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Continuous Hybrid Multistep Methods with Legendre Basis Function for Direct Treatment of Second Order Stiff ODEs

Olabode B. T.*, Momoh A. L.

Department of Mathematical Sciences, Federal University of Technology, Akure, Nigeria

Abstract This article proposed continuous hybrid multistep methods with Legendre polynomial as basis functions for the direct solution of system of second order ordinary differential equations. This was achieved by constructing a continuous representation of hybrid multistep schemes via interpolation of the approximate solution and collocation of derivative function with Legendre polynomial as basis functions. The discrete schemes were obtained from the continuous scheme as a by-product and applied in block form as simultaneous numerical integrators to solve initial value problems (IVPs). The resultant schemes are self-starting, do not need the development of separate predictors, consistent, zero-stable and convergent. The performance of the methods was demonstrated on some numerical examples to show accuracy and efficiency advantages. The numerical results compared favourably with existing method.

Keywords Continuous, Hybrid, Multistep method, Legendre polynomial, Stiff equations, Initial Value Problems (IVPs), Ordinary Differential Equations (ODEs)

1. Introduction

The mathematical modeling of physical phenomena in science and engineering field especially in mechanical systems with several springs attached in series or dissipation, control theory, celestial mechanics, series circuits lead to a system of differential equations (see Landau and Lifshitz (1965), Liboff (1980)). Realistically, the analytical solutions of most differential equations are not easily obtainable. This necessitated the need for approximate solution by the application of numerical techniques.

The techniques for the derivation of continuous linear multistep methods (LMMs) for direct solution of initial value problems in ordinary differential equations have been discussed in literature over the years and these include, among others collocation, interpolation, integration and interpolation polynomials. Basis functions such as, power series, Chebyshev polynomials, trigonometric functions, the monomials x^r, the canonical polynomial $(Q_r(x), \quad r \geq 0)$ of the Lanczos Tau method in a perturbed collocation approach have been employed for this purpose (see Abualnaja (2015); Adeyefa *et al.*, (2014); Awoyemi and Idowu (2005); Lambert (1991)).

* Corresponding author:
olabodebola@yahoo.com (Olabode B. T.)
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Moreover, power series has also being extensively used in literature for the same purpose. Sirisena *et al.*, (2004) proposed a new Butcher type two-step block hybrid multistep method for accurate and efficient parallel solution of order ordinary differential equations. Awoyemi and Idowu (2005) developed a class of hybrid collocation methods for third order ordinary differential equations with power series as the basis functions and were implemented in predictor –corrector mode. Ehigie *et al.*, (2010) worked on generalized two-step continuous linear multistep method of hybrid type for the direct integration of second order ordinary differential equations. Fudzial *et al.*, (2009) constructed the explicit and implicit 3-point-1- block (I3P1B) for solving special second order ordinary equations directly. Awari (2013) considered the derivation and application of six-point linear multistep numerical method for the solution of second order initial value problems which was implemented in block mode. Yusuph and Onumanyi (2002) demonstrated a successful application of multiple finite difference methods through multistep collocation for the second order ordinary differential equations.

Furthermore, Abualnaja (2015) constructed a block procedure with linear multistep methods using Legendre polynomial for solving first order ordinary differential equations. The method depends on the perturbed collocation approximation with Legendre as perturbation term for the solution of first order ordinary differential equations. In the work of Yakusak *et al.*, (2015), uniform order Legendre approach for continuous hybrid block methods were proposed for the solution of first order ordinary differential equations.

In this paper, we propose the choice of Legendre polynomial without perturbation as basis functions for the construction of continuous schemes, which simultaneously generate solution of (1). They are self-starting and do not need any predictors.

Preliminaries

A central notion in this work concerns the choice of Legendre polynomial as basis functions in the derivation of the continuous schemes, the implementation strategies employed (in block mode) and the stability analysis of the methods. For convenience of the reader, we recall the definitions as follows:

Definition 1.1: The block method is said to be zero-stable if the roots λ_j, $j = 1, 2, \ldots s$ of the characteristic polynomial $\rho(\lambda)$ defined by $\rho(\lambda) = \left|\sum_{i=0}^{s} A^i \lambda^{s-i}\right| = 0$ satisfies $|\lambda_j| \leq 1$ and for those roots with $|\lambda_j| = 1$, the multiplicity must not exceed the order of the differential equation. (see Fatunla (1994)).

Definition 1.2: The set of W equals

$\tau \in C$; all roots $\xi_i(\tau)$ of the characteristic equation

satisfy $|\xi_i(\tau)|$, multiple roots satisfy $|\xi_i(\tau)| < 1$

is called the stability region or region of absolute stability of the method (Hairer and Wanner (1996)).

Definition: 1.3 (Widlund (1967))

A method is said to be A(α)-stable if the sector

$$S_\alpha = \left\{ z : |\arg(-z)| \leq \alpha, z \neq 0 \right\}$$

is contained in the stability region.

