixed product $\vec{A}\otimes\vec{B} = \vec{A}.\vec{B} + i\vec{A}\times\vec{B}$, the mixed number Lorentz transformations [3– 6] can be written as

$$\begin{aligned} t' &= \gamma(t - \vec{r}.\vec{V}) \\ \vec{r}' &= \gamma(\vec{r} - t\vec{V} - i\vec{r}\times\vec{V}) \end{aligned} \tag{5}$$

and the inverse mixed number Lorentz transformation[3– 6], can be written as

$$\begin{aligned} t &= \gamma(t' + \vec{r}'\vec{V}) \\ \vec{r} &= \gamma(\vec{r}' + t'\vec{V} + i\vec{r}'\times\vec{V}) \end{aligned} \tag{6}$$

Let, $r_x = x$, $r_y = y$, $r_z = z$, $r'_x = x'$, $r'_y = y'$, $r'_z = z'$. If V_x, V_y, and V_z denote the components of the velocity of the system S' relative to S then equation (5) and (6) can be written as

$$\begin{aligned} x' &= \gamma\{x - tv_x - i(yv_z - zv_y)\} \\ y' &= \gamma\{y - tv_y - i(zv_x - xv_z)\} \\ z' &= \gamma\{z - tv_z - i(xv_y - yv_x)\} \\ t' &= \gamma(t - xv_x - yv_y - zv_z) \end{aligned} \tag{7}$$

and

$$\begin{aligned} x &= \gamma\{x' + t'v_x + i(y'v_z - z'v_y)\} \\ y &= \gamma\{y' + t'v_y + i(z'v_x - x'v_z)\} \\ z &= \gamma\{z' + t'v_z + i(x'v_y - y'v_x)\} \\ t &= \gamma(t' + x'v_x + y'v_y + z'v_z) \end{aligned} \tag{8}$$

1.4. Quaternion Lorentz Transformation

In this case the velocity $\vec{V}$ of S' with respect to S has also three components, V_x, V_y, and V_z as the most general Lorentz transformation. Let in this case $\vec{r}$ and $\vec{r}'$ be the space parts in S and S' frames respectively. Then using the quaternion product $\underline{AB} = -\vec{A}.\vec{B} + \vec{A}\times\vec{B}$ the quaternion Lorentz transformation [7-10] can be written as

$$\begin{aligned} t' &= \gamma(t + \vec{r}.\vec{V}) \\ \vec{r}' &= \gamma(\vec{r} - t\vec{V} - \vec{r}\times\vec{V}) \end{aligned} \tag{9}$$

and the inverse quaternion Lorentz transformation[7-10] can be written as

$$\begin{aligned} t &= \gamma(t' - \vec{r}'\vec{V}) \\ \vec{r} &= \gamma(\vec{r}' + t'\vec{V} + \vec{r}'\times\vec{V}) \end{aligned} \tag{10}$$

Let, $r_x = x$, $r_y = y$, $r_z = z$, $r'_x = x'$, $r'_y = y'$, $r'_z = z'$. If V_x, V_y, and V_z denote the components of the velocity of the system S' relative to S then equation (9) and (10) can be written as

$$\begin{aligned} x' &= \gamma\{x - tv_x - (yv_z - zv_y)\} \\ y' &= \gamma\{y - tv_y - (zv_x - xv_z)\} \\ z' &= \gamma\{z - tv_z - (xv_y - yv_x)\} \\ t' &= \gamma(t + xv_x + yv_y + zv_z) \end{aligned} \tag{11}$$

and

$$\begin{aligned} x &= \gamma\{x' + t'v_x + (y'v_z - z'v_y)\} \\ y &= \gamma\{y' + t'v_y + (z'v_x - x'v_z)\} \\ z &= \gamma\{z' + t'v_z + (x'v_y - y'v_x)\} \\ t &= \gamma(t' - x'v_x - y'v_y - z'v_z) \end{aligned} \tag{12}$$

1.5. Geometric Product Lorentz Transformation

In this case the velocity $\vec{V}$ of S' with respect to S has also three components, V_x, V_y, and V_z as the Most general Lorentz transformation. Let in this case $\vec{r}$ and $\vec{r}'$ be the space parts in S and S' frames respectively. Then using the geometric product of two vectors $\vec{A}\vec{B} = \vec{A}.\vec{B} + \vec{A}\times\vec{B}$ the geometric product Lorentz transformation [11, 12] can be written as

$$\begin{aligned} t' &= \gamma(t - \vec{r}.\vec{V}) \\ \vec{r}' &= \gamma(\vec{r} - t\vec{V} - \vec{r}\times\vec{V}) \end{aligned} \tag{13}$$

and the inverse geometric product Lorentz transformation, [11, 12] can be written as

$$\begin{aligned} t &= \gamma(t' + \vec{r}'\vec{V}) \\ \vec{r} &= \gamma(\vec{r}' + t'\vec{V} + \vec{r}'\times\vec{V}) \end{aligned} \tag{14}$$

Let, $r_x = x$, $r_y = y$, $r_z = z$, $r'_x = x'$, $r'_y = y'$, $r'_z = z'$. If V_x, V_y, and V_z denote the components of the velocity of the system S' relative to S then equation (13) and (14) can be written as

$$\begin{aligned} x' &= \gamma\{x - tv_x - (yv_z - zv_y)\} \\ y' &= \gamma\{y - tv_y - (zv_x - xv_z)\} \\ z' &= \gamma\{z - tv_z - (xv_y - yv_x)\} \\ t' &= \gamma(t - xv_x - yv_y - zv_z) \end{aligned} \tag{15}$$

and

$$\begin{aligned} x &= \gamma\{x' + t'v_x + (y'v_z - z'v_y)\} \\ y &= \gamma\{y' + t'v_y + (z'v_x - x'v_z)\} \\ z &= \gamma\{z' + t'v_z + (x'v_y - y'v_x)\} \\ t &= \gamma(t' + x'v_x + y'v_y + z'v_z) \end{aligned} \tag{16}$$

Now we are going to derive the formulae for relativistic aberration, relativistic Doppler's effect and reflection of light ray by a moving mirror of different types of lorentz transformations respectively.

