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On the Approximation of Symmetric Periodic Solutions of the Sitnikov Problem

Everestus Obinwanne Eze*, Toochukwu Ogbonnia Oko, Chinenye Okorafor Goodluck

Department of Mathematics, Michael Okpara University of Agriculture Umudike, Abia State, Nigeria

Abstract In this paper, an approximate solution of the Sitnikov problem is investigated using both the Euler and fourth-order Runge-Kutta methods. The various values of eccentricities were obtained and demonstrated by simulations using MATCAD which showed that the range for the search of eccentricities can be narrowed down at different values of eccentricities, different sinusoidal frequencies were obtained.

Keywords Symmetric Periodic Solution, Sitnikov Problem, Fourth-order Runge-Kutta method

1. Introduction

The Sitnikov problem describes the motion of a particle of negligible mass attracted by two equal masses $m_1 = m_2 = \frac{1}{2}$. The primaries $m_1\ and\ m_2$ move on the plane (x, y), following an elliptic motion with eccentricity $e \in [0,1]$, while the massless body m_3 performs motion along an axis perpendicular to the primary orbit plane through the barycentre of the primaries. The minimal period of the elliptic motion is 2π if the gravitational constant is assumed to be $G = 1$. If z denotes the position of the particle m_3 in a coordinate system with origin at the centre of mass of the primaries, then the equation of motion of the Sitnikov problem becomes

$$\ddot{z} + \frac{z}{(z^2 + r(t,e)^2)^{3/2}} = 0 \tag{1.1}$$

where z is the distance from the center of the orbit to m_3, $\ddot{z}$ is acceleration, e is eccentricity and $r(t,e)$ is the distance of the primaries to their center of mass and it is given by

$$r(t,e) = \frac{1 - e\cos\ u(t)}{2} \tag{1.2}$$

which is a circular or an elliptic solution of the Kepler problem

$$\ddot{r} = \frac{1-e^2}{16r^3} - \frac{1}{8r^2} \tag{1.3}$$

with eccentricity $e = 0\ or\ 0 < e < 1$, respectively. Here $u(t)$ is the eccentricity anomaly which is a function of time through Kepler equation

$$u - e\ sin u(t) = t \tag{1.4}$$

without loss of generality, when $0 < e < 1$, we take the origin of time in such a way that at $t = 0$ the primaries are at the pericenter of the ellipse. We note that system (1.1) depends on one parameter, the eccentricity $e \in [0,1]$. when the eccentricity e is zero (that is, the primaries move on the circular orbit $r(t) = \frac{1}{2}$ of the Kepler problem (1.3)), (1.1) becomes the equation of motion

$$\ddot{z} = -\frac{z}{\left(z^2 + \frac{1}{4}\right)^{3/2}} \tag{1.5}$$

for the circular Sitnikov problem.

More information can be found in [5] and in the more recent [1]. The existence of symmetric (even or odd) periodic solutions has been discussed in [2-4-6-7]. In [2] methods of local analysis were employed, and they got results which are valid only for small eccentricity e. The papers [4, 6] considered arbitrary eccentricity from a theoretical perspective by using the global continuation method due to Leray and Schauder, and [6] found families of symmetric periodic solutions bifurcating from the equilibrium at the center of mass. These families were labelled according to the number of zeros in the same fashion as it occurs in the work by Rabinowitz [9] for other non-linearities. [7] combines Shooting arguments with Sturm comparison theory to prove the existence of odd periodic solutions with a prescribed number of zeros. While [3] presents a very complete description of the set of symmetric periodic solutions based on numerical computations. [8] discussed on the circular Sitnikov problem as a subsystem of the three-dimensional circular restricted three-body problem, where they used elliptic functions to give the analytical expressions for the solutions of the circular Sitnikov problem and for the period function of its family of periodic orbits. They also analyzed the qualitative and quantitative behaviour of the period function. The purpose of this note is to show that it is also possible to

* Corresponding author:
obinwanneeze@gmail.com (Everestus Obinwanne Eze)
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obtain numerical results for all values of the eccentricity using only very elementary tool, the fourth-order Runge-Kutta method. This paper is divided into sections. Section 2 is the definition and theorems which was used in the result while section 3 is the derivation of the fourth-order Runge-Kutta method, section 4 is the result obtained with numerical simulations and section 5 is conclusion.