Definition 1.4 (Ehle (1969)): A method is called L-stable if it is A-stable and if in addition $\lim_{z\to\infty} R(z) = 0$.

Definition 1.5 (Olagunju *et al.,* (2012)): Legendre polynomial is special case of the Legendre function which satisfy the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad n > o, \quad |x| < 1.$$

The general solution can be expressed as:

$$y = AP_n(x) + BQ_n(x), \quad |x| < 1.$$

$P_n(x)$ and $Q_n(x)$ are respectively the Legendre functions of the first-and second-kind of the order n. The nth order polynomial $P_n(x)$ is generally given by the following equation:

$$P_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \left(\frac{n!}{2n!} \right) \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}$$

where n is the order of the Legendre polynomials, $\left\lfloor \frac{n}{2} \right\rfloor$ signifies the integer part of $\frac{n}{2}$. Legendre polynomials are orthogonal to each other with respect to weight function w(x) = 1 on [-1,1]. The first two polynomials are always the same in all cases but the higher orders are created with recursive formula:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n = 1, 2, \ldots$$

with initial conditions: $p_0(x) = 1$, $p_1(x) = x$. The first four terms of the polynomial are; $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$, $p_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$.

The paper is organized as follows. Section 1 is of an introductory nature. The materials and methods are described in Section 2. Stability analysis of the methods is discussed in Section 3. In Section 4, some numerical experiments and results showing the relevance of the new methods are discussed. Finally, in Section 5 some conclusions are drawn.

2. Mathematical Formulation

Consider the second-order initial value problem:

$$y'' = f(x, y, y'), y(a) = \eta_0, y'(a) = \eta_1 \quad (1)$$

where $f \in R$ is sufficiently differentiable and satisfies a Lipschitz condition, sufficiently smooth, $f: \mathbb{R}^{m+1} \to \mathbb{R}^m$,y is an m-dimensional vector and x is a scalar variable and a set of equally spaced points on the integration interval also given by

$$a = x_0 < x_1 < \ldots < x_n < \cdots < x_{n+k} < x_N = b, \quad (2)$$

with a specified positive integer step number k greater than zero, h can be variable or constant step-size given by

$$h = x_{n+1} - x_n, n = 1, \ldots, N; hN = b - a.$$

Assuming an approximate solution to (1) by taking the partial sum of Legendre polynomial of the form:

$$y(x) = \sum_{r=0}^{t+s-1} a_r P_r(x), x_n \leq x \leq x_{n+r} \quad (3)$$

where x can be used only after certain transformation. The second derivative of (3) gives

$$y''(x) = \sum_{r=0}^{t+s-1} a_r P_r''(x) \quad (4)$$

Substituting (4) into (1) gives

$$\sum_{r=0}^{t+s-1} a_r P_r''(x) = f(x, y(x), y'(x)), x_n \leq x \leq x_{n+r} \quad (5)$$

where $P_r(x)$ is the Legendre polynomial of degree r, valid in $x_n \leq x \leq x_{n+r}$ and a_r's are real unknown parameters to be determined and ($t+s-1$) is the sum number of collocation and interpolation points. The well-known Legendre polynomials are defined on the interval [-1,1].

2.1. Derivation of the Continuous Hybrid Multistep Methods

Our objective here is to construct a continuous formulation of the general linear multistep method $\overline{y}(x)$ of degree $r=t+s-1 \quad t>0, s>0$. Two cases were considered one-step and two-step methods.

CASE 1: One-Step Continuous Hybrid Multistep Method (OSCHMM).

Collocating (5) at points, and interpolating (3) at points $x=x_{n+s}, s=0,\frac{1}{4},$ respectively lead to a system of equations expressed in matrix form:

$$MD=U \tag{6}$$

where,

$$M=\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{8} & \frac{7}{16} & -\frac{37}{128} & -\frac{23}{256} & \frac{331}{1024} \\ 0 & 0 & 12 & -60 & 180 & -420 & 840 \\ 0 & 0 & 12 & -30 & \frac{45}{2} & \frac{105}{4} & -\frac{2415}{32} \\ 0 & 0 & 12 & 0 & -30 & 0 & \frac{105}{2} \\ 0 & 0 & 12 & 30 & \frac{45}{2} & -\frac{105}{4} & -\frac{2415}{32} \\ 0 & 0 & 12 & 60 & 180 & 420 & 840 \end{pmatrix}$$

$D=[a_0,a_1,a_2,a_3,a_4,a_5,a_6,]^T$ and $U=[y_n,y_{n+1/4},f_n,f_{n+1/4},f_{n+1/2},f_{n+3/4},f_{n+1}]^T$.

