Modified King's Methods with Optimal Eighth-order of Convergence and High Efficiency Index

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Abstract In this paper, based on King's methods, a new family of eighth-order methods for solving nonlinear equations is derived. This family of methods includes method given in [9] as a particular case. The optimal choice of the iteration parameter allows us to accelerate and improve the convergence of iterations. At each iteration of these methods requires three evaluation of the function and one evaluation of its first derivative, which has optimal efficiency index 1.682, according to Kung and Traube's conjecture. Numerical comparisons are made to show the performance of the presented methods.

Keywords Nonlinear equations, Modified King's method, Convergence order, Efficiency index

1. Introduction

In this work, we consider iterative methods to find a simple root x^* of a nonlinear equation

$$f(x) = 0, \qquad (1)$$

where $f : I \subset \mathsf{R} \to \mathsf{R}$ is a scalar function on the open interval I.

The constructing of higher-order convergence methods has attracted a lot of attention from both theoretical as well as practical point of view, see for example [1-9] and references therein.

King in [3] developed a one-parameter family of fourth-order methods, which is written as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)} \frac{f(y_n)}{f'(x_n)}, \quad (2)$$

where $\beta \in \mathsf{R}$ is a parameter. In particular, the famous Ostrowski's method is a member of this family when $\beta = 0$.

Recently, based on King's or Ostrowski's method, some optimal order iterative methods have been proposed and analyzed for solving nonlinear equations [1, 2, 5, 6, 8, 9]. In particular, Bi et al [1] presented a new family of eighth-order methods:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n) - \frac{1}{2} f(y_n)}{f(x_n) - \frac{5}{2} f(y_n)} \frac{f(y_n)}{f'(x_n)}, \qquad (3)$$

$$x_{n+1} = z_n - H(\mu_n) \frac{f(z_n)}{f[z_n, y_n] + (z_n - y_n) f[z_n, x_n, x_n]},$$

where $\mu_n = \dfrac{f(z_n)}{f(x_n)}$ and $H(\mu_n)$ represents a real-valued function, satisfying the conditions $H(0) = 1$, $H'(0) = 2$ and $|H''(0)| < \infty$ and divided differences are denoted by $f[,]$.

Wang and Liu developed in [9] a new modified Ostrowski's method:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \qquad (4)$$

$$x_{n+1} = z_n - \frac{f(z_n)}{2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] + (y_n - z_n) f[y_n, x_n, x_n]}.$$

Another modifications of Ostrowski's method were developed in [2, 5, 6]. The first two steps are the same as in (4) and the third step is different in these methods.

All the above mentioned optimal eighth-order methods have the efficiency index $8^{\frac{1}{4}} \approx 1.682$. In this paper based on King's method (2), we derive a new family of eighth-order methods for any $\beta \in \mathbb{R}$. It has a form:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)} \frac{f(y_n)}{f'(x_n)}, \qquad (5)$$

$$x_{n+1} = z_n - \frac{f(z_n)}{2f[x_n,z_n]+f[y_n,z_n]-2f[x_n,y_n]+(y_n-z_n)f[y_n,x_n,x_n]},$$

which includes the method (4) proposed in [9] as a particular case when $\beta = 0$. It requires, as in (3), (4) the evaluations of only three function and one first-order derivative per iteration. Therefore it has also efficiency index $8^{\frac{1}{4}} \approx 1.682$.

2. Construction of a Family of Iterative Methods and Convergence Analysis

We consider the following iterative methods

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)}\Delta_n, \quad \Delta_n = \frac{f(x_n)+\beta f(x_n)}{f(x_n)+(\beta-2)f(x_n)}, \qquad (6)$$

$$x_{n+1} = y_n + t(z_n - y_n), \qquad (7)$$

where $t > 0$ is a real parameter to be determined.

Note that the two first steps in (6) and (7) is King's iteration and the third step is a linear combination of the two approximations y_n and z_n obtained by preceding steps. We call this iteration (6) and (7) as a modified King's iterations.

It is well known that the King's method prescribed by the two first steps in (6) and (7) has a optimal fourth order convergence [4]. The third step in (6) and (7) can be considered as a accelerating procedure for King's method.

