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Right Ideals and Generalized Reverse Derivations on Prime Rings

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Abstract Let R be a prime ring and d be a reverse derivation on R. If f is a generalized reverse derivation on R such that f is commuting and centralizing on a right ideal I of R, then R is a commutative.

Keywords Prime rings, Right ideals, Reverse derivations, Generalized reverse derivations, Centralizing and Commuting generalized reverse derivations

1. Introduction

Let R be a ring with center Z. R is said to be prime if aRb = 0 implies that either a = 0 or b = 0. A mapping f is said to be commuting on a right ideal I of R if

[x, f(x)] = 0 for all $x \in I$ and f is said to be centralizing if [x, f(x)] $\in$ Z (R) for all $x \in I$. An additive mapping d: R $\rightarrow$R is called a derivation if d (xy) = d (x) y + xd (y) for all x, y $\in$ R, and d is called a reverse derivation if d (xy) =d(y)x+ yd(x) for all

x, y $\in$ R. The notion of reverse derivation has been introduced by Bresar and Vukman [3], and the reverse derivations on semi prime rings have been studied by Samman and Alyamani [6]. An additive mapping f: R $\rightarrow$R is said to be a generalized derivation on R if f (xy) = f (x) y + xd (y) for all x, y $\in$ R, where d is a derivation on R, and f is said to be a generalized reverse derivation on R if

f (xy)= f (y) x + yd (x) for all x, y $\in$ R, where d is a reverse derivation on R.

A. Aboubakr and S. Gonzalez [1] studied the relationship between generalized reverse derivation and generalized derivation on an ideal in semi prime rings, and in [4] the authors proved that in case R is a prime ring with a non-zero right reverse derivation d and U is the left ideal of R then R is commutative.

In this paper we prove that a prime ring R is commutative if f is a generalized reverse derivation on R with a non zero derivation d on R such that f is centralizing and commuting on a right ideal I of R.

2. Preliminaries

Throughout, R will represent a prime ring with center Z.

Let [x, y] = x y - y x with the important identity [x, yz] = y[x, z] + [x, y] z, and

[xy, z] = [x, z] y + x[y, z], for all x, y, z $\in$ R.

In order to prove the main results, we being with following preliminary results.

Remark (2.1) [2]: Let R be a prime ring. For a nonzero element a $\in$ Z (R), if

ab $\in$ Z(R), then b $\in$ Z (R).

Lemma (2.2): Let R be a prime ring, and d be a reverse derivation on R. For an element a $\in$ R, if a d(r) = 0 for all r $\in$ R, then either a = 0 or d = 0.

Proof: For a $\in$ R, let a d(r) = 0 for all r $\in$ R. (1)

Replace r by sr in (1), we have a d (sr) = 0, then

$$a\ d(r)s + a\ rd(s) = 0, \text{ for all } r, s \in R. \quad (2)$$

By used (1) in (2), we have a r d(s) = 0 for all r, s $\in$ R.

If d(s) $\neq$ 0 for some s $\in$ R, then a = 0 by definition of prime ring. Hence proved.

Lemma (2.3): Let I be a nonzero right ideal of a prime ring R. If R has a zero reverse derivation d on I, then d is also zero reverse derivation on R.

Proof: Let I $\neq$ {0} is a right ideal of R.

We assume that d (I) =0. (3)

Since IR $\subseteq$ R we have:

$$d\ (IR)= d(R)I+ Rd(I) = 0. \quad (4)$$

By substitute (3) in (4), we have d(R) I= {0}.

Since I $\neq$ {0}, then by lemma (2.2), d(R) = 0.

Lemma (2.4) [5]: Let R be a prime ring and I a non zero right ideal of R. If I is commutative, then R is also commutative.

3. Main Results

Theorem (3.1): Let R be a prime ring, and I be a non zero right ideal of R. If d is a non zero reverse derivation on R,

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such that d is a centralizing on I, then R is commutative.

Proof: Let d be a centralizing on I, then we have

$$[a, d(a)] \in Z(R), \text{ for all } a \in I. \quad (5)$$

Replacing a by a^2 in (5), we get

$$[a^2, d(a)a + ad(a)] \in Z(R). \quad (6)$$

By add and subtract a d(a) in (6), we get
$[a^2, 2ad(a) - [a, d(a)]] \in Z(R)$, for all $a \in I$. That's equal to $2[a^2, ad(a)] = 4a^2[a, d(a)] \in Z(R)$.
Thus, $4[a^2[a, d(a)], d(a)] = 0$, for all $a \in I$.

$$\text{And } 2[a, d(a)] = 0, \text{ for all } a \in I. \quad (7)$$

$$\text{Also } [a^2, d(a)] = 0, \text{ for all } a \in I. \quad (8)$$

Now, linearizing both (5) and (7), we get
$[a, d(b)] + [b, d(a)] \in Z(R)$, and $2[a, d(b)] + [b, d(a)] = 0$, for all $a, b \in I$.