Solving (6) using Gaussian Elimination method in Maple soft environment produces the following values of $a_r{}'s$:

$$a_0=-y_n+2y_{n+\frac{1}{4}}+h^2\left(\frac{407}{80640}f_n+\frac{131}{4480}f_{n+\frac{1}{4}}+\frac{1231}{20160}f_{n+\frac{1}{2}}+\frac{181}{20160}f_{n+\frac{3}{4}}-\frac{13}{80640}f_{n+1}\right) \tag{7}$$

$$a_1=-2y_n+2y_{n+\frac{1}{4}}+h^2\left(\frac{727}{80640}f_n+\frac{37}{640}f_{n+\frac{1}{4}}+\frac{1999}{20160}f_{n+\frac{1}{2}}+\frac{437}{20160}f_{n+\frac{3}{4}}-\frac{13}{80640}f_{n+1}\right) \tag{8}$$

$$a_2=h^2\left(\frac{1}{1512}f_n+\frac{4}{189}f_{n+\frac{1}{4}}+\frac{5}{126}f_{n+\frac{1}{2}}+\frac{4}{126}f_{n+\frac{3}{4}}-\frac{1}{1512}f_{n+1}\right) \tag{9}$$

$$a_3=h^2\left(-\frac{1}{1080}f_n+\frac{4}{189}f_{n+\frac{1}{4}}-\frac{2}{135}f_{n+\frac{3}{4}}+\frac{1}{1080}f_{n+1}\right) \tag{10}$$

$$a_4=h^2\left(\frac{13}{13860}f_n+\frac{4}{693}f_{n+\frac{1}{4}}-\frac{31}{2310}f_{n+\frac{1}{2}}+\frac{4}{693}f_{n+\frac{3}{4}}-\frac{13}{13860}f_{n+1}\right) \tag{11}$$

$$a_5=h^2\left(-\frac{1}{945}f_n+\frac{2}{945}f_{n+\frac{1}{4}}-\frac{2}{945}f_{n+\frac{3}{4}}-\frac{1}{945}f_{n+1}\right) \tag{12}$$

$$a_6=h^2\left(\frac{4}{10395}f_n-\frac{16}{10395}f_{n+\frac{1}{4}}+\frac{8}{3465}f_{n+\frac{1}{2}}+-\frac{16}{10395}f_{n+\frac{3}{4}}+\frac{16}{10395}f_{n+1}\right) \tag{13}$$

Substituting (7)-(13) into equation (3) and after some manipulation gives the continuous scheme

$$
\begin{aligned}
\bar{y}(x) = &\left(1-4\left(\frac{x-x_n}{h}\right)\right)y_n + 4\left(\frac{x-x_n}{h}\right)y_{n+\frac{1}{4}} \\
&+\frac{h^2}{5760}\left(-367\left(\frac{x-x_n}{h}\right)+2880\left(\frac{x-x_n}{h}\right)^2-8000\left(\frac{x-x_n}{h}\right)^3+11200\left(\frac{x-x_n}{h}\right)^4-7680\left(\frac{x-x_n}{h}\right)^5+2048\left(\frac{x-x_n}{h}\right)^6\right)f_n \\
&-\frac{h^2}{1440}\left(135\left(\frac{x-x_n}{h}\right)-3840\left(\frac{x-x_n}{h}\right)^3+8320\left(\frac{x-x_n}{h}\right)^4-6912\left(\frac{x-x_n}{h}\right)^5+2048\left(\frac{x-x_n}{h}\right)^6\right)f_{n+\frac{1}{4}} \\
&+\frac{h^2}{960}\left(47\left(\frac{x-x_n}{h}\right)-1920\left(\frac{x-x_n}{h}\right)^3+6080\left(\frac{x-x_n}{h}\right)^4-6144\left(\frac{x-x_n}{h}\right)^5+2048\left(\frac{x-x_n}{h}\right)^6\right)f_{n+\frac{1}{2}} \\
&-\frac{h^2}{1440}\left(29\left(\frac{x-x_n}{h}\right)-1280\left(\frac{x-x_n}{h}\right)^3+4480\left(\frac{x-x_n}{h}\right)^4-5376\left(\frac{x-x_n}{h}\right)^5-2448\left(\frac{x-x_n}{h}\right)^6\right)f_{n+\frac{3}{4}}
\end{aligned} \tag{14}
$$

Evaluating (14) at $x = x_{n+1}$ gives the discrete scheme

$$y_{n+1} = 4y_{n+\frac{1}{4}} - 3y_n + \frac{h^2}{1920}\left(27f_n + 332f_{n+\frac{1}{4}} + 222f_{n+\frac{1}{2}} + 132f_{n+\frac{3}{4}} + 7f_{n+1}\right) \tag{15}$$

The discrete scheme (15) is consistent, zero-stable and of order $p = 5$ with error constant $C_{p+2} = \frac{1}{1966080}$.