2. Preliminary

Theorem 1. Precision of the Runge-Kutta methods

Assume that $y(t)$ is the solution of the problem

$$y^n = f^{'}(t, y) \tag{2.1}$$

If $y(t) \in C^3, [t_0, b]\ and\ (t_i, y_i)^m = 0$ is the sequence of approximations generated by the Runge-Kutta method of order 2, then

$$|E_i| = |y(t_i) - y_i| = 0(h^2)$$

$$|E_{i+1}| = |y(t_{i+1}) - y_i - hTN(t, y_i)| = 0(h^3)$$

Given the interval $[t_0, b]$, we satisfy that

$$E\left(y^{(b)}, h\right) = \left[y^{(b)}, y^m\right] = 0(h^2)$$

Theorem 2. Assume the existence of such a solution $y(t)$ is guaranteed and unique, provided $f(t, y)$,

(i) is continuous in the infinite strip

$$R = \{x_0 \le x \le T, |y| < \infty\}$$

(ii) is more specifically Lipchitz continuous in the dependent variable y over the same region R, that is there exist a positive constant L such that for all $(t, y), (t, y^{'}) \in R$

$$|f(t, y) - f(t, y^{'})| < L|y - y^{'}|.$$

Theorem 3. Suppose that σ is a nonempty, closed and bounded limit set of a planar flow, then one of the following holds:

- σ is an equilibrium point
- σ is a periodic solution
- σ consists of a set of equilibria and connecting orbits between these equilibria.

Proof

We consider $\sigma = w(x)$ for some $x \in \mathbb{R}^2$. The argument in the case of an $\alpha - limit$ set is similar.

Let $y \in w(x)\ and\ z \in w(y)$. If z is not an equilibrium point, then $w(y)$ must be a periodic solution and if $w(y)$ is a periodic solution then $w(x) = w(y)$ and thus $w(x)$ is also a periodic solution.

Now we assume that $z \in w(y)$ is an equilibrium point. Then $w(y)$ must consist entirely of equilibria since if there is a point $z \in w(y)$ that is not an equilibrium, then we know that $w(y)$ is a periodic solution (and in particular contains no equilibrium). We note that since $y \in w(x)$ it follows that $\{\emptyset^t(y)\}_{t\in\mathbb{R}} \subset w(x)$, where $\emptyset^t$ denotes the time-t flow. Hence $\propto(y) \in w(x)$ and for the same reasons as before $\propto(y)$ must be an equilibrium, since otherwise $w(y)(and\ w(x))$ must be a periodic solution. Hence, we find that either y is an equilibrium point, or that y lies in the intersection between the stable and unstable manifolds of the equilibria $w(y)\ and \propto(y)$ (that is on a connecting orbit between equilibria).

Theorem 4. For each integer $m \ge 1$, there exists a unique solution $z(t)$ of (1.1) satisfying the conditions,

$$z(t + 2mt) = z(t), z(-t) = -z(t), t \in \mathbb{R} \tag{2.2}$$

$$z(t) > 0, t \in [0, m\pi] \tag{2.3}$$

The variational equation at the center of mass $z = 0$ will play an important role; it is the equation of Hill's type

$$\ddot{\xi} + \frac{1}{r(t,e)^3}\xi = 0 \tag{2.4}$$

Theorem 3. Assume that $m \ge 1\ and\ N \ge 0$ are given integers. Then the following statements are equivalent:

i) there exist a solution of (1.1) satisfying the condition in (2.2) and having exactly N zero in the interval $[0, m\pi]$

ii) the solution $\xi(t)$ of (2.4) with initial conditions $\xi(0) = 0, \dot{\xi}(0) = 1$ has more than N zero in $[0, m\pi]$

3. Derivation of Fourth-order Runge-Kutta Method

The simple Euler method comes from using just one term from the Taylor series for $y(x)$ expanded about $x = x_0$. The modified Euler method can also be derived from using terms

$$y(x_0 + h) = y(x_0) + y^{'}(x_0) * h + y^{''}(x_0) * \frac{h^2}{2} \tag{3.1}$$

If we replace the second derivative with a backward-difference approximation,

$$\begin{aligned} y(x_0 + h) &= y(x_0) + y^{'}(x_0) * h \\ &+ \left[\frac{(y^{'}(x_0 + h) - y^{'}(x_0))}{h}\right] * \frac{h^2}{2} \\ &= y(x_0) + \frac{y^{'}(x_0) + y^{'}(x_0 + h)}{2} h \end{aligned} \tag{3.2}$$

We get the formula for the modified method. What if we use more terms of the Taylor series? Two German mathematicians, Runge and Kutta, developed algorithms from using more than two terms of the series. We will consider only fourth-order formula. The modified Euler method is a second-order Runge-Kutta method.

Second-order Runge-Kutta methods are obtained by using a weighted average of two increments to $y(x_0), k_1\ and\ k_2$. For the equation $dy/_{dx} = f(x, y)$

$$\begin{aligned} y_{n+1} &= y_n + ak_1 + bk_2, \\ k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + \alpha h, y_n + \beta k_1). \end{aligned} \tag{3.3}$$

We can think of the values $k_1\ and\ k_2$ as estimates of the change in y when x advances by h, because they are the product of the change in x and a value for the slope of the curve, $\frac{dy}{dx}$.