2. Aberration

The speed of light is independent of the medium of transmission but the direction of light rays depends on the motion of the source emitting light and observer [10]

2.1. Relativistic Aberration of Special Lorentz Transformation

The earth moves round the sun in its orbit. Consider sun is to be the system S and the earth is in the system S' is moving with velocity V relative to the system S along positive direction of common X-axis. Let the star P observed from the observers O and O' in system S and S' where the frame S' is moving along X-axis with velocity V with respect to S frame. Let the angles made by the light ray in X-Y plane from the star P at any instant in two systems at O and O' be φ $(\varphi = \angle POX)$ and φ' $(\varphi' = \angle PO'X')$ respectively [Figure 1].

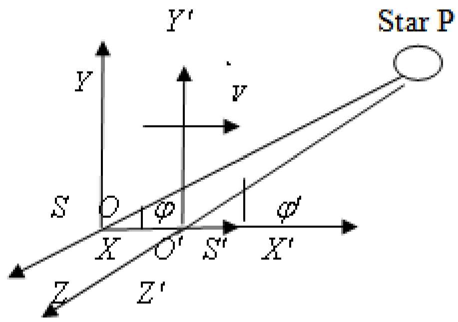

Figure 1. Direction of light rays observed from a fixed frame S and a moving frame S' (the frame S' is moving along common X-axis with velocity V with respect to S frame)

It can be shown that [13]

$$\tan\varphi' = \frac{\tan\varphi\sqrt{1-\beta^2}}{1-\beta\sec\varphi}$$

where

$$\beta = v/c \qquad (17)$$

and

$$\tan\varphi = \frac{\tan\varphi'\sqrt{1-\beta^2}}{1-\beta\sec\varphi'} \qquad (18)$$

It is clear from equation (17) and (18), φ and φ' are not same in the two systems; they depend upon the motion of source and observer. Hence equation (17) describes the relativistic aberration of special Lorentz transformation.

2.2. Relativistic Aberration of Most General Lorentz Transformation

Consider the light from a star P observed by the observers O and O' in system S and S' where the frame S' is moving along any arbitrary direction with velocity $\vec{V}$ with respect to S frame as shown in figure -2. Let the angles made by the light ray in X-Y plane from the star P at any instant in two systems at O and O' be φ $(\varphi = \angle POX)$ and φ' $(\varphi' = \angle PO'X')$ respectively. Here the angle φ is same in Fig. 1 and 2. But the angle φ' is different Fixed Moving

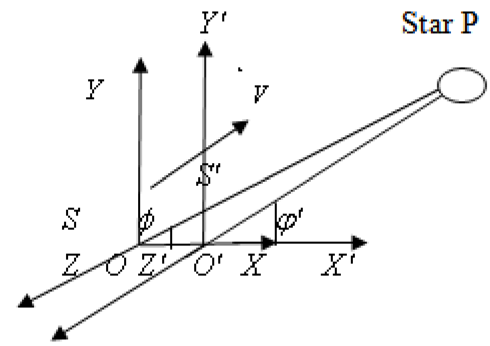

Figure 2. Direction of light rays observed from a fixed frame S and a moving frame S' (the frame S' is moving along any arbitrary direction with velocity V with respect to S frame)

It also can be shown that for two dimension case, the Relativistic aberration of most general Lorentz transformation is clear which has been described [10] by the following relationship

$$\tan\varphi' = \frac{(\gamma-1)\frac{v_y v_x}{v^2} + \left\{1+(\gamma-1)\frac{v_y^2}{v^2}\right\}\tan\varphi + v_y\gamma\sec\varphi}{\left\{1+(\gamma-1)\frac{v_x^2}{v^2}\right\} + (\gamma-1)\frac{v_y v_x}{v^2}\tan\varphi + v_x\sec\varphi} \qquad (19)$$

From equation (19), we have φ and φ' are not same in the two systems. i.e. the direction of light rays depends on the motion of the source emitting light and observer. Equation (19) gives the relativistic formula for aberration of most general Lorentz transformation.

2.3. Relativistic Aberration of Mixed Number Lorentz Transformation

Consider the same situation as described in Figure 2. Let, $r_x = x,\ \ r_y = y,\ r_z = z,\ r'_x = x',$ $r'_y = y',\ r'_z = z'$,then using equation (5) and (6) we can easily find out the aberration formula for two dimension case of the mixed number Lorentz transformation[10]

$$\tan\varphi' = \frac{\tan\varphi + v_y\sec\varphi}{1+v_x\sec\varphi} \qquad (20)$$

From equation (20), we have φ and φ' are not same in the two systems. i.e. the direction of light rays depends on the motion of the source emitting light and observer. Equation (20) gives the relativistic formula for aberration of mixed number Lorentz transformation.