Here, it is our intention to get additional discrete schemes, so, we evaluated (14) at the points $x = x_{n+i}, i = \frac{3}{4}, \frac{1}{2}$ to obtain:

$$y_{n+\frac{3}{4}} = 3y_{n+\frac{1}{4}} - 2y_n + \frac{h^2}{3840}\left(37f_n + 432f_{n+\frac{1}{4}} + 222f_{n+\frac{1}{2}} + 32f_{n+\frac{3}{4}} - 3f_{n+1}\right) \tag{16}$$

$$y_{n+\frac{1}{2}} = 4y_{n+\frac{1}{4}} - 4y_n + \frac{h^2}{5760}\left(367f_n + 540f_{n+\frac{1}{4}} - 282f_{n+\frac{1}{2}} + 116f_{n+\frac{3}{4}} - 21f_{n+1}\right) \tag{17}$$

The first derivative of (14) is found and evaluated at points $x = x_{n+i}, i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ yields the following derivative schemes:

$$hy'_n = 4y_{n+\frac{1}{4}} - 4y_n - \frac{h^2}{5760}\left(367f_n + 540f_{n+\frac{1}{4}} - 282f_{n+\frac{1}{2}} + 116f_{n+\frac{3}{4}} - 21f_{n+1}\right) \tag{18}$$

$$hy'_{n+\frac{1}{4}} = 4y_{n+\frac{1}{4}} - 4y_n + \frac{h^2}{5760}\left(135f_n + 752f_{n+\frac{1}{4}} - 246f_{n+\frac{1}{2}} + 96f_{n+\frac{3}{4}} - 17f_{n+1}\right) \tag{19}$$

$$hy'_{n+\frac{1}{2}} = 4y_{n+\frac{1}{4}} - 4y_n + \frac{h^2}{5760}\left(97f_n + 1444f_{n+\frac{1}{4}} + 666f_{n+\frac{1}{2}} - 52f_{n+\frac{3}{4}} + 5f_{n+1}\right) \tag{20}$$

$$hy'_{n+\frac{1}{4}} = 4y_{n+\frac{1}{4}} - 4y_n + \frac{h^2}{5760}\left(81f_n + 1508f_{n+\frac{1}{4}} + 1050f_{n+\frac{1}{2}} + 1932f_{n+\frac{3}{4}} + 469f_{n+1}\right) \tag{21}$$

$$hy'_{n+1} = 4y_{n+\frac{1}{4}} - 4y_n + \frac{h^2}{5760}\left(81f_n + 1508f_{n+\frac{1}{4}} + 1050f_{n+\frac{1}{2}} + 1932f_{n+\frac{3}{4}} + 469f_{n+1}\right) \tag{22}$$

2.2. Implementation of the One-Step Continuous Hybrid Multistep Method (OSCHMM).

In this section, the implementation strategy of this work is discussed. Following Fatunla (1991, 1994), the general discrete block formula is given as:

$$A^0 Y_m = e y_n + h^{\mu} dF(Y_m) + h^{\mu} BF(Y_m) \tag{23}$$

where e, d are vectors, B are RxR matrix and A^0 identity matrix, μ is the order of differential equation. Expressing equations (20) - (23) and (24) in form of (23) and solving with matrix inversion method gives:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-\frac{1}{4}} \\ y_{n-\frac{1}{2}} \\ y_{n-\frac{3}{4}} \\ y_n \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y'_{n-\frac{1}{4}} \\ y'_{n-\frac{1}{2}} \\ y'_{n-\frac{3}{4}} \\ y'_n \end{pmatrix} +$$
$$h^2 \begin{pmatrix} 0 & 0 & 0 & \frac{367}{23040} \\ 0 & 0 & 0 & \frac{53}{1440} \\ 0 & 0 & 0 & \frac{147}{2560} \\ 0 & 0 & 0 & \frac{7}{90} \end{pmatrix} \begin{pmatrix} f_{n-\frac{1}{4}} \\ f_{n-\frac{1}{2}} \\ f_{n-\frac{3}{4}} \\ f_n \end{pmatrix} + h^2 \begin{pmatrix} \frac{3}{128} & -\frac{47}{3840} & \frac{29}{5760} & -\frac{7}{7680} \\ \frac{1}{10} & -\frac{1}{48} & \frac{1}{90} & -\frac{1}{480} \\ \frac{117}{640} & \frac{27}{1280} & \frac{3}{128} & -\frac{9}{2560} \\ \frac{4}{15} & \frac{1}{15} & \frac{4}{45} & 0 \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{pmatrix} \tag{24}$$

Writing (24) explicitly

$$y_{n+\frac{1}{4}} = y_n - \frac{h}{4} y'_n + \frac{h^2}{23040}\left(367 f_n + 540 f_{n+\frac{1}{4}} - 282 f_{n+\frac{1}{2}} + 116 f_{n+\frac{3}{4}} - 21 f_{n+1}\right) \tag{25}$$

$$y_{n+\frac{1}{2}} = y_n - \frac{h}{2} y'_n + \frac{h^2}{1440}\left(53 f_n + 144 f_{n+\frac{1}{4}} - 30 f_{n+\frac{1}{2}} + 16 f_{n+\frac{3}{4}} - 3 f_{n+1}\right) \tag{26}$$

$$y_{n+\frac{3}{4}} = y_n - \frac{3h}{4} y'_n + \frac{3h^2}{2560}\left(49 f_n + 156 f_{n+\frac{1}{4}} + 18 f_{n+\frac{1}{2}} + 20 f_{n+\frac{3}{4}} - 3 f_{n+1}\right) \tag{27}$$

$$y_{n+1} = y_n - h y'_n + \frac{h^2}{90}\left(7 f_n + 24 f_{n+\frac{1}{4}} + 6 f_{n+\frac{1}{2}} + 8 f_{n+\frac{3}{4}}\right) \tag{28}$$