The Runge-Kutta methods always use the simple Euler estimate as the first of Δy; the other estimate is taken with $x\ and\ y$ stepped up by the fractions $\alpha\ and\ \beta\ of\ h$ and of the earlier estimate of $\Delta y, k_1$. Our problem is to devise a scheme of choosing the four parameters, a, b, α, β. We do also by making equation (3.3) agree as well as possible with the Taylor-series expansion, in which the $y - derivatives$ are written in terms of f, from $\frac{dy}{dx} = f(x, y)$,

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2} f'(x_n, y_n) + \cdots$$

An equivalent form, because $df/_{dx} = f_x + f_y\, dy/_{dx} = f_x + f_y f$, is

$$y_{n+1} = y_n + hf_n + h^2\left(\frac{1}{2}f_x + \frac{1}{2}f_y f\right)_n \tag{3.4}$$

[All the derivatives in equation (3.4) are calculated at the point (x_n, y_n).] we now rewrite equation (3.4) by substituting the definitions of $k_1\ and\ k_2$.

$$y_{n+1} = y_n + ahf(x_n, y_n) + bhf[x_n + \alpha h, y_n + \beta hf(x_n, y_n)] \tag{3.5}$$

To make the last term of equation (3.5) comparable to equation (3.4), we expand $f(x, y)$ in a Taylor series in terms of x_n, y_n remembering that f is a function of two variables, retaining only first derivative terms:

$$f[x_n + \alpha h, y_n + \beta hf(x_n, y_n)] \approx \left(f + f_x \alpha h + f_y \beta hf\right)_n \tag{3.6}$$

On the right side of both equations (3.4) and (3.6), f and its partial derivatives are all to be evaluated at (x_n, y_n).

Substituting from equation (3.6) into equation (3.5), we have

$$y_{n+1} = y_n + (a + b)hf_n + h^2\left(\alpha b f_x + \beta b f_y f\right)_n \tag{3.7}$$

Equation (3.7) will be identical to equation (3.4) if

$$a + b = 1, \alpha b = \frac{1}{2}, \beta b = \frac{1}{2}.$$

Note that only three equations need to be satisfied by the four unknowns. We can choose one value arbitrary (with minor restrictions); hence, we have a set of second-order methods.

One choice can be $a = 0, b = 1$; $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$. this gives the midpoint method.

Another choice can be $a = \frac{1}{2}, b = \frac{1}{2}$; $\alpha = 1, \beta = 1$, which give the modified Euler.

Still another possibility is $a = \frac{1}{3}, b = \frac{2}{3}; \alpha = \frac{3}{4}, \beta = \frac{3}{4}$; this is said to give a minimum bound to the error. All of these are special cases of second-order of Runge-Kutta methods.

Fourth-order Runge-Kutta methods are most widely used and are derived in similar fashion.

Greater complexity results from having to compare terms through h^4, and this gives a set of 11 equations in 13 unknowns. The set of 11 equations can be solved with 2 unknowns being chosen arbitrarily. The most commonly used set of values leads to the procedure;

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1), \\ k_3 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2), \\ k_4 &= hf(x_n + h, y_n + k_3). \end{aligned} \tag{3.8}$$

This Runge-Kutta method will be used to solve equation (1.1) in section 4. Numerically, we shall use Euler method and fourth-order Runge-Kutta method.

4. Results

Considering equation (1.1);

Let $\dot{z} = \varphi$, such that $\ddot{z} = \dot{\varphi}$.

Therefore; equation (1.1) becomes;

$$\dot{\varphi} = -\frac{z}{\left(z^2 + r(t,e)^2\right)^{3/2}}$$

But from theorem 2, equation (3.5) states;

$$\dot{\varphi} = -\frac{1}{r(t,e)^3} y, \text{ at } \xi = y,$$

which is linear. (Hill's type of equation at $z = 0$).

Euler method

$$y_{n+1} = y_n + hf(t, e),$$
$$\dot{\varphi} = -\frac{y}{r(t,e)^3}.$$

Given $r(t, e) = \frac{1}{2}(1 - e\cos u(t)), let\ u(t) = 0, e \in [0,1[, h = 0.02, y(0) = 0, f_0 = 1$.

Table 1

e	y_n	$\dot{\varphi} = f_n$
0	0	1
0.02	0.02	−0.169997
0.04	0.016600	−0.150101
0.06	0.013598	−0.130973
0.08	0.010979	−0.112795
0.10	0.008723	−0.095726
0.12	0.006808	−0.079921
0.14	0.005210	−0.065529