2.4. Relativistic Aberration of Quaternion Lorentz Transformation

Consider the same situation as described in Figure 2. Let,

$$r_x = x,\quad r_y = y,\quad r_z = z,\quad r'_x = x,\quad r'_y = y,\ r'_z = z'$$

then using equation (9) and (10) we can easily find out the aberration formula for two dimension case of the quaternion Lorentz transformation [14]

$$\tan\varphi' = \frac{\tan\varphi + v_y \sec\varphi}{1 + v_x \sec\varphi} \tag{21}$$

2.5. Relativistic Aberration of Geometric Product Lorentz Transformation

Consider the same situation as described in figure 2. Let,

$$r_x = x,\quad r_y = y,\quad r_z = z,\quad r'_x = x,\quad r'_y = y,\ r'_z = z'$$

Equation (13) can be written as

$$\begin{aligned} x' &= \gamma\{x - tv_x - (yv_z - zv_y)\} \\ y' &= \gamma\{y - tv_y - (zv_x - xv_z)\} \\ z' &= \gamma\{z - tv_z - (xv_y - yv_x)\} \\ t' &= \gamma(t - xv_x - yv_y - zv_z) \end{aligned} \tag{22}$$

Differentiating both sides of equation (22) we get

$$\begin{aligned} dx' &= \gamma\{dx - v_x dt - (v_z dy - v_y dz)\} \\ dy' &= \gamma\{dy - v_y dt - (v_x dz - v_z dx)\} \\ dt' &= \gamma(dt - v_x dx - v_y dy - v_z dz \end{aligned}$$

from the above equation we have

$$\frac{dx'}{dt'} = \frac{\gamma\{dx - v_x dt - (v_z dy - v_y dz)\}}{\gamma(dt - v_x dx - v_y dy - v_z dz)}$$

$$\text{or,}\quad u'_x = \frac{\left\{\frac{dx}{dt} - v_x - \left(v_z \frac{dy}{dt} - v_y \frac{dz}{dt}\right)\right\}}{1 - v_x \frac{dx}{dt} - v_y \frac{dy}{dt} - v_z \frac{dz}{dt}}$$

$$\text{or,}\quad u'_x = \frac{\{u_x - v_x - (v_z u_y - v_y u_z)\}}{1 - v_x u_x - v_y u_y - v_z u_z} \tag{23}$$

and

$$\frac{dy'}{dt'} = \frac{\gamma\{dy - v_y dt - (v_x dz - v_z dx)\}}{\gamma(dt - v_x dx - v_y dy - v_z dz}$$

$$\text{or,}\quad u'_y = \frac{\left\{\frac{dy}{dt} - v_y - \left(v_x \frac{dz}{dt} - v_z \frac{dx}{dt}\right)\right\}}{1 - v_x \frac{dx}{dt} - v_y \frac{dy}{dt} - v_z \frac{dz}{dt}}$$

$$\text{or,}\quad u'_y = \frac{\{u_y - v_y - (u_z v_x - v_z u_x)\}}{1 - v_x u_x - v_y u_y - v_z u_z} \tag{24}$$

Dividing equation (23) and (24) we get

$$\frac{u'_y}{u'_x} = \frac{\{u_y - v_y - (u_z v_x - v_z u_x)\}}{\{u_x - v_x - (v_z u_y - v_y u_z)\}}$$

If we consider two dimension case, then we can write

$$\frac{u'_y}{u'_x} = \frac{u_y - v_y}{u_x - v_x} \tag{25}$$

In this case the star light travelling in x-y plane with velocity c, has component $c\cos(\pi+\varphi)$ and $c\cos(\pi+\varphi')$ along positive direction of X-axis in system S and S' respectively. Also those $c\sin(\pi+\varphi)$ and $c\sin(\pi+\varphi')$ are along positive direction of Y-axis in system S and S' respectively. Thus we have

$$\begin{aligned} u_x &= c\cos(\pi+\varphi) = -c\cos\varphi \\ u_y &= c\sin(\pi+\varphi) = -c\sin\varphi \\ u'_x &= c\cos(\pi+\varphi') = -c\cos\varphi' \\ u'_y &= c\sin(\pi+\varphi') = -c\sin\varphi' \end{aligned} \tag{26}$$

From Equation (25) and (26) we have

$$\frac{-c\sin\varphi'}{-c\cos\varphi'} = \frac{-c\sin\varphi - v_y}{-c\cos\varphi - v_x}$$

$$\text{or,}\quad \tan\varphi' = \frac{c\sin\varphi + v_y}{c\cos\varphi + v_x}$$

Considering $c = 1$ we have

$$\tan\varphi' = \frac{\sin\varphi + v_y}{\cos\varphi + v_x}$$

$$\text{or,}\quad \tan\varphi' = \frac{\tan\varphi + v_y \sec\varphi}{1 + v_x \sec\varphi} \tag{27}$$

From equation (27), we have φ and φ' are not same in the two systems. i.e. the direction of light rays depends on the motion of the source emitting light and observer. Equation (27) gives the relativistic formula for aberration of geometric product Lorentz transformation.

3. Relativistic Doppler's Effect

The Doppler's effect [15] is an apparent change in frequency of a wave that results from the motion of source or observer or both. The relativistic Doppler's effect [16] is the change in frequency and wavelength of light, caused by the relative motion of the source and observer such as in the regular Doppler's effect, when taking into account effects of special theory of relativity. The relativistic Doppler's effect is different from the true (non – relativistic) Doppler's effect as the equations include the time dilation effect of the special relativity. They described the total difference in the frequencies and possess the Lorentz symmetry.