The block method is of uniform order $p = (5,\ 5,\ 5,\ 5)^T$ with error constant $C_{p+2} = \left(\frac{107}{165150720}, \frac{1}{645120}, \frac{9}{3670016}, \frac{1}{322560}\right)^T$.

Substituting (25) into (19)-(22) yields

$$y'_{n+\frac{1}{4}} = y'_n + \frac{h}{2880}\left(251 f_n + 646 f_{n+\frac{1}{4}} - 264 f_{n+\frac{1}{2}} + 106 f_{n+\frac{3}{4}} - 19 f_{n+1}\right) \tag{29}$$

$$y'_{n+\frac{1}{2}} = y'_n + \frac{h}{360}\left(29 f_n + 124 f_{n+\frac{1}{4}} + 24 f_{n+\frac{1}{2}} + 4 f_{n+\frac{3}{4}} - f_{n+1}\right) \tag{30}$$

$$y'_{n+\frac{3}{4}} = y'_n + \frac{3h}{320}\left(9 f_n + 34 f_{n+\frac{1}{4}} + 24 f_{n+\frac{1}{2}} + 14 f_{n+\frac{3}{4}} - f_{n+1}\right) \tag{31}$$

$$y'_{n+1} = y'_n + \frac{h}{90}\left(7f_n + 32f_{n+\frac{1}{4}} + 12f_{n+\frac{1}{2}} + 32f_{n+\frac{3}{4}} + 7f_{n+1}\right) \quad (32)$$

Equations (25)-(32) are then applied in block form as simultaneous numerical integrators to solve (1).

CASE 2: Two-Step Continuous Hybrid Multistep Method (TSCHMM).

Similarly, collocating (5) at points, and interpolating (3) at points $x = x_{n+s}, s = 0, \frac{1}{2}$ lead to a system of equations of form (6) where,

$$M = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -\frac{1}{2} & \frac{1}{8} & -\frac{7}{16} & -\frac{37}{128} & -\frac{23}{256} & \frac{331}{1024} \\ 0 & 0 & 3 & -15 & 45 & -105 & 210 \\ 0 & 0 & 3 & -\frac{15}{2} & \frac{45}{8} & \frac{105}{16} & -\frac{2415}{128} \\ 0 & 0 & 3 & 0 & -\frac{15}{2} & 0 & \frac{105}{8} \\ 0 & 0 & 3 & \frac{15}{2} & \frac{45}{8} & -\frac{105}{16} & -\frac{2415}{128} \\ 0 & 0 & 3 & 15 & 45 & 105 & 210 \end{pmatrix}$$

$D = [a_0, a_1, a_2, a_3, a_4, a_5, a_6,]^T$ and $U = [y_n, y_{n+1/2}, f_n, f_{n+1/2}, f_{n+1}, f_{n+3/2}, f_{n+2}]^T$. The continuous scheme is as follows :

$$\begin{aligned}
\overline{y}(x) &= \left(1 - 2\left(\frac{x-x_n}{h}\right)\right)y_n + 2\left(\frac{x-x_n}{h}\right)y_{n+\frac{1}{4}} \\
&+ \frac{h^2}{210240}\left(-26791\left(\frac{x-x_n}{h}\right) + 105120\left(\frac{x-x_n}{h}\right)^2 - 14400\left(\frac{x-x_n}{h}\right)^3 + 102200\left(\frac{x-x_n}{h}\right)^4 - 35040\left(\frac{x-x_n}{h}\right)^5 + 4672\left(\frac{x-x_n}{h}\right)^6\right)f_n \\
&- \frac{h^2}{720}\left(135\left(\frac{x-x_n}{h}\right) - 960\left(\frac{x-x_n}{h}\right)^3 + 1040\left(\frac{x-x_n}{h}\right)^4 - 432\left(\frac{x-x_n}{h}\right)^5 + 64\left(\frac{x-x_n}{h}\right)^6\right)f_{n+\frac{1}{2}} \\
&+ \frac{h^2}{480}\left(47\left(\frac{x-x_n}{h}\right) - 480\left(\frac{x-x_n}{h}\right)^3 + 760\left(\frac{x-x_n}{h}\right)^4 - 384\left(\frac{x-x_n}{h}\right)^5 + 64\left(\frac{x-x_n}{h}\right)^6\right)f_{n+1} \\
&+ \frac{h^2}{720}\left(29\left(\frac{x-x_n}{h}\right) - 320\left(\frac{x-x_n}{h}\right)^3 + 560\left(\frac{x-x_n}{h}\right)^4 - 336\left(\frac{x-x_n}{h}\right)^5 + 64\left(\frac{x-x_n}{h}\right)^6\right)f_{n+\frac{3}{2}} \\
&+ \frac{h^2}{2880}\left(21\left(\frac{x-x_n}{h}\right) - 2880\left(\frac{x-x_n}{h}\right)^3 + 440\left(\frac{x-x_n}{h}\right)^4 - 288\left(\frac{x-x_n}{h}\right)^5 + 64\left(\frac{x-x_n}{h}\right)^5\right)f_{n+2}
\end{aligned} \quad (33)$$