0.16	0.003899	-0.052627
0.18	0.002846	-0.041294
0.20	0.002020	-0.031563
0.22	0.001389	-0.023416
0.24	0.000921	-0.016785
0.26	0.000585	-0.011549
0.28	0.000354	-0.007587
0.30	0.000202	-0.004711
0.32	0.000108	-0.002748
0.34	0.000053	-0.001476
0.36	2.35×10^{-5}	-7.18×10^{-4}
0.38	9.16×10^{-6}	-3.07×10^{-4}
0.40	3.02×10^{-6}	-1.12×10^{-4}
0.42	7.80×10^{-7}	-3.20×10^{-5}
0.44	1.40×10^{-7}	-6.38×10^{-6}
0.46	1.24×10^{-8}	-6.30×10^{-7}
0.48	-2.00×10^{-10}	1.14×10^{-8}
0.50	2.8×10^{-11}	-1.79×10^{-9}
0.52	-7.8×10^{-12}	5.64×10^{-10}
0.54	3.48×10^{-12}	-2.86×10^{-10}
0.56	-2.24×10^{-12}	2.10×10^{-10}
0.58	1.96×10^{-12}	-2.12×10^{-10}
0.60	-2.28×10^{-12}	2.85×10^{-10}
0.62	3.42×10^{-12}	-4.99×10^{-10}
0.64	-6.56×10^{-12}	1.12×10^{-9}
0.66	1.58×10^{-11}	-3.22×10^{-9}
0.68	-4.86×10^{-11}	1.19×10^{-8}
0.70	1.89×10^{-10}	-5.60×10^{-8}
0.72	-9.31×10^{-10}	3.39×10^{-7}
0.74	5.85×10^{-9}	-2.66×10^{-6}
0.76	-4.74×10^{-8}	2.74×10^{-5}
0.78	5.01×10^{-7}	-3.76×10^{-4}
0.80	-7.02×10^{-6}	7.02×10^{-3}
0.82	1.41×10^{-3}	-1.934156
0.84	-0.037273	72.798828
0.86	1.418704	-4136.163265
0.88	-81.304561	376410.0046
0.90	7446.895531	-59575164.25
0.92	-1184056.389	1.85×10^{10}
0.94	368815943.6	-1.37×10^{13}
0.96	-2.74×10^{11}	3.43×10^{16}
0.98	6.86×10^{14}	-6.86×10^{20}
1.00	-1.37×10^{19}	...

The table above shows the results.

Fourth-order Runge-Kutta methods:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

Using the same parameters, we obtain the results in the table below;

