

New Fifth-order Simpson-type Method for Solving Nonlinear Equations

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Abstract The objective of this paper is to present further development of the Simpson-type methods for finding zeros of a nonlinear equation. The new Simpson-type method is shown to converge of the order five. Per iteration the new method requires the same amount of evaluations of the function and therefore the new method has an efficiency index better than the other similar Simpson-type methods. We examine the effectiveness of the new fifth-order Simpson-type method by approximating the simple root of a given nonlinear equation. Numerical comparisons are made to show the performance of the new iterative method and thus verifies the theoretical results.

Keywords Simpson method, Newton method, Simple root, Nonlinear equation, Root-finding, Order of convergence

1. Introduction

In this paper, we present a new fifth-order iterative method to find a simple root of the nonlinear equation. It is well known that the techniques to solve nonlinear equations have many applications in science and engineering. We shall utilise two well-known techniques, namely the classical Newton method and the classical Simpson method, for their simplicity with convergence order of two and three respectively [1-13]. The new Simpson-type method requires the same amount of evaluations of the function as the classical Simpson method. Hence, we shall prove that the new Simpson-type iterative method has a better efficiency index than the other similar Simpson-type methods [1, 2, 4-7, 10, 12]. The prime motive for the development of the new fifth-order Simpson-type method was to increase the order of convergence of the recently introduced fourth-order Simpson-type methods.

2. The Method and Convergence Analysis

Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α . The order of convergence p is given by

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = \omega \neq 0, \quad (1)$$

where $p \in \mathbb{R}^+$ and ω is the asymptotic error constant. Also, let $e_k = x_k - \alpha$ be the error in the k th iteration. Then the relation

$$e_{k+1} = \zeta e_k^p + O(e_k^{p+1}), \quad (2)$$

is the error equation. If the error equation exists, then p is the order of convergence of the iterative method [3, 8, 9, 13].

Furthermore, the efficiency of the method is measured by the concept of efficiency index, given by

$$EI(r, p) = \sqrt[p]{r}, \quad (3)$$

where r is the number of function evaluations of the method and p is the order of the convergence of the method [8]. In order to approximate the order of convergence of the method we use the following formula. Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1). Then the computational order of convergence may be approximated by

$$\text{COC} \approx \frac{\ln|\lambda_n \div \lambda_{n-1}|}{\ln|\lambda_{n-1} \div \lambda_{n-2}|}, \quad (4)$$

where $\lambda_i = f(x_i) \div f'(x_i)$, [11].

Before we define the new fifth-order Simpson-type method we state essentially the classical third-order Simpson method and the recently introduced fourth-order Simpson-type method.

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2.1. The Classical Simpson Third-order Method

Since this method is well established, we shall state the essential expressions used in order to calculate the root of the given nonlinear equation. Hence the Simpson third-order method is given as

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n) + 4f'(z_n) + f'(y_n)}, \quad (5)$$

where

$$u_n = \frac{f(x_n)}{f'(x_n)}, \quad (6)$$

$$y_n = x_n - u_n, \quad (7)$$

$$z_n = x_n - 2^{-1}u_n, \quad (8)$$

x_0 is the initial approximation and provided that the denominator of (5) is not equal to zero.

2.2. The Simpson-type Fourth-order Methods

In this section we state the recently introduced fourth-order Simpson-type method to find simple root of a nonlinear equation. The improvement of the classical Simpson method was made by introducing a new factors in (5). Hence, the improved fourth-order Simpson-type methods is given by

$$x_{n+1}^{(i)} = x_n - \frac{6f(x_n)}{f'(x_n) + 4f'(z_n) + f'(y_n)} - 2^{-2} \left(\frac{\delta_i}{f'(x_n)} \right)^2 u_n^3 \quad (9)$$

where $m \in \mathbb{R}$,

$$\delta_1 = \frac{2(f'(y_n) - f'(z_n))}{u_n}, \quad (10)$$

$$\delta_2 = \frac{2(f'(x_n) - f'(z_n))}{u_n}, \quad (11)$$

$$\delta_3 = \frac{5f'(x_n) - 4f'(z_n) - f'(y_n)}{3u_n}, \quad (12)$$

$$\delta_4 = \frac{4f'(z_n) - f'(y_n) - 3f'(x_n)}{(m+1)w_n + mv_n - 2mu_n}, \quad (13)$$

$$v_n = \frac{f(x_n)}{f'(y_n)}, \quad (14)$$

$$w_n = \frac{f(x_n)}{f'(z_n)}. \quad (15)$$

Further details of the above methods are given in [12].