The discrete scheme is obtained as:

$$y_{n+2} = 4y_{n+\frac{1}{2}} - 3y_n + \frac{h^2}{1920}\left(27f_n + 332f_{n+\frac{1}{2}} + 222f_{n+1} + 132f_{n+\frac{3}{2}} + 7f_{n+2}\right) \quad (34)$$

The discrete scheme (34) is consistent, zero-stable and of order $p = 5$, with the error constant $C_{p+2} = \frac{1}{15360}$.

Evaluating (33) at the points $x = x_{n+i}, i = \frac{3}{2}, 1$, we obtained the following discrete schemes:

$$y_{n+\frac{3}{2}} = 3y_{n+\frac{1}{2}} - 2y_n + \frac{h^2}{3840}\left(37f_n + 432f_{n+\frac{1}{2}} + 222f_{n+1} + 32f_{n+\frac{3}{2}} - 3f_{n+2}\right) \tag{35}$$

$$y_{n+1} = 2y_{n+\frac{1}{2}} - y_n + \frac{h^2}{5760}\left(19f_n + 204f_{n+\frac{1}{2}} + 14f_{n+1} + 4f_{n+\frac{3}{2}} - f_{n+2}\right) \tag{36}$$

The first derivative of (33) is found and evaluated at points $x = x_{n+i}, i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ yield the following derivative schemes:

$$hy'_n = 2y_{n+\frac{1}{2}} - 2y_n - \frac{h^2}{2880}\left(367f_n + 540f_{n+\frac{1}{2}} - 282f_{n+1} + 116f_{n+\frac{3}{2}} - 21f_{n+2}\right) \tag{37}$$

$$hy'_{n+\frac{1}{2}} = 2y_{n+\frac{1}{2}} - 2y_n + \frac{h^2}{2880}\left(135f_n + 752f_{n+\frac{1}{2}} - 246f_{n+1} + 96f_{n+\frac{3}{2}} - 17f_{n+2}\right) \tag{38}$$

$$hy'_{n+1} = 4y_{n+\frac{1}{2}} - 4y_n + \frac{h^2}{2880}\left(97f_n + 1444f_{n+\frac{1}{2}} + 666f_{n+1} - 52f_{n+\frac{3}{2}} + 5f_{n+2}\right) \tag{39}$$

$$hy'_{n+\frac{3}{2}} = 2y_{n+\frac{1}{2}} - 2y_n + \frac{h^2}{2880}\left(119f_n + 1296f_{n+\frac{1}{2}} + 1578f_{n+1} + 640f_{n+\frac{3}{2}} - 33f_{n+2}\right) \tag{40}$$

$$hy'_{n+1} = 2y_{n+\frac{1}{2}} - 2y_n + \frac{h^2}{2880}\left(81f_n + 1508f_{n+\frac{1}{2}} + 1050f_{n+1} + 1932f_{n+\frac{3}{2}} + 469f_{n+2}\right) \tag{41}$$

The implementation of Two-Step Continuous Hybrid Multistep Method (TSCHMM) is as follows:
Combining (34), (35), (36), (37) and solving by using matrix inversion method gives:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n-\frac{1}{2}} \\ y_{n-1} \\ y_{n-\frac{3}{2}} \\ y_n \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} y'_{n-\frac{1}{2}} \\ y'_{n-1} \\ y'_{n-\frac{3}{2}} \\ y'_n \end{pmatrix} +$$

$$h^2\begin{pmatrix} 0 & 0 & 0 & \frac{367}{57600} \\ 0 & 0 & 0 & \frac{53}{360} \\ 0 & 0 & 0 & \frac{147}{640} \\ 0 & 0 & 0 & \frac{14}{45} \end{pmatrix}\begin{pmatrix} f_{n-\frac{1}{2}} \\ f_{n-1} \\ f_{n-\frac{3}{2}} \\ f_n \end{pmatrix} + h^2\begin{pmatrix} \frac{3}{32} & -\frac{47}{960} & \frac{29}{1440} & -\frac{7}{1920} \\ \frac{2}{5} & -\frac{1}{12} & \frac{2}{45} & -\frac{1}{120} \\ \frac{117}{160} & \frac{27}{320} & \frac{3}{32} & -\frac{9}{640} \\ \frac{16}{15} & \frac{4}{15} & \frac{16}{45} & 0 \end{pmatrix}\begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix} \tag{42}$$

The above block method is of uniform order $p = (5, 5, 5, 5)^T$ with the error constant $C_{p+2} = \left(\frac{107}{1290240}, \frac{1}{5040}, \frac{9}{28672}, \frac{1}{2520}\right)^T$.
Equations (38) -(41) with (42) are applied in block form as simultaneous numerical integrators to solve (1).