Table 2

e	y_n	k_1	k_2	k_3	k_4
0	0	0	0	0	0
0.02	0.002825	0.02	-0.001649	1.4×10^{-4}	-2.3×10^{-5}
0.04	0.002371	-4.8×10^{-4}	-4.5×10^{-4}	-4.6×10^{-4}	-4.3×10^{-4}
0.06	0.001920	-4.3×10^{-4}	-4.0×10^{-4}	-4.1×10^{-4}	-6.6×10^{-4}
0.08	0.001573	-3.7×10^{-4}	-3.5×10^{-4}	-3.5×10^{-4}	-3.2×10^{-4}
0.10	0.001272	-3.2×10^{-4}	-3.0×10^{-4}	-3.0×10^{-4}	-2.8×10^{-4}
0.12	0.001014	-2.8×10^{-4}	-2.6×10^{-4}	-2.6×10^{-4}	-2.4×10^{-4}
0.14	7.95×10^{-4}	-2.4×10^{-4}	-2.2×10^{-4}	-2.2×10^{-4}	-2.0×10^{-4}
0.16	6.13×10^{-4}	-2.0×10^{-4}	-1.8×10^{-4}	-1.8×10^{-4}	-1.7×10^{-4}
0.18	4.64×10^{-4}	-1.7×10^{-4}	-1.5×10^{-4}	-1.5×10^{-4}	-1.4×10^{-4}
0.20	3.43×10^{-4}	-1.4×10^{-4}	-1.2×10^{-4}	-1.2×10^{-4}	-1.1×10^{-4}
0.22	2.50×10^{-4}	-1.1×10^{-4}	-9.0×10^{-5}	-9.3×10^{-5}	-8.4×10^{-5}
0.24	1.76×10^{-4}	-8.4×10^{-5}	-7.3×10^{-5}	-7.5×10^{-5}	-6.4×10^{-5}
0.26	1.20×10^{-4}	-6.4×10^{-5}	-5.5×10^{-5}	-5.6×10^{-5}	-4.7×10^{-5}
0.28	7.96×10^{-5}	-4.7×10^{-5}	-4.0×10^{-5}	-4.1×10^{-5}	-3.4×10^{-5}
0.30	5.09×10^{-5}	-3.4×10^{-5}	-2.8×10^{-5}	-2.9×10^{-5}	-2.4×10^{-5}
0.32	3.13×10^{-5}	-2.4×10^{-5}	-1.9×10^{-5}	-2.0×10^{-5}	-1.6×10^{-5}
0.34	1.84×10^{-5}	-1.6×10^{-5}	-1.2×10^{-5}	-1.3×10^{-5}	-9.9×10^{-6}
0.36	1.03×10^{-5}	-1.0×10^{-5}	-7.8×10^{-6}	-8.5×10^{-6}	-6.1×10^{-6}
0.38	5.44×10^{-6}	-6.3×10^{-6}	-4.6×10^{-6}	-5.1×10^{-6}	-3.5×10^{-6}
0.40	2.70×10^{-6}	-3.7×10^{-6}	-2.5×10^{-6}	-2.9×10^{-6}	-1.9×10^{-6}
0.42	1.24×10^{-6}	-2.0×10^{-6}	-1.3×10^{-6}	-1.6×10^{-6}	-9.1×10^{-7}
0.44	5.26×10^{-7}	-1.0×10^{-6}	-6.3×10^{-7}	-7.9×10^{-7}	-4.0×10^{-7}
0.46	2.04×10^{-7}	-4.8×10^{-7}	-2.8×10^{-7}	-3.7×10^{-7}	-1.6×10^{-7}
0.48	7.20×10^{-8}	-2.1×10^{-7}	-1.1×10^{-7}	-1.6×10^{-7}	-4.9×10^{-8}
0.50	2.28×10^{-8}	-8.2×10^{-8}	-3.7×10^{-8}	-6.4×10^{-8}	-9.9×10^{-9}
0.52	1.36×10^{-8}	-2.9×10^{-8}	-1.1×10^{-8}	-2.3×10^{-8}	4.3×10^{-8}
0.54	3.68×10^{-9}	-1.9×10^{-8}	-5.8×10^{-9}	-1.7×10^{-8}	4.8×10^{-9}
0.56	1.03×10^{-9}	-6.1×10^{-9}	-1.2×10^{-9}	-5.4×10^{-9}	3.3×10^{-9}
0.58	3.46×10^{-10}	-1.9×10^{-9}	-1.3×10^{-10}	-1.9×10^{-9}	1.9×10^{-9}
0.60	1.72×10^{-10}	-7.5×10^{-10}	6.4×10^{-11}	-8.8×10^{-10}	1.3×10^{-9}
0.62	1.78×10^{-10}	-4.3×10^{-10}	1.4×10^{-10}	-7.6×10^{-10}	1.7×10^{-9}
0.64	3.08×10^{-10}	-5.2×10^{-10}	2.6×10^{-10}	-9.7×10^{-10}	2.7×10^{-9}
0.66	1.13×10^{-9}	-1.1×10^{-9}	8.3×10^{-10}	-2.7×10^{-9}	9.7×10^{-9}
0.68	9.15×10^{-9}	-4.6×10^{-9}	5.2×10^{-9}	-1.7×10^{-8}	7.6×10^{-8}
0.70	1.74×10^{-7}	-4.5×10^{-8}	7.1×10^{-8}	-2.4×10^{-7}	1.4×10^{-6}
0.72	8.05×10^{-6}	-1.0×10^{-6}	2.2×10^{-6}	-8.5×10^{-6}	6.1×10^{-5}
0.74	9.54×10^{-4}	-5.9×10^{-5}	1.7×10^{-4}	-7.7×10^{-4}	6.9×10^{-3}
0.76	0.763069	-8.7×10^{-3}	0.034769	-0.188087	1.077431
0.78	721.703614	-8.831817	48.063684	-326.248829	4890.845379
0.80	2165610.343	-10844.53214	81210.45169	-714010.5297	14265776.52
0.82	2.28×10^{10}	-43312206.86	454655036.4	-5353389789	1.5×10^{11}
0.84	9.61×10^{14}	-6.26×10^{11}	9.45×10^{12}	-1.55×10^{14}	6.1×10^{15}
0.86	1.95×10^{20}	-3.8×10^{16}	8.4×10^{17}	-2.0×10^{19}	1.2×10^{21}
0.88	2.19×10^{26}	-1.1×10^{22}	4.0×10^{23}	-1.5×10^{25}	1.3×10^{27}
0.90	1.90×10^{33}	-2.0×10^{28}	1.2×10^{30}	-7.2×10^{31}	1.2×10^{34}
0.92	1.87×10^{41}	-3.0×10^{35}	3.3×10^{37}	-3.6×10^{39}	1.1×10^{42}
0.94	3.88×10^{50}	-5.8×10^{43}	1.4×10^{46}	-3.2×10^{48}	2.3×10^{51}
0.96	4.91×10^{61}	-2.9×10^{53}	1.8×10^{56}	-1.2×10^{59}	2.9×10^{62}
0.98	3.60×10^{75}	-1.2×10^{65}	3.6×10^{68}	-1.1×10^{72}	2.2×10^{76}
1.00	…	-7.2×10^{79}	2.9×10^{86}	-2.3×10^{91}	…

MATHCAD SIMULATION

SIMULATION OF $\ddot{z} + z\left[z^2 + \gamma(t,e)^2\right]^{-3/2}$;

$z(0)=0,\ \dot{z}(0)=1,\ \ \gamma(t,e)=0.5\left(1-e\cos u(t)\right).$

u(t) := 0 e := 0.5

Define a function that determines a vector of derivative values at any solution point (t,Z):

$$D(t,Z) := \left[\begin{array}{c} Z_1 \\ \dfrac{-Z_0}{\left[\left(Z_0\right)^2+\left[0.5\cdot\left(1-e\cdot\cos\left(u(t)\right)\right)\right]^2\right]^{1.5}} \end{array}\right]$$

Define additional arguments for the ODE solver:

t0 := 0 Initial value of independent variable

t1 := 150 final value of independent variable

$Z0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Vector of initial function values

N := 1500 Number of solution values on [t0, t1]

Solution matrix:

$S := \mathrm{Rkadapt}(Z0, t0, t1, N, D)$

$t := S^{\langle 0\rangle}$ Independent variable values

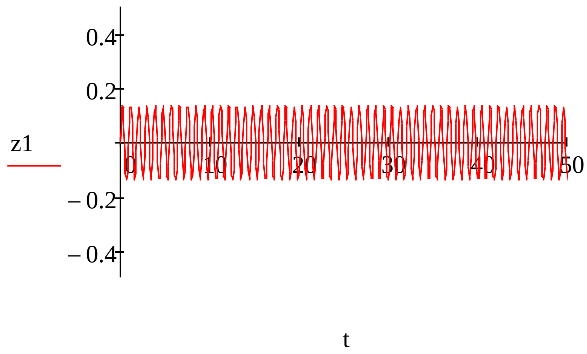
$z1 := S^{\langle 1\rangle}$ First solution function values

$z2 := S^{\langle 2\rangle}$ Second solution function values

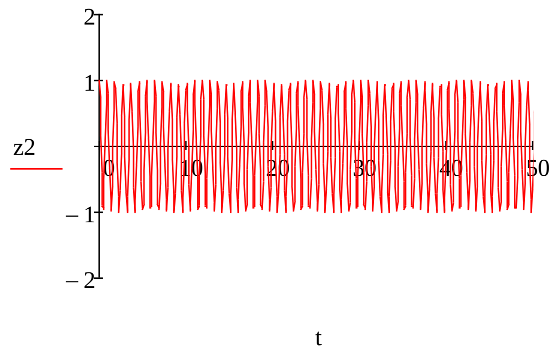
S =

	0	1	2
0	0	0	1
1	0.1	0.09	0.724
2	0.2	0.136	0.17
3	0.3	0.123	-0.417
4	0.4	0.056	-0.9
5	0.5	-0.042	-0.944
6	0.6	-0.116	-0.502
7	0.7	-0.138	0.081
8	0.8	-0.101	0.649
9	0.9	-0.015	0.993
10	1	0.079	0.794
11	1.1	0.133	0.26
12	1.2	0.129	-0.33
13	1.3	0.069	-0.844
14	1.4	-0.027	-0.977
15	1.5	-0.108	...