We can analyse the Doppler's effect [17] of light by considering a light source as a clock that ticks ν_0 times per second and emits a wave of light with each tick. In the case of observer receding from the light source, the observer travels the distance Vt away from the source between the ticks, which mean that the light wave from a given tick takes $\frac{Vt}{c}$ longer to reach him than the previous one. Hence the total time between the arrivals of successive waves is $T = t + \frac{Vt}{c} = t_0\sqrt{\frac{1+\frac{V}{c}}{1-\frac{V}{c}}}$ and the observed frequency is

$$\nu = \nu_0\sqrt{\frac{1-\frac{V}{c}}{1+\frac{V}{c}}}$$

In the case of observer approaching the light source, the observed frequency is

$$\nu = \nu_0\sqrt{\frac{1+\frac{V}{c}}{1-\frac{V}{c}}} \tag{28}$$

3.1. Relativistic Doppler's Effect of Special Lorentz Transformation

Consider two frames of references S and S' where S' is moving along common X-axis with velocity V with respect to S frame. Let the transmitter and receiver be situated at origins O and O' of frames S and S' respectively. Let two light signals or pulses be transmitted at time $t = 0$ and $t = T$, T being the true period of light pulses. Let $\Delta t'$ be the interval between the receptions of the pulses by the receiver in the frame S'. Since the observer and receiver is at rest in frame S', $\Delta t'$ is the proper time interval T' between these pulses. Since the observer continues to be at O' all the time, the distance $\Delta r'$ covered by him in the frame S' during the reception of the two pulses is zero.

Using the inverse Lorentz transformation we can write,

$$\Delta x = \frac{VT'}{\sqrt{(1-V^2)}} \;\&\; \Delta t = \frac{T'}{\sqrt{(1-V^2)}} \tag{29}$$

$$\text{But} \quad \Delta t = T + \Delta x \quad \text{where} \quad c = 1 \tag{30}$$

Using the equations (29) and (30) we can write [13]

$$\nu' = \nu\sqrt{\frac{1-\vec{V}}{1+\vec{V}}} \tag{31}$$

If ν and ν' be the actual and observed frequencies of light pulses, respectively .This equation (31) is the formula of the relativistic Doppler's effect of Special Lorentz transformation for light.

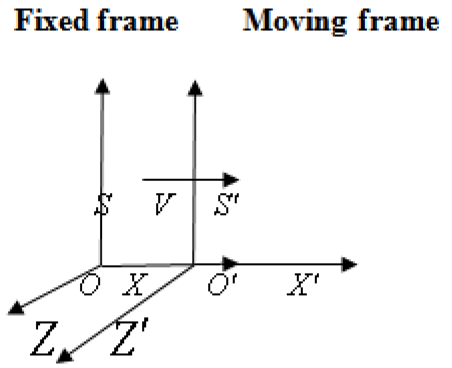

Figure 3. The frame S' is moving along common X-axis with velocity V with respect to S frame

3.2. Relativistic Doppler's Effect of most General Lorentz Transformation

Consider two frames of references S and S' the velocity $\vec{V}$ of S' with respect to S is not along X-axis i.e. the velocity $\vec{V}$ has three components V_x, V_y and V_z. Let the transmitter and receiver be situated at origins O and O' of frames S and S' respectively. Let two light signals or pulses be transmitted at time $t = 0$ and $t = T$, T being the true period of light pulses. Let $\Delta t'$ be the interval between the receptions of the pulses by the receiver in the frame S'. Since the observer and receiver is at rest in frame S', $\Delta t'$ is the proper time interval T' between these pulses. Since the observer continues to be at O' all the time, the distance $\Delta r'$ covered by him in the frame S' during the reception of the two pulses is zero.

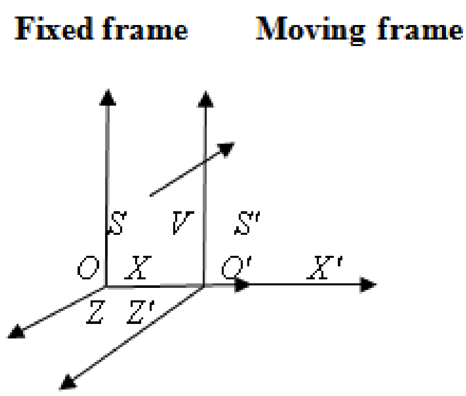

Figure 4. The frame S' is moving along any arbitrary direction with velocity $\vec{V}$ with respect to S frame

Using the space part and time part of inverse most general Lorentz transformation [2] any one can get the formula of relativistic Doppler's effect [18] in the most general Lorentz transformation which is

$$\nu' = \nu\sqrt{\frac{1-\vec{V}}{1+\vec{V}}} \tag{32}$$

This formula of the relativistic Doppler's effect coincides with the formula of Special Lorentz transformation

3.3. Relativistic Doppler's Effect of Mixed Number Lorentz Transformation

Consider the same situation described in Figure 4. Using the similar way of most general Lorentz transformation [2] we can find the formula of Relativistic Doppler's Effect of mixed number Lorentz transformation [18] which is given by

$$\nu' = \nu\sqrt{\frac{1-\vec{V}}{1+\vec{V}}} \tag{33}$$

3.4. Relativistic Doppler's Effect of Quaternion Lorentz Transformation

Consider two frames of references S and S' as described in Figure 4. If ν and ν' be the actual and observed frequencies of light pulses, respectively, then the formula for the relativistic Doppler's Effect of quaternion Lorentz transformation [14] can be found as

$$\nu' = \gamma\nu\left(1-\vec{V}\right) \tag{34}$$

3.5. Relativistic Doppler's Effect of Geometric Product Lorentz Transformation

Consider two frames of references S and S'. The velocity $\vec{V}$ of S' with respect to S is not along X-axis i.e. the velocity $\vec{V}$ has three components V_x, V_y and V_z. Let the transmitter and receiver be situated at origins O and O' of frames S and S' respectively. Let two light signals or pulses be transmitted at time $t = 0$ and $t = T$, T being the true period of light pulses. Let $\Delta t'$ be the interval between the receptions of the pulses by the receiver in the frame S'. Since the observer and receiver is at rest in frame S', $\Delta t'$ is the proper time interval T' between these pulses. Since the observer continues to be at O' all the time, the distance $\Delta r'$ covered by him in the frame S' during the reception of the two pulses is zero.