2.3. New Simpson-type Fifth-order Method

Here we present a new scheme to find simple root of a nonlinear equation. In order to increase the order of convergence of the classical Simpson method and the fourth-order Simpson-type method, we shall introduce a new different type of factor in (5). Hence, the scheme of the new fifth-order Simpson method is given by

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n) + 4f'(z_n) + f'(y_n)} - 2^{-2} \left(\frac{\phi}{f'(x_n)} \right)^2 u_n^3 + 2^{-1} \left(\frac{\phi}{f'(x_n)} \right)^3 u_n^4, \quad (16)$$

alternatively the new formula maybe expressed as

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n) + 4f'(z_n) + f'(y_n)} - 2^{-2} \left(\frac{\phi}{f'(x_n)} \right)^2 u_n^3 \left[1 + 2 \left(\frac{\phi}{f'(x_n)} \right) u_n \right], \quad (17)$$

where u_n is given by (6) and

$$\phi = \frac{f'(x_n) - f'(y_n)}{u_n}. \quad (18)$$

Theorem 1

Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval D . If the initial guess x_0 is sufficiently close to α , then the order of convergence of the new Simpson-type method defined by (16) is five.

Proof

Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the error is expressed as

$$e = x - \alpha. \quad (19)$$

Using Taylor expansion, we have

$$f(x_n) = f(\alpha) + f'(\alpha)e_n + 2^{-1}f''(\alpha)e_n^2 + \dots \quad (20)$$

Taking $f(\alpha) = 0$ and simplifying, expression (20) becomes

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + \dots]. \quad (21)$$

where

$$c_k = \frac{f^{(k)}(\alpha)}{(k!)f'(\alpha)} \text{ for } k = 2, 3, 4, \dots \quad (22)$$

Furthermore, we have

$$f'(x_n) = f'(\alpha) \left[1 + 2c_2e_n + 3c_3e_n^2 + \dots \right]. \quad (23)$$

Dividing (21) by (23), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + \dots \quad (24)$$

and hence, we have

$$y_n = e_n - \frac{f(x_n)}{f'(x_n)} = c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - \dots \quad (25)$$

The expansion of $f'(y_n)$ about α is given as

$$f'(y_n) = f'(\alpha) \left[c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - \dots \right]. \quad (26)$$

Since from (8) we have

$$z_n = x_n - 2^{-1} \left(\frac{f(x_n)}{f'(x_n)} \right) = \frac{e_n}{2} + \frac{c_2e_n^2}{2} + \dots \quad (27)$$

Taylor expansion of $f'(z_n)$ about α is

$$f'(z_n) = f'(\alpha) \left[c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - \dots \right]. \quad (28)$$

The denominator of (5) is given as

$$\begin{aligned} & f'(x_n) + 4f'(z_n) + f'(y_n) \\ &= 6 + 6c_2e_n + 6(c_2^2 + c_3)e_n^2 + \dots \end{aligned} \quad (29)$$

Dividing the numerator by denominator of (5), we get

$$e_n - c_2^2e_n^3 + 3(c_2^3 - c_2c_3)e_n^4 + \dots \quad (30)$$

It is well established that the error equation of (5) is

$$e_{n+1} = c_2^2e_n^3 - 3(c_2^3 - c_2c_3)e_n^4 + \dots \quad (31)$$

The improvement factors introduce in (16) are

$$\phi = \frac{f'(x_n) - f'(y_n)}{u_n} = 2c_2 + 3c_3e_n + \dots, \quad (32)$$

$$2^{-2} \left(\frac{\phi}{f'(x_n)} \right)^2 u_n^3 = c_2^2e_n^3 + (3c_2c_3 - 7c_2^3)e_n^4 + \dots, \quad (33)$$

$$2^{-1} \left(\frac{\phi}{f'(x_n)} \right)^3 u_n^4 = 4c_2^3e_n^4 + 2(9c_2^2c_3 - 20c_2^4)e_n^5 + \dots \quad (34)$$

Substituting appropriate expressions in (16), we obtain

$$e_{n+1} = 2^{-2} (52c_2^4 - c_3^2)e_n^5 + \dots \quad (35)$$

The error equation (35) establishes the fifth-order convergence of the new iterative method defined by (16).

3. Numerical Test Examples

Table 1. Test functions and their simple roots

Functions	Parameter	Simple Root	Initial point
$f_1(x) = x^{15} + x^4 - 4x^2 + 2$	$m = -2$	$\alpha = -0.7614\dots$	$x_0 = -1$
$f_2(x) = (x^3 - 1)^{-1} - 2$	$m = -3$	$\alpha = 1.1447\dots$	$x_0 = 1.1$
$f_3(x) = (x - 2)(x^{10} + x + 1)\exp(-x - 1)$	$m = -4$	$\alpha = 2$	$x_0 = 1.99$
$f_4(x) = (x + 1)\exp(\sin(x)) - x^2\exp(\cos(x)) - 1$	$m = 10$	$\alpha = 0$	$x_0 = 0.125$
$f_5(x) = \cos(x)^2 - x^2 + 3$	$m = -70$	$\alpha = 1.7402\dots$	$x_0 = 2$
$f_6(x) = \exp(1 - x) - \cos(x)$	$m = 60$	$\alpha = 4.7362\dots$	$x_0 = 5$
$f_7(x) = \ln(x^2 + x + 2) - x + 11$	$m = 300$	$\alpha = 16.6951\dots$	$x_0 = 16$
$f_8(x) = x^5 - x + 2$	$m = 10$	$\alpha = -1.2671\dots$	$x_0 = -1.2$
$f_9(x) = x^2 - \exp(x) - 3x + 2$	$m = 60$	$\alpha = 0.2575\dots$	$x_0 = 0.5$
$f_{10}(x) = 10x\exp(-x^2) - 1$	$m = 17$	$\alpha = 0.1010\dots$	$x_0 = 0.25$