By combining these results with (7), we can show that

$$[ab + ba, d(a)] + [a^2, d(b)] = 0, \text{ for all } a, b \in I. \quad (9)$$

Replacing b by ba in (9), and using (8) and (9), we get
$[ab + ba, d(a)]a - a[ab + ba, d(a)] + d(a)[a^2, b] = 0$, for all $a, b \in I$.

Then, we get

$$[[b, d(a)], a^2] + d(a)[a^2, b] = 0, \text{ for all } a, b \in I. \quad (10)$$

Replacing $[b, d(a)]$ by a^2 in (10), we get

$$d(a)[a^2, b] = 0, \text{ for all } a, b \in I. \quad (11)$$

Replacing d(a) by d(a) r in (11), we get

$$d(a)\, r\, [a^2, b] = 0, \text{ for all } a, b \in I \text{ and } r \in R.$$

Since R is a prime, we have $d(a) = 0$ or $[a^2, b] = 0$.

If $d(a) = 0$ for all $a \in I$ then by lemma (2.2), $d(R) = 0$ this is a contradiction.

So $[a^2, b] = 0$, for all $a, b \in I$, that's mean I is commutative and hence by lemma (2.4), R is commutative.

Theorem (3.2): Let R be a prime ring, and I be a right ideal of R. If f is a generalized reverse derivation on R with a reverse derivation d on R, such that f is centralizing on I, then for all $a \in I \cup Z(R)$, $f(a) \in Z(R)$.

Proof: Since f is centralizing on I, we have

$$[a, f(a)] \in Z(R), \text{ for all } a \in I. \quad (12)$$

By linearizing (12) for all $a, b \in I$, we have

$$[a, f(b)] + [b, f(a)] \in Z(R). \quad (13)$$

$$\text{If } a \in Z(R), \text{ this implies that } [b, f(a)] \in Z(R). \quad (14)$$

Replacing b by b f(a) in (14), we get

$$[b, f(a)]\, f(a) \in Z(R), \text{ for all } a, b \in I.$$

If $[b, f(a)] = 0$, then $f(a) \in C_R(I)$, the centralizer of I in R, and hence $f(a) \in Z(R)$. On the other hand if $[b, f(a)] \neq 0$, then by remark (2.1), we get $f(a) \in Z(R)$.

Theorem (3.3): Let I be a nonzero right ideal of a prime ring R, and f is a generalized reverse derivation on R with a non zero reverse derivation d on R. If f is commuting on I, then R is commutative.

Proof: Let f is commuting on I, then for all $a \in I$,

$$\text{we have } [a, f(a)] = 0. \quad (15)$$

Replacing a by a+b in (15), we get

$$[a, f(b)] + [b, f(a)] = 0. \quad (16)$$

Substituting b = ba in (16), and using (15), we get

$$f(a)[a, b] + a[a, d(b)] + [b, f(a)]a = 0, \text{ for all } a, b \in I. \quad (17)$$

Replacing a by b in (17) and using (15), we get

$$b[b, d(b)] = 0, \text{ for all } a, b \in I. \quad (18)$$

Now we substituting $d(b) = d(b)r$ in (17), and using (18), we get

$$b\, d(b)\, [b, r] = 0, \text{ for all } b \in I, \text{ and } r \in R. \quad (19)$$

Replacing r by rs in (19), and using (19), we get

$$b\, d(b)\, r\, [b, s] = 0, \text{ for all } b \in I, \text{ and } r, s \in R.$$

Since R is a prime ring, and $b\, d(b) \neq 0$, then $[b, s] = 0$, for all $b \in I$, and $s \in R$. Therefore $b \in Z(R)$ and so $I \subseteq Z(R)$, which implies that I is commutative and by lemma (2.4), R is commutative.

Theorem (3.4): Let R be a prime ring, and I be a right ideal of R such that
$I \cap Z(R) \neq 0$. Let f be generalized reverse derivations on R with a non zero reverse derivation d on R. If f is commuting on I, then R is commutative.

Proof: Let we take $Z(R) \neq 0$, since f is commuting on I then the proof is complete.

Now, by equation (13), we have

$$[a, f(b)] + [b, f(a)] \in Z(R), \text{ for all } a, b \in I.$$

If we replacing b by ar, where $0 \neq r \in Z(R)$, we get

$$[a, f(r)]a + r[a, d(a)] + [a, f(a)]r \in Z(R),$$
$$\text{for all } a \in I, \text{ and } r \in R. \quad (20)$$

By using lemma (2.2) in (20) we have $f(r) \in Z(R)$, and since f is centralizing on I,

$$\text{we get } r[a, d(a)] \in Z(R), \text{ for all } a \in I, \text{ and } r \in R. \quad (21)$$

By using remark (2.1) in (21), we get $[a, d(a)] \in Z(R)$, for all $a \in I$. And hence by theorem (3.1), R is commutative.

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