Using the space part of the inverse geometric product Lorentz transformation of equation (13)

$$\Delta\vec{r} = \gamma\left(\Delta r' + \Delta t'\vec{V} + i\Delta r' \times \vec{V}\right) \tag{35}$$

Since $\Delta\vec{r}' = 0$ and $\Delta t' = T'$ then equation (35) can be written as

$$\Delta\vec{r} = \gamma\,\Delta t'\vec{V}$$

or,

$$\Delta\vec{r} = \gamma T'\vec{V} \tag{36}$$

This equation (36) shows the second pulse has to travel this extra distance $\Delta\vec{r}$ then the first pulse in the frame S to be able to reach at origin O' in the moving frame S'.

Using the time part of the geometric product Lorentz transformation of equation (13)

$$\Delta t = \gamma\left(\Delta t' + \Delta r'.\vec{V}\right) \tag{37}$$

Since $\Delta\vec{r}' = 0$ and $\Delta t' = T'$ then equation (37) can be written as

$$\Delta t = \gamma T' \tag{38}$$

This relation includes both the actual time period T of the pulses and the time taken $\frac{\Delta\vec{r}}{c}$ by the second pulse to cover the extra distance $\Delta\vec{r}$ in the frame S, i.e.

$$\Delta t = T + \frac{\Delta\vec{r}}{c} \tag{39}$$

Using $c = 1$ the equation (39) can be written as

$$\Delta t = T + \Delta\vec{r} \tag{40}$$

Using (36), (38) and (40) we can write

$$T'\gamma = T + \vec{V}T'\gamma$$

$$\text{or, } T = T'\sqrt{\frac{\left(1-\vec{V}\right)^2}{\left(1-V^2\right)}} \tag{41}$$

Using equation (23), the equation (41) can be written as

$$T = T'\sqrt{\frac{1-\vec{V}}{1+\vec{V}}} \tag{42}$$

If ν and ν' be the actual and observed frequencies of light pulses, respectively, we have $\nu = \frac{1}{T}$ and $\nu' = \frac{1}{T'}$ then equation (42) gives

$$\nu' = \nu\sqrt{\frac{1-\vec{V}}{1+\vec{V}}} \tag{43}$$

this equation (43) is the formula of the relativistic Doppler's effect of geometric product Lorentz transformation for light.

4. Reflection of Light by a moving Mirror

4.1. Reflection of light by a moving mirror in Special Lorentz Transformation

Consider two frames of references S and S' where S' is moving along common X-axis with velocity V with respect to S frame. Consider a mirror M to be fixed in the $Y'-Z'$ plane of system S'. Consider the mirror is perfectly reflecting, moving along the direction of its normal relative to S'. Let a ray of light in $X'-Y'$ be incident at angle ϕ' at O' in system S'. As mirror M is stationary in system S' ordinary laws of reflection hold good and so the angle of reflection will be ϕ' and the reflected ray lies in $X'-Y'$ plane. Let the angles of incidence and reflection are measured in system S be ϕ_1 & ϕ_2, we get [2]

$$\frac{\sin\phi_1}{\cos\phi_1+\beta}=\frac{\sin\phi_2\sqrt{1-\beta^2}}{(\cos\phi_2-\beta)} \tag{44}$$

Using $\gamma=\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}=\frac{1}{\sqrt{1-\beta^2}}, c=1$ & $\beta=\frac{V}{c}$

we get from (44)

$$\frac{\sin\phi_1}{\cos\phi_1+V}=\frac{\sin\phi_2}{\gamma(\cos\phi_2-V)} \tag{45}$$

This equation (45) is the law of reflection of light by a moving mirror [2] in the case of special Lorentz transformation

4.2. Reflection of Light by a moving Mirror in Most General Lorentz Transformation

Consider two frames of references S and S'. The velocity $\vec{V}$ of S' with respect to S is not along X-axis i.e. the velocity $\vec{V}$ has two components V_x, and V_y. Consider a mirror M to be fixed in the $Y'-Z'$ plane of system S'. Consider the mirror is perfectly reflecting, moving along the direction of its normal relative to S'. Let a ray of light in $X'-Y'$ be incident at angle ϕ' at O' in system S'.

As mirror M is stationary in system S' ordinary laws of reflection hold good and so the angle of reflection will be ϕ' and the reflection ray lies in $X'-Y'$ plane.

Let the angles of incidence and reflection are measured in system S be ϕ_1 & ϕ_2, we get [3]