3. Stability Analysis

In the spirit of Sommeijer *et al.,* (1992), the linear stability of block method can be investigated by applying the method to the test equation $y'' = \lambda y$. This leads to a recursion of the form:

$$Y_{n+2} = M(z)Y_n,$$
$$M(z) := [I - zD]^{-1}[A + zB], \quad z := \lambda h$$

M is called the amplification matrix and its eigenvalues the amplification factors. By requiring the elements of the diagonal matrix D to be positive, the matrix $I - zD$ is nonsingular for all z on the negative real axis. Therefore, in the sequel, we assume that the (diagonal) elements of D are positive. We shall use the result on the power of a matrix N (Varga 1962),

$$\left\|N^n\right\| = o(n^{q-1}[\rho(N)]^n) \text{ as } n \to \infty,$$

where $\|.\|$ and $\rho(N)$ are the spectra norm and the radius of N and where all diagonal sub-matrices of the Jordan normal form of N which have spectral radius $\rho(N)$ are at most qxq. If the spectra radius $\rho(N)$<1, then N is called power bounded. The region where the amplification matrix M(z) is power bounded is called the stability region of the block method. If the stability region contains the origin, then the method is called the zero- stable. Below are the graphical representations of stability of OSCHMM and TSCHMM respectively.

4. Numerical Experiments and Results

In this section, we applied the new methods to some problems: the first is Undamped Duffing's equation of Fang and Wu (2008), two body problem of Fatunla (1990), stiff problem, linear second order initial value problem, Stiefel and Bettis problem and Implicit 3-point 1-block (I3P1B) of Fadzial (2009).

Problem 1: The Undamped Duffing's equation:

$$y'' + y = -y^3 + (\cos t + \varepsilon \sin 10t)^3 - 99\varepsilon \sin 10t$$
$$y(t_0) = 1, \; y'(t_0) = 10\varepsilon, \; \varepsilon = 10^{-10}.$$

The exact solution $y(t) = \cos t + \varepsilon \sin 10t$.

It describes a periodic motion of low frequency with a small perturbation of high frequency. The numerical results are shown on tables 1and 2 below.

Problem 2: Consider the given two-body problem

$$y_1'' = \frac{-y_1}{r}, \quad y_1(0) = 1, \; y_1'(0) = 0,$$
$$y_2'' = \frac{-y_2}{r}, \quad y_2(0) = 0, \; y_2'(0) = 1,$$
$$r = \sqrt{\left(y_1^2 + y_2^2\right)} \quad x \in [0, 15\pi]$$

Theoretical solution: $y_1(x) = \cos x; \; y_2(x) = \sin x$

Problem 3: Consider the stiff problem

$$y'' + 1001y' + 1000y = 0, \; \text{y(0)=1, y}'\text{(0)=1, h = 0.1}$$

Theoretical solution: $y(x) = \exp(-x)$.

Problem 4:

$$y'' - 4y' + 8y = x^3, \; \text{y(0)=2, y}'(0) = 4, x \in [0,1],$$

Theoretical solution:

$$y(x) = e^{2x}(2\cos(2x) - \frac{3}{64}\sin(2x)) + \frac{3x}{32} + \frac{3x^2}{16} + \frac{x^2}{8}$$

Problem 5: Consider the system of equations of Stiefel and Bettis problem:

$$y_1'' + y_1 = 0.001\cos(x), y_1(0) = 1, \; y_1'(0) = 0$$
$$y_2'' + y_2 = 0.001\sin(x), y_2(0) = 1, \; y_2'(0) = 0.9995,$$
$$x \in [0, 40\pi],$$

The exact solutions are given as:

$$y_1(x) = \cos(x) + 0.0005x\sin x,$$
$$y_2(x) = \sin(x) - 0.0005x\cos x.$$

Table 1. One-step method for problem 1 Undamped Duffing's Equation

X	y-exact solution	y-approximate	Error
0.0025	0.9999968750041270	0.9999968750041270	0.000000e+000
0.0050	0.9999875000310390	0.9999875000310390	1.110223e-016
0.0075	0.9999718751393280	0.9999718751393290	8.881784e-016
0.0100	0.9999500004266480	0.9999500004266490	7.771561e-016
0.0125	0.9999218760297140	0.9999218760297150	4.440892e-016
0.0150	0.9998875021243030	0.9998875021243040	9.992007e-016
0.0175	0.9998468789252480	0.9998468789252500	1.665335e-015
0.0200	0.9998000066864440	0.9998000066864470	2.775558e-015
0.0225	0.9997468857008410	0.9997468857008460	5.440093e-015
0.0250	0.9996875163004430	0.9996875163004500	7.216450e-015
0.0275	0.9996218988563060	0.9996218988563160	9.436896e-015