From (7) it clear that x_{n+1} belongs to interval connecting y_n and z_n under condition $0 \le t \le 1$ and does not belong to this interval when $t > 1$. Our aim is to find optimal value $t = t_{opt}$ in (7) such that the new approximation x_{n+1} will be situated more close to x^* as compared with y_n and z_n. From this we deduce that

$$t_{opt} > 1 \quad if \quad f(y_n)f(z_n) > 0, \qquad (8)$$

and

$$t_{opt} < 1 \quad if \quad f(y_n)f(z_n) < 0. \qquad (9)$$

Now we proceed to analyse the convergence of iteration (6) and (7). For convenience, we rewrite x_{n+1} as

$$x_{n+1} = z_n + (t-1)(z_n - y_n). \qquad (10)$$

We use Taylor expansion of $f \in C^2$ around z_n:

$$f(x_{n+1}) = f(z_n) + f'(z_n)(t-1)(z_n - y_n) + O\big(((t-1)(z_n - y_n))^2\big) \qquad (11)$$

Now we approximate $f'(z_n)$ using already computed function values. This can be done by the method of undetermined coefficients, such that

$$f'(z_n) = a_n f(x_n) + b_n f(y_n) + c_n f(z_n) + d_n f'(x_n) + O\big(f^4(x_n)\big) \qquad (12)$$

By using the Taylor expansions of $f(x_n)$, $f(y_n)$, $f'(x_n)$ around z_n, we obtain the following linear system of equations

$$a_n + b_n + c_n = 0,$$

$$a_n(x_n - z_n) + b_n(y_n - z_n) + d_n = 1,$$

$$a_n(x_n - z_n)^2 + b_n(y_n - z_n)^2 + 2d_n(x_n - z_n) = 0, \quad (13)$$

$$a_n(x_n - z_n)^3 + b_n(y_n - z_n)^3 + 2d_n(x_n - z_n)^2 = 0,$$

which has a unique solution:

$$a_n = \frac{\gamma_n(2\gamma_n - 3\omega_n)}{\omega_n(\gamma_n - \omega_n)^2}, \quad b_n = \frac{\omega_n^2}{\gamma_n(\gamma_n - \omega_n)^2},$$

$$c_n = -\frac{2\gamma_n + \omega_n}{\gamma_n \omega_n}, \quad d_n = -\frac{\gamma_n}{\gamma_n - \omega_n}, \qquad (14)$$

where

$$\omega_n = x_n - z_n = \frac{f(x_n)}{f'(x_n)}(1 + \theta_n \Delta_n),$$

$$\gamma_n = y_n - z_n = \frac{f(y_n)}{f'(x_n)}\Delta_n = \frac{f(x_n)}{f'(x_n)}\Delta_n \theta_n, \quad (15)$$

$$\Delta_n = \frac{1 + \beta\theta_n}{1 + (\beta - 2)\theta_n}, \quad \theta_n = \frac{f(y_n)}{f(x_n)}.$$

A simple calculations give us

$$a_n = -\frac{f'(x_n)}{f(x_n)} \frac{\theta_n \Delta_n (3 + \theta_n \Delta_n)}{1 + \theta_n \Delta_n},$$

$$b_n = \frac{f'(x_n)}{f(x_n)} \frac{(1 + \theta_n \Delta_n)^2}{\theta_n \Delta_n}, \tag{16}$$

$$c_n = -\frac{f'(x_n)}{f(x_n)} \frac{1 + 3\theta_n \Delta_n}{\theta_n \Delta_n (1 + \theta_n \Delta_n)}, \quad d_n = \theta_n \Delta_n.$$

Substituting (16) into (12) we get

$$f'(z_n) = A_n + O(f^4(x_n)), \tag{17}$$

where

$$A_n = f'(x_n)\left(\frac{-2\theta_n \Delta_n}{1 + \theta_n \Delta_n} + \frac{(1 + \theta_n \Delta_n)^2}{\Delta_n} - \frac{1 + 3\theta_n \Delta_n}{\Delta_n(1 + \theta_n \Delta_n)}\frac{f(z_n)}{f(y_n)}\right) \neq 0. \tag{18}$$

Substituting (17) into (11) we get

$$f(x_{n+1}) = (f(z_n) + A_n(t-1)(z_n - y_n))$$
$$+ O((t-1)^2(z_n - y_n)^2) \tag{19}$$
$$+ O((t-1)(z_n - y_n)f^4(x_n)).$$

Now we are ready to state the following convergence theorem for the family of methods (6) and (7).

Theorem 1 *Assume that the function $f(x)$ is sufficiently differentiable and $f(x)$ has a simple zero $x^* \in I$. If the initial point x_0 is sufficiently close to x^* and the parameter t is chosen by*

$$t - 1 = -\frac{f(z_n)}{A_n(z_n - y_n)}. \tag{20}$$

Then the method defined by (6) and (7) converges to x^* with eighth-order.