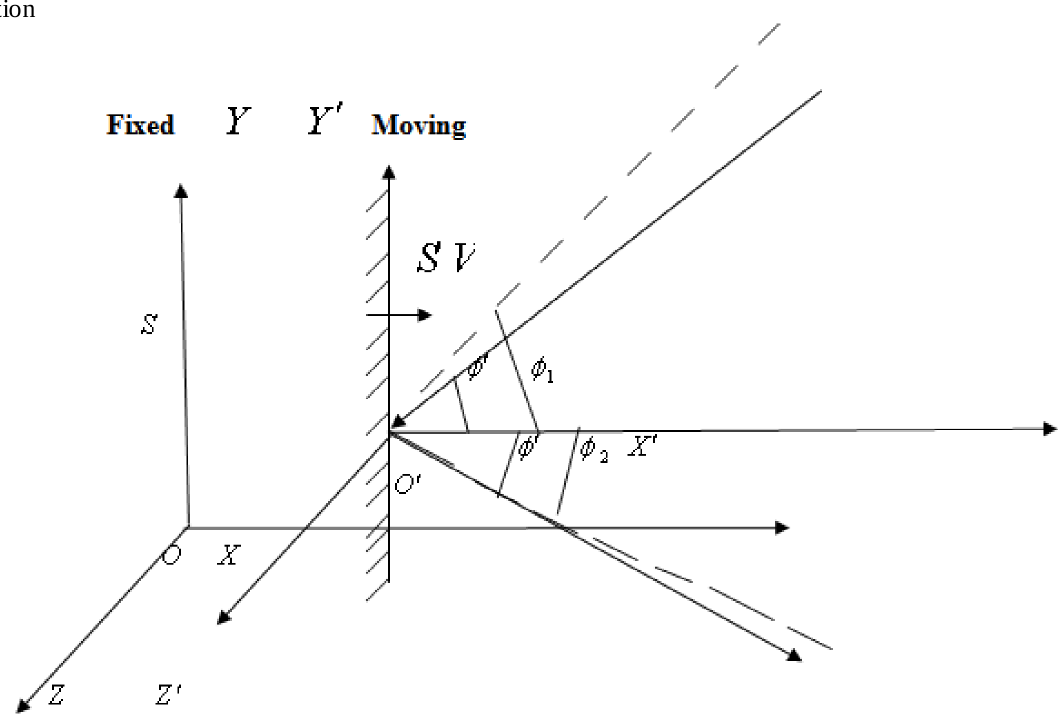

Figure 5. The reflection of light by a moving mirror where M is fixed in the $Y'-Z'$ plane of system S'

Let the incident ray in system S and S' be represented by ψ and ψ' where

$$\psi=\frac{A}{\gamma}\exp\left[2\pi i\nu\left\{t-\frac{x\cos(\pi+\phi_1)+y\sin(\pi+\phi_1)}{c}\right\}\right] \tag{46}$$

and

$$\psi'=\frac{A'}{\gamma}\exp\left[2\pi i\nu'\left\{t'-\frac{x'\cos(\pi+\phi')+y'\sin(\pi+\phi')}{c}\right\}\right] \tag{47}$$

As phase is a Lorenz invariant quantity, using equations (46) and (47) we must have

$$\nu\left\{t-\frac{x\cos(\pi+\phi_1)+y\sin(\pi+\phi_1)}{c}\right\}=\nu'\left\{t'-\frac{x'\cos(\pi+\phi')+y'\sin(\pi+\phi')}{c}\right\}$$

$$\text{or, }\ \nu\left\{t+\frac{x\cos\phi_1+y\sin\phi_1}{c}\right\}=\nu'\left\{t'+\frac{x'\cos\phi'+y'\sin\phi'}{c}\right\} \tag{48}$$

Using $c=1$ we get from equation (48)

$$\nu\{t+x\cos\phi_1+y\sin\phi\}=\nu'\{t'+x'\cos\phi'+y'\sin\phi'\} \tag{49}$$

Using equations (3), (4) and (49) and considering the velocity V has two components V_x and V_y, we can get the formula [19] of the reflection of light ray by a moving mirror in most general Lorentz transformation

$$\frac{\gamma V_yV^2+\sin\phi_1V^2+(\gamma-1)V_xV_y\cos\phi_1+(\gamma-1)V_y^2\sin\phi_1}{\gamma V_xV^2+\cos\phi_1V^2+(\gamma-1)V_xV_y\sin\phi_1+(\gamma-1)V_x^2\cos\phi_1}$$
$$=\frac{-\gamma V_yV^2-\sin\phi_2V^2+(\gamma-1)V_xV_y\cos\phi_2-(\gamma-1)V_y^2\sin\phi_2}{\gamma V_xV^2+\cos\phi_2V^2-(\gamma-1)V_xV_y\sin\phi_2+(\gamma-1)V_x^2\cos\phi_2}$$