Table 2. Errors occurring in the estimates of the simple root by the methods described

f_i	(5)	(9): $i=1$	(9): $i=2$	(9): $i=3$	(9): $i=4$	(16)
f_1	0.785e-31	0.807e-56	0.791e-75	0.537e-97	0.230e-113	0.503e-138
f_2	0.139e-41	0.620e-89	0.574e-106	0.445e-102	0.588e-148	0.146e-202
f_3	0.720e-113	0.473e-296	0.115e-308	0.292e-306	0.208e-312	0.679e-685
f_4	0.318e-90	0.959e-216	0.932e-253	0.149e-243	0.194e-319	0.315e-509
f_5	0.130e-160	0.280e-344	0.386e-331	0.367e-347	0.139e-435	0.532e-748
f_6	0.865e-147	0.181e-311	0.129e-289	0.537e-304	0.221e-336	0.291e-716
f_7	0.172e-205	0.441e-567	0.192e-545	0.107e-556	0.252e-571	0.650e-1318
f_8	0.689e-74	0.880e-166	0.124e-181	0.148e-178	0.804e-253	0.174e-364
f_9	0.751e-138	0.319e-325	0.248e-351	0.517e-374	0.192e-406	0.342e-908
f_{10}	0.130e-93	0.176e-274	0.745e-200	0.274e-210	0.871e-284	0.259e-521

Table 3. Performance of computational order of convergence

f_i	(5)	(9): $i=1$	(9): $i=2$	(9): $i=3$	(9): $i=4$	(16)
f_1	3.0003	4.0000	4.0000	4.0000	4.0000	5.0000
f_2	3.0001	4.0000	4.0000	4.0000	4.0000	5.0000
f_3	3.0000	4.0000	4.0000	4.0000	4.0000	5.0000
f_4	3.0000	4.0000	4.0000	4.0000	4.0000	5.0000
f_5	3.0000	4.0000	4.0000	4.0000	4.0000	4.9995
f_6	3.0000	4.0000	4.0000	4.0000	4.0000	4.9994
f_7	3.0000	4.0000	4.0000	4.0000	4.0000	4.9998
f_8	3.0000	4.0000	4.0000	4.0000	4.0000	5.0000
f_9	3.0000	4.0000	4.0000	4.0000	4.0000	5.0000
f_{10}	3.0000	4.0000	4.0000	4.0000	4.0000	4.9989

The new fifth-order Simpson-type method given by (16) is employed to solve nonlinear equations with simple root. To demonstrate the performance of the new iterative method, ten particular nonlinear equations are used. The estimates are given of the approximate solutions produced by the methods considered and list the errors obtained by each of the methods. To determine the efficiency index of the new method, formula (3) will be used. Hence, the efficiency index of the new iterative method given by (16) is $\sqrt[4]{5} \approx 1.4953$, whereas the efficiency index of the fourth-order Simpson-type method given by (9) is $\sqrt[4]{4} \approx 1.4142$ and the efficiency index of the classical Simpson third-order method is given by (5) is $\sqrt[4]{3} \approx 1.3161$. Ten particular test functions are displayed in Table 1. The difference between the simple root and the approximation

x_n for test functions with initial guess x_0 are displayed in Table 2. In fact, x_n is calculated by using the same total number of function evaluations for all methods. Furthermore, we display the computational order of convergence approximations in Table 3. From the tables we observe that the computational order of convergence perfectly coincides with the theoretical result.

4. Conclusions

In this paper, we have demonstrated the performance of the new iterative method, namely the Simpson-type fifth-order iterative method. The prime motive of the development of the new fifth-order method was to establish a higher order of convergence method than the classical

Simpson third-order iterative method and recently introduced the Simpson-type fourth-order iterative methods. It is shown that the efficiency index of the new fifth-order method is much better than the fourth-order Simpson-type method and the classical Simpson third-order method. Numerical comparisons are made to demonstrate the performance of the derived method. We have examined the effectiveness of the new fifth-order iterative method by showing the accuracy of the simple root of a nonlinear equation.

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