Proof. The first term in brackets in (19) vanishes under choice (20). From (6) we see that

$$f(y_n) = O(f^2(x_n)),$$

$$z_n - y_n = O(f(y_n)) = O(f^2(x_n)). \tag{21}$$

Since the King's method has a fourth order of convergence, we also have

$$f(z_n) = O(f^4(x_n)). \tag{22}$$

If we take (18) and (21), (22) into account, then from (20) we get

$$t - 1 = O(f^2(x_n)). \tag{23}$$

Using (21), (22) and (23) in (19) we arrive

$$f(x_{n+1}) = O(f^8(x_n)).$$

It means that there exists $M < \infty$ such that

$$|f(x_{n+1})| \leq M |f(x_n)|^8. \tag{24}$$

On the other hand, by mean-value theorem we have

$$f(x_{n+1}) - f(x^*) = f'(\xi)(x_{n+1} - x^*), \tag{25}$$

where ξ - some point lying between x_{n+1} and x^*.

Since $f'(x) \neq 0$ on the interval I, there exist the constants m_1 and M_1 such that

$$0 < m_1 \leq |f'(x)| < M_1 < \infty, \quad \forall x \in I. \tag{26}$$

Then using (24) and (26) in (25) we obtain

$$|x_{n+1} - x^*| \leq \frac{|f(x_{n+1})|}{m_1} \leq \frac{M}{m_1}|f(x_n) - f(x^*)|^8$$

$$\leq \frac{MM_1^8}{m_1}|x_n - x^*|^8 = \hat{M}|x_n - x^*|^8, \quad \hat{M} = \frac{MM_1^8}{m_1},$$

which completes the proof of Theorem 1.

According to (10) and (20) the third step (7) reads as

$$x_{n+1} = z_n - \frac{f(z_n)}{A_n}. \tag{27}$$

It is easy to show that

$$A_n = 2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n]$$
$$+ (y_n - z_n)f[y_n, x_n, x_n]. \tag{28}$$

Then the proposed iterative methods (6) and (7) can be rewritten as (11).

We call the value t given by (20), the optimal one.

Using (6) and (18) in (20) we obtain

$$t - 1 = \frac{f(z_n)}{f(y_n)} \frac{1 + \theta_n \Delta_n}{(1 + \theta_n \Delta_n)^3 - 2\theta_n \Delta_n^2 - (1 + 3\theta_n \Delta_n)\frac{f(z_n)}{f(y_n)}}, \tag{29}$$

where

$$\theta_n = \frac{f(y_n)}{f(x_n)} = O(f(x_n)), \quad \Delta_n = \frac{1 + \beta \theta_n}{1 + (\beta - 2)\theta_n}.$$

By virtue of (21) and (22) we have

$$\frac{f(z_n)}{f(y_n)} = O(f^2(x_n)).$$

Then from (29) we have

$$t - 1 = \frac{f(z_n)}{f(y_n)}(1 + O(f(x_n))). \tag{30}$$

The factor $1 + O(f(x_n)) > 0$ because of $O(f(x_n))$ is small enough. Thus the inequalities (8) and (9) will be hold for the approximations y_n and z_n sufficiently close to x^*.

Table 1.