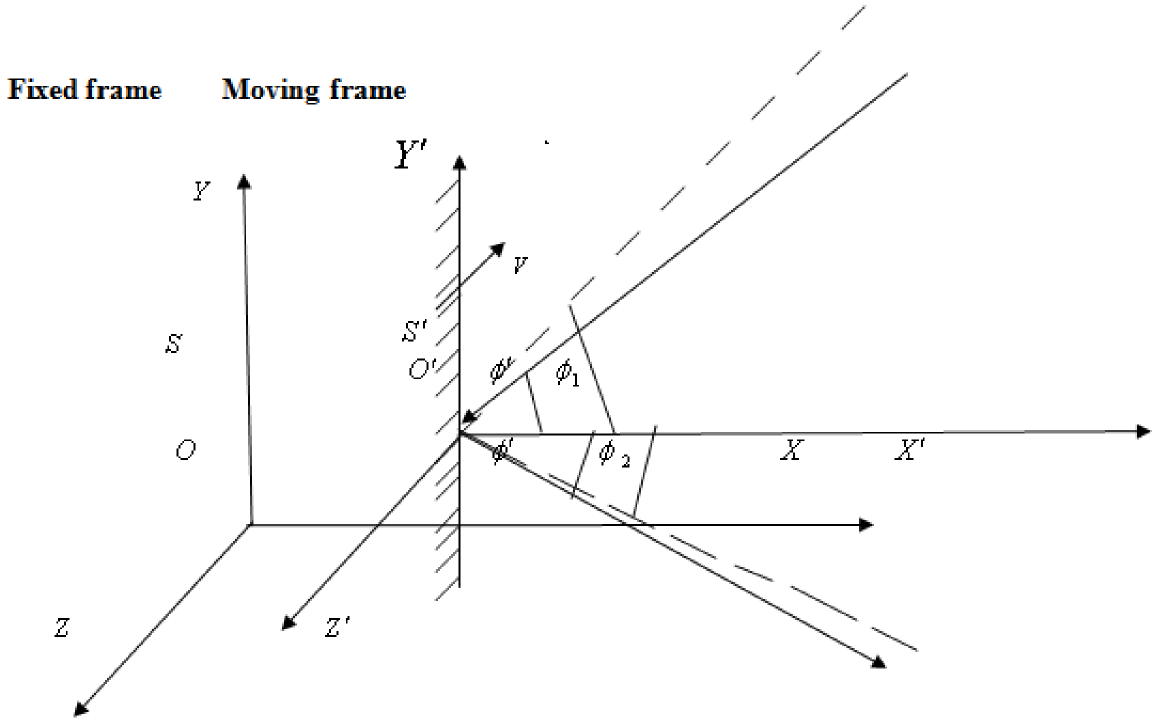

Figure 6. The frame S' is moving along any arbitrary direction with velocity $\vec{V}$ with respect to S frame where a mirror M is fixed in the $Y'-Z'$ plane of system S'

4.3. Reflection of Light by a moving Mirror in mixed number Lorentz Transformation

Consider two frames of reference S and S'. The velocity $\vec{V}$ of S' with respect to S is not along X-axis i.e. the velocity $\vec{V}$ has three components V_x, V_y and V_z. Consider same situation as described in the Figure 6.Using equation (7) and (49) we have considering the velocity V has two components V_x and V_y

$$\nu\{t+x\cos\phi_1+y\sin\phi\}=\nu'\gamma t-\nu'\gamma\{xV_x-yV_y+(x-tV_x)\cos\phi'+(y-tV_y)\sin\phi'\} \tag{50}$$

$$[\because V_z=0\ \ \&\ z=0]$$

Equating the coefficient of t of equation (50) we get,

$$\nu = \nu'\gamma'\left(1 - V_x \cos\phi' - V_y \sin\phi'\right) \tag{51}$$

Equating the coefficient of x of equation (50) we get,

$$\nu \cos\phi_1 = \nu'\gamma'\left(\cos\phi' - V_x\right)$$

or,

$$\nu'\gamma'\left(1 - V_x \cos\phi' - V_y \sin\phi'\right)\cos\phi_1 = \nu'\gamma'\left(\cos\phi' - V_x\right)$$

or, $$\cos\phi_1 = \frac{\left(\cos\phi' - V_x\right)}{\left(1 - V_x \cos\phi' - V_y \sin\phi'\right)} \tag{52}$$

Equating the coefficient of y of equation (50) we get,

$$\nu \sin\phi_1 = \nu'\gamma'\left(\sin\phi' - V_y\right)$$

or,

$$\nu'\gamma'\left(1 - V_x \cos\phi' - V_y \sin\phi'\right)\sin\phi_1 = \nu'\gamma'\left(\sin\phi' - V_y\right)$$

or, $$\sin\phi_1 = \frac{\left(\sin\phi' - V_y\right)}{\left(1 - V_x \cos\phi' - V_y \sin\phi'\right)} \tag{53}$$

Now from equation (52) and (53) we get,

$$\tan\phi_1 = \frac{\sin\phi' - V_y}{\cos\phi' - V_x} \tag{54}$$

Using equation (8) and (49) we have considering the velocity V has two components V_x and V_y

$$\begin{aligned}&\nu\gamma\left\{t' + x'V_x + y'V_y + \left(x' + t'V_x\right)\cos\phi_1 + \left(y' + t'V_y\right)\sin\phi_1\right\}\\ &= \nu'\left\{t' + x'\cos\phi' + y'\sin\phi'\right\}\end{aligned} \tag{55}$$

Equating the coefficient of t' of equation (55) we get,

$$\nu\gamma\left(1 + V_x \cos\phi_1 + V_y \sin\phi_1\right) = \nu' \tag{56}$$

Equating the coefficient of x' of equation (55) we get,

$$\nu\gamma\left(V_x + \cos\phi_1\right) = \nu'\cos\phi'$$

or,

$$\nu\gamma\left(V_x + \cos\phi_1\right) = \nu\gamma\left(1 + V_x \cos\phi_1 + V_y \sin\phi_1\right)\cos\phi'$$

or, $$\cos\phi' = \frac{V_x + \cos\phi_1}{1 + V_x \cos\phi_1 + V_y \sin\phi_1} \tag{57}$$

Equating the coefficient of y' of equation (55) we get,

$$\nu\gamma\left(V_y + \sin\phi_1\right) = \nu'\sin\phi'$$

or,

$$\nu\gamma\left(V_y + \sin\phi_1\right) = \nu\gamma\left(1 + V_x \cos\phi_1 + V_y \sin\phi_1\right)\sin\phi'$$

or, $$\sin\phi' = \frac{V_y +_y \sin\phi_1}{1 + V_x \cos\phi_1 + V_y \sin\phi_1} \tag{58}$$

Now from equation (57) and (58) we get,

$$\tan\phi' = \frac{V_y + \sin\phi_1}{V_x + \cos\phi_1} \tag{59}$$

Similarly, for reflected ray the relation between angle of reflection in system S and S' can be obtained as

$$\tan\phi' = \frac{-V_y - \sin\phi_2}{V_x + \cos\phi_2} \tag{60}$$

From equation (59) and (60), we get,

$$\frac{V_y + \sin\phi_1}{V_x + \cos\phi_1} = \frac{-V_y - \sin\phi_2}{V_x + \cos\phi_2} \tag{61}$$

This equation (61) is the law of reflection of light ray by a moving mirror in mixed number Lorentz transformation.