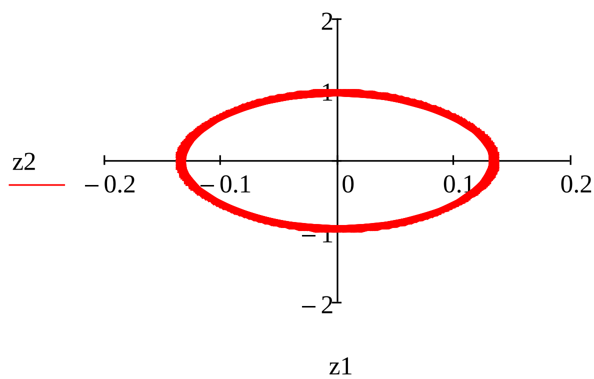
Solution matrix



Trajectory $z(t)$ as a function of time

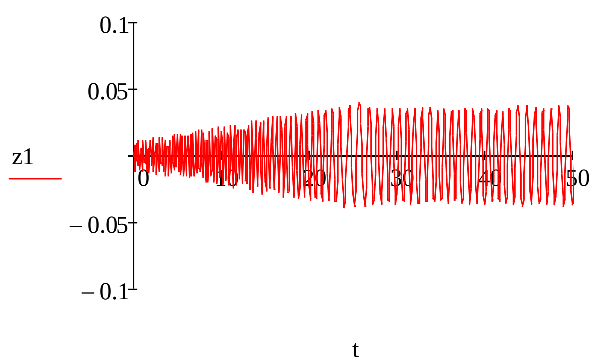


Velocity $\dot{z}(t)$ as a function of time

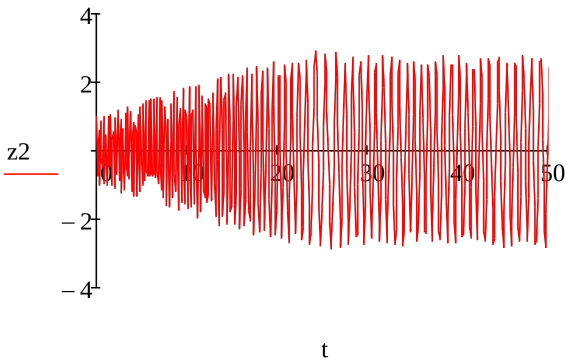


Phase portrait

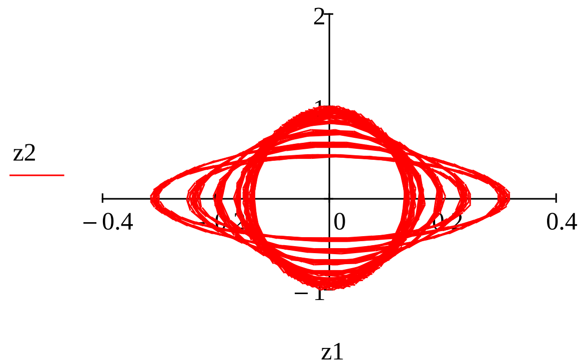
u(t) := 0 e := 0.9



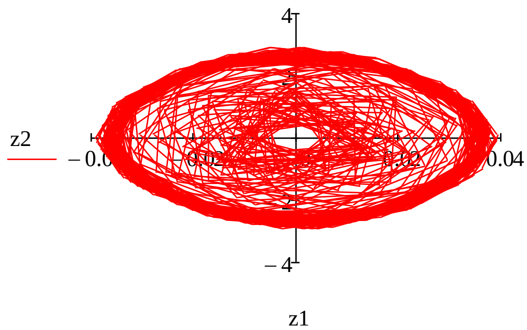
Trajectory $z(t)$ as a function of time



Velocity $\dot{z}(t)$ as a function of time

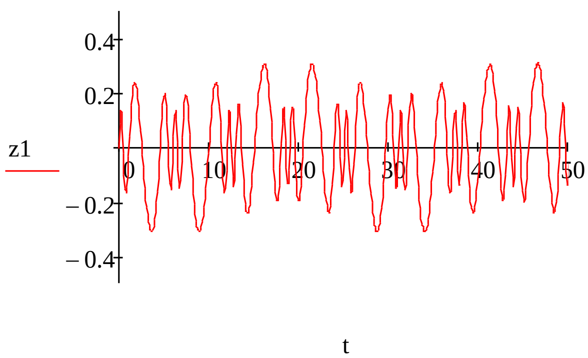


Phase portrait

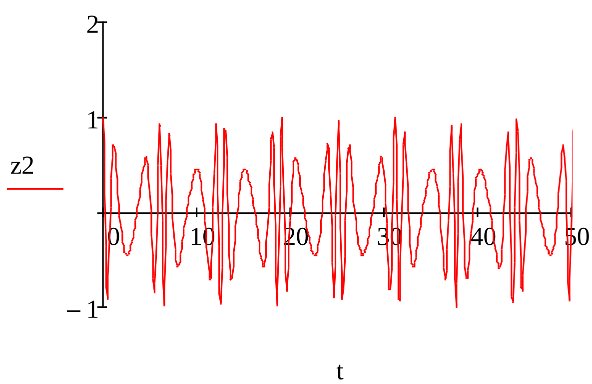


Phase portrait

u(t) := t e := 0.5

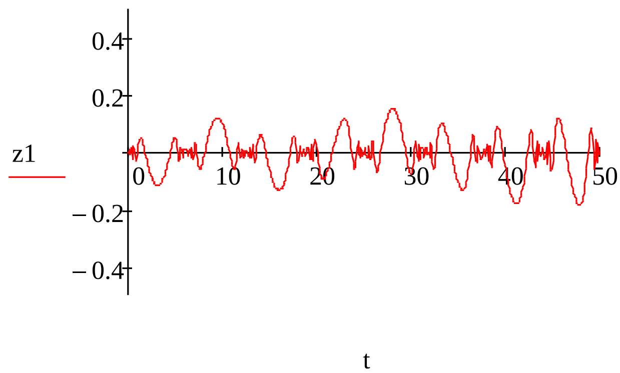


Trajectory $z(t)$ as a function of time

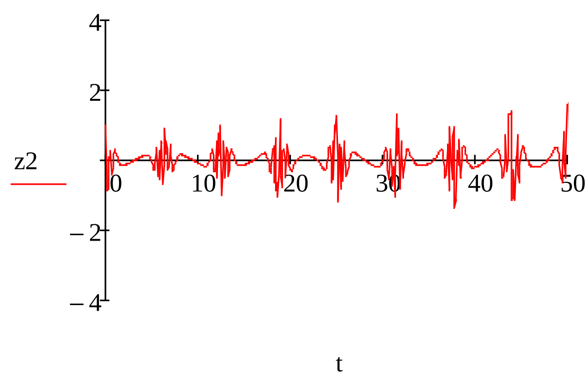


Velocity $\dot{z}(t)$ as a function of time

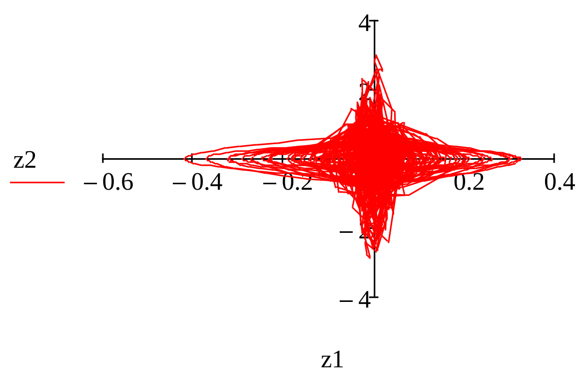
u(t) := t e := 0.9



Trajectory $z(t)$ as a function of time

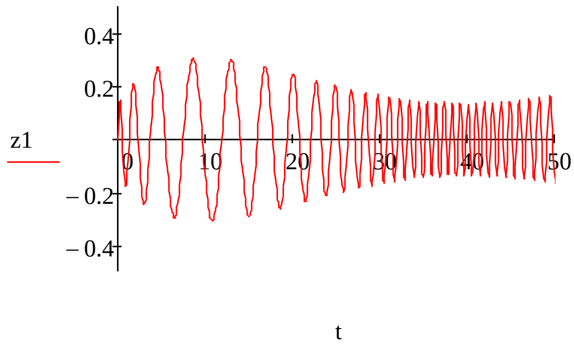


Velocity $\dot{z}(t)$ as a function of time

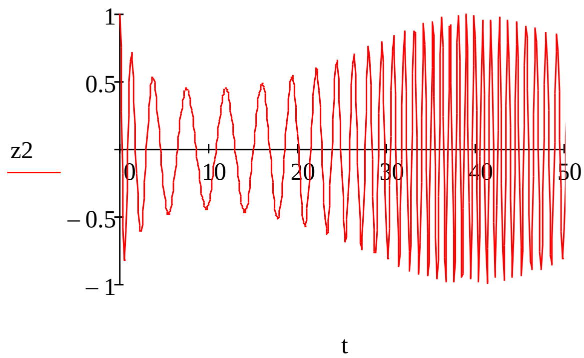


Phase portrait

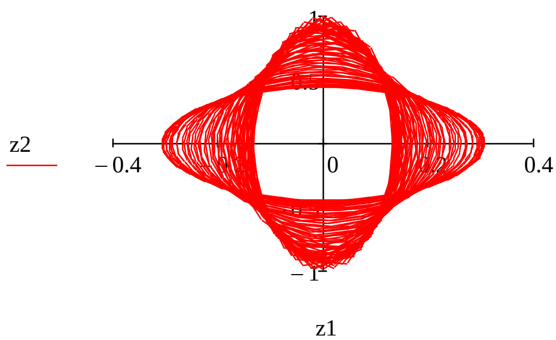