Methods		$\|x^* - x_n\|$	$\|f(x_n)\|$	COC	$\|x^* - x_n\|$	$\|f(x_n)\|$	COC
		$f_1(x)$, $x_0 = 3.1$			$f_2(x)$, $x_0 = -1.3$		
BM8 method (4) in [9]		2.024(-87)	2.631(-86)	7.99417	2.326(-399)	4.724(-398)	8.00000
BM8-2 method (5) in [9]		3.784(-26)	4.919(-25)	7.99846	5.936(-409)	1.205(-407)	8.00000
KT method (21) in [9]		3.768(-132)	4.899(-131)	7.99916	5.234(-434)	1.063(-432)	8.00000
method (6) and (7)	$\beta = 2$	1.360(-102)	1.769(-101)	7.99512	7.033(-363)	1.428(-361)	8.00000
	$\beta = 1$	2.458(-116)	3.195(-116)	7.99788	2.929(-393)	5.949(-392)	8.00000
	$\beta = \frac{1}{2}$	2.834(-130)	3.684(-129)	7.99904	1.691(-422)	3.434(-421)	7.99999
	$\beta = 0$	8.451(-164)	1.098(-162)	7.99992	2.356(-506)	4.785(-505)	8.00000
	$\beta = -\frac{1}{2}$	5.881(-169)	7.645(-168)	8.00041	1.992(-535)	4.046(-534)	8.00000
	$\beta = -1$	3.565(-86)	4.635(-85)	8.00482	8.672(-413)	1.761(-411)	8.00000
		$f_3(x)$, $x_0 = 2.0$			$f_4(x)$, $x_0 = 1.0$		
BM8 method (4) in [9]		5.306(-164)	1.466(-163)	7.99612	7.921(-145)	2.934(-143)	7.99929
BM8-2 method (5) in [9]		3.711(-213)	1.025(-212)	7.99961	2.168(-189)	8.034(-188)	7.99993
KT method (21) in [9]		3.874(-140)	1.070(-139)	7.99881	7.612(-145)	2.820(-143)	7.99940
method (6) and (7)	$\beta = 2$	4.135(-57)	1.142(-56)	7.90175	4.038(-36)	1.496(-34)	7.81218
	$\beta = 1$	4.685(-111)	1.294(-109)	7.99371	1.410(-96)	5.224(-95)	7.99365
	$\beta = \frac{1}{2}$	7.163(-155)	1.979(-154)	7.99906	7.402(-142)	2.742(-140)	7.99924
	$\beta = 0$	1.233(-256)	3.410(-256)	7.99998	1.318(-244)	4.884(-243)	7.99999
	$\beta = -\frac{1}{2}$	3.959(-217)	1.094(-216)	7.99950	1.925(-197)	7.134(-196)	7.99985
	$\beta = -1$	4.819(-160)	1.332(-158)	7.99920	1.273(-146	4.718(-145)	7.99952
		$f_5(x)$, $x_0 = 1.9$			$f_6(x)$, $x_0 = 2.4$		
BM8 method (4) in [9]		4.851(-245)	2.911(-244)	7.99998	3.850(-508)	5.361(-507)	8.00000
BM8-2 method (5) in [9]		2.489(-281)	1.493(-280)	7.99999	2.360(-515)	3.286(-514)	8.00000
KT method (21) in [9]		7.548(-236)	4.528(-235)	7.99998	8.135(-451)	1.132(-449)	8.00000
method (6) and (7)	$\beta = 2$	8.704(-143)	5.222(-142)	7.99901	1.541(-417)	2.146(-416)	8.00000
	$\beta = 1$	9.894(-201)	5.936(-200)	7.99992	5.671(-447)	7.897(-446)	8.00000
	$\beta = \frac{1}{2}$	8.030(-243)	4.818(-242)	7.99998	8.058(-473)	1.122(-471)	7.99999
	$\beta = 0$	2.717(-331)	1.630(-330)	7.99999	1.396(-523)	1.944(-522)	8.00000
	$\beta = -\frac{1}{2}$	3.042(-308)	1.825(-307)	7.99999	5.944(-591)	8.278(-590)	8.00000
	$\beta = -1$	1.361(-251)	8.168(-251)	7.99999	6.457(-485)	8.992(-484)	8.00000

3. Numerical Experiments

We takes six examples from [9],

(a) $f_1(x) = \exp(x^2 + 7x - 30) - 1$,
(b) $f_2(x) = x \exp(x^2) - \sin^2(x) + 3\cos(x) + 5$,
(c) $f_3(x) = 10x \exp(-x^2) - 1$,
(d) $f_4(x) = x^5 + x^4 + 4x^2 - 15$,
(e) $f_5(x) = (x - 1)^6 - 1$,
(f) $f_6(x) = x^3 - 10$,

All numerical calculations were performed using Maple 18 system. To study the convergence of iterations, we compute the computational order of convergence (COC) of d_{x_n} using the formulae

$$d_{x_n} = \frac{\ln(\|x_n - x^*\| / \|x_{n-1} - x^*\|)}{\ln(\|x_{n-1} - x^*\| / \|x_{n-2} - x^*\|)}, \quad (31)$$

where x_n, x_{n-1}, x_{n-2} are three consecutive approximations of iterations.

In this Table 1, we present the results of some numerical tests to compare the efficiencies of the methods. We employed, the BM8 method with $H(\mu_n) = \dfrac{1 + 3\mu_n}{1 + \mu_n}$

((4) in [9]), the BM8-2 method with $h(\mu_n) = \left(\dfrac{1}{1 - 3\mu_n}\right)^{\frac{2}{3}}$

((5) in [9]), the method (12) in [9], the KT method ((21) in [9]) and new method (6) and (7). In Table 1, $f_i(x) = 0$, $(i = 1, 2, \ldots, 6)$ are the test functions, x_0 is the initial approximation.

The factor l in the brackets denotes 10^l. From tables we see that the COC perfectly coincides with the theoretical order and new method (6) and (7) is comparable with other optimal order methods.

4. Conclusions

We propose a new family of eighth–order methods based on King's methods. The high order of convergence is obtained by acceleration procedure. Numerical results clearly demonstrate the theoretical analysis (speed of convergence). Moreover, our acceleration procedure can also be applied to any iteration, for which will be devoted forth coming paper.

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