4.4. Reflection of Light by a moving Mirror in Quaternion Lorentz Transformation

Consider two frames of reference S and S'. The velocity $\vec{V}$ of S' with respect to S is not along X-axis i.e. the velocity $\vec{V}$ has three components V_x, V_y and V_z. Consider same situation as described in the Figure 6.

Using equation (11) and (49) we have considering the velocity V has two components V_x and V_y we can get by the similar process of mixed number Lorentz transformation, the same formula for the reflection of light ray by a moving mirror in quaternion Lorentz transformation

$$\frac{V_y + \sin\phi_1}{V_x + \cos\phi_1} = \frac{-V_y - \sin\phi_2}{V_x + \cos\phi_2} \tag{62}$$

4.5. Reflection of Light by a moving Mirror in Geometric Product Lorentz Transformation

Consider two frames of reference S and S'. The velocity $\vec{V}$ of S' with respect to S is not along X-axis i.e. the velocity $\vec{V}$ has three components V_x, V_y and V_z. Consider same situation as described in the Figure 6.

Using equation (15) and (49), we have considering the velocity V has two components V_x and V_y then after a little bit lengthy calculation according to the similar process of mixed number Lorentz transformation we can get the same formula for the reflection of light ray by a moving mirror in geometric product Lorentz transformation

$$\frac{V_y + \sin\phi_1}{V_x + \cos\phi_1} = \frac{-V_y - \sin\phi_2}{V_x + \cos\phi_2} \tag{63}$$

5. Comparison of the Study

5.1. Comparison of Relativistic Aberrations of Special, Most General, Mixed Number, Quaternion and Geometric Product Lorenz Transformations

Names of Lorentz transformations	Relativistic aberration formula
Special Lorentz transformation	$\tan\varphi' = \dfrac{\tan\varphi\sqrt{1-\beta^2}}{1-\beta\sec\varphi}$
Most general Lorentz transformation	$\tan\varphi' = \dfrac{(\gamma-1)\,v_y v_x/v^2 + \left\{1+(\gamma-1)\,v_y^{2}/v^2\right\}\tan\varphi + v_y\gamma\sec\varphi}{\left\{1+(\gamma-1)\,v_x^{2}/v^2\right\}+(\gamma-1)\dfrac{v_y v_x}{v^2}\tan\varphi + v_x\sec\varphi}$
Mixed Number Lorentz transformation	$\tan\varphi' = \dfrac{\tan\varphi + v_y\sec\varphi}{1+v_x\sec\varphi}$
Quaternion Lorentz transformation	$\tan\varphi' = \dfrac{\tan\varphi + v_y\sec\varphi}{1+v_x\sec\varphi}$
Geometric product Lorentz transformation	$\tan\varphi' = \dfrac{\tan\varphi + v_y\sec\varphi}{1+v_x\sec\varphi}$

5.2. Comparison of Relativistic Doppler's Effect of Special, Most General, Mixed Number, Quaternion and Geometric Product Lorenz Transformations

Names of Lorentz transformations	Relativistic Doppler's effect formula
Special Lorentz transformation	$\nu' = \nu\sqrt{\dfrac{1-V}{1+V}}$
Most general Lorentz transformation	$\nu' = \nu\sqrt{\dfrac{1-\vec{V}}{1+\vec{V}}}$
Mixed number Lorentz transformation	$\nu' = \nu\sqrt{\dfrac{1-\vec{V}}{1+\vec{V}}}$
Quaternion Lorentz transformation	$\nu' = \gamma\nu\left(1-\vec{V}\right)$
Geometric product Lorentz transformation	$\nu' = \nu\sqrt{\dfrac{1-\vec{V}}{1+\vec{V}}}$

5.3. Comparison of the Formulae of the Reflection of Light by a Moving Mirror in Special, Most General, Mixed Number, Quaternion and Geometric Product Lorenz Transformations

Names of Lorentz transformations	The formulae of reflection of light by a moving mirror
Special Lorentz transformation	$\frac{\sin f_1}{\cos f_1 + V} = \frac{\sin f_2}{g(\cos f_2 - V)}$
Most general Lorentz transformation	$\frac{g V_y V^2 + \sin f_1 V^2 + (g-1)V_x V_y \cos f_1 + (g-1)V_y{}^2 \sin f_1}{g V_x V^2 + \cos f_1 V^2 + (g-1)V_x V_y \sin f_1 + (g-1)V_x{}^2 \cos f_1} = \frac{-g V_y V^2 - \sin f_2 V^2 + (g-1)V_x V_y \cos f_2 - (g-1)V_y{}^2 \sin f_2}{g V_x V^2 + \cos f_2 V^2 - (g-1)V_x V_y \sin f_2 + (g-1)V_x{}^2 \cos f_2}$
Mixed number Lorentz transformation	$\frac{V_y + \sin f_1}{V_x + \cos f_1} = \frac{-V_y - \sin f_2}{V_x + \cos f_2}$
Quaternion product Lorentz transformation	$\frac{V_y + \sin f_1}{V_x + \cos f_1} = \frac{-V_y - \sin f_2}{V_x + \cos f_2}$
Geometric Lorentz transformation	$\frac{V_y + \sin f_1}{V_x + \cos f_1} = \frac{-V_y - \sin f_2}{V_x + \cos f_2}$

6. Conclusions

The Relativistic aberration, Doppler's effect and the reflection of light ray by a moving mirror formulae of special, most general, mixed number, quaternion and geometric product Lorentz transformations have been derived clearly. We have found that the formulae for Relativistic aberration and for the reflection of light ray by a moving mirror of mixed number, quaternion and geometric product Lorenz transformations are simpler than the formula of most general Lorenz transformation .

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