u(t) := √t e := 0.5



Trajectory $z(t)$ as a function of time

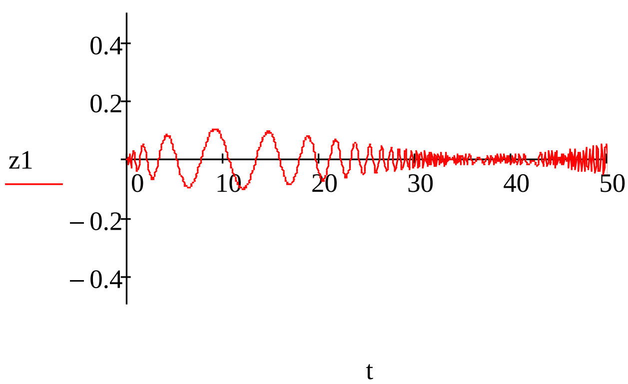


Velocity $\dot{z}(t)$ as a function of time

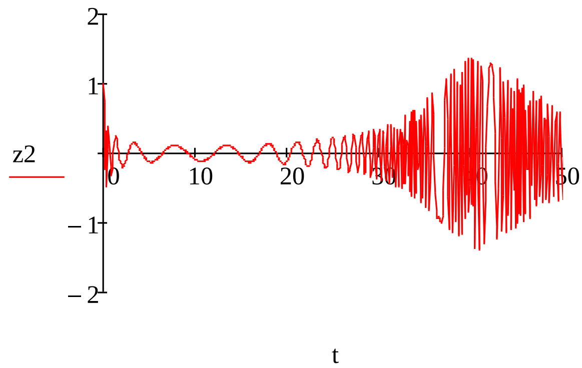


Phase portrait

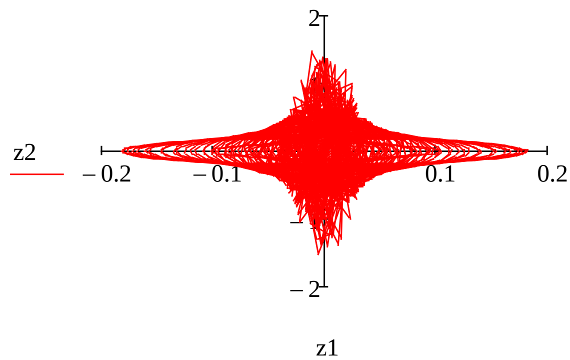
u(t) := √t e := 0.9



Trajectory $z(t)$ as a function of time



Velocity $\dot{z}(t)$ as a function of time



Phase portrait

5. Conclusions

An approximate solution of the Sitnikov problem has been investigated using both the Euler and fourth-order Runge-Kutta methods. The fourth-order Runge-Kutta method gave us more accurate results than Euler method. The various values of eccentricities were obtained and demonstrated by simulations using MATCAD. The simulations reveal the behaviour of the solutions at any given eccentricity, this showed that the range for the search of eccentricities can be narrowed down at different values of eccentricities, different sinusoidal frequencies were obtained.

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