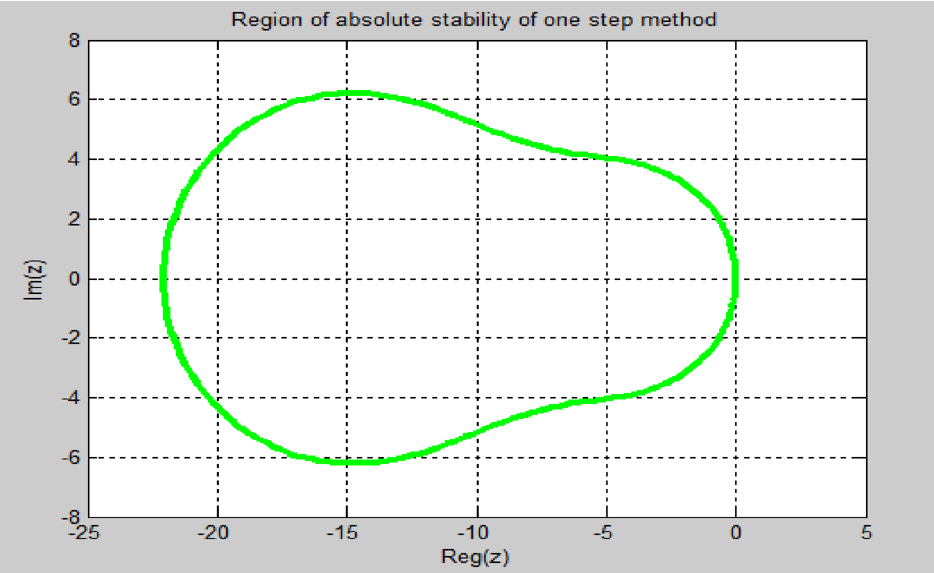

Figure 1. Stability Domain of Block of OSCHMM which is A(α)-stable by Definition 1.3

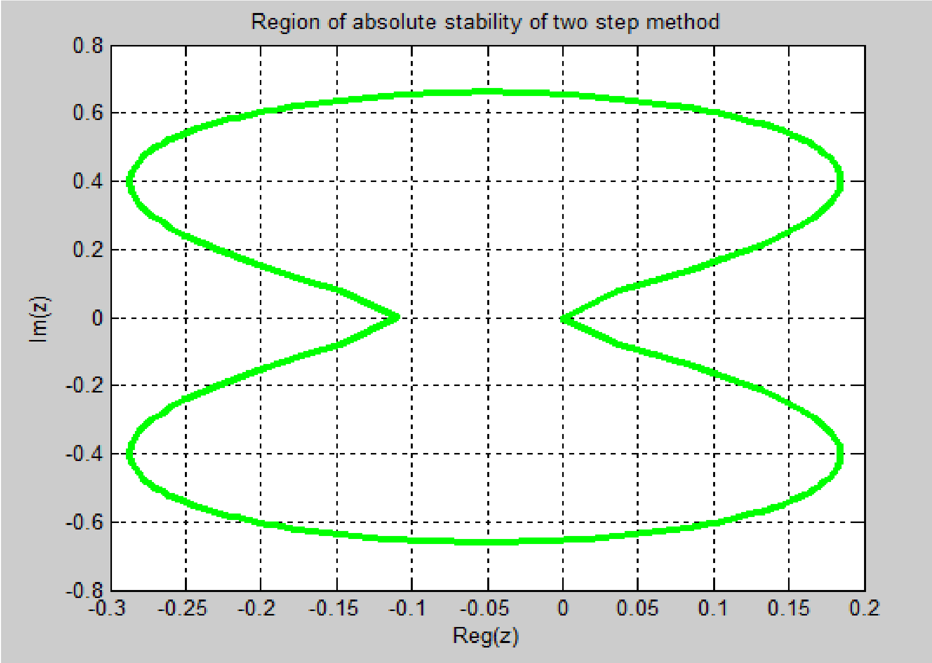

Figure 2. Stability Domain of Block TSCHMM which is L(α)-stable by definition 1.3 and 1.4

Table 2. The y-exact, y-approximate and error of TSCHMM for problem 1

Two-step method, h = 0.01 problem 1 Undamped Duffing's Equation			
X	**y-exact solution**	**y-approximate**	**Error**
0.0050	0.9999875000310390	0.9999875000310390	1.110223e-016
0.0100	0.9999500004266480	0.9999500004266540	5.551115e-015
0.0150	0.9998875021243030	0.9998875021243660	6.328271e-014
0.0200	0.9998000066864440	0.9998000066864920	4.740652e-014
0.0250	0.9996875163004430	0.9996875163005230	8.015810e-014
0.0300	0.9995500337785390	0.9995500337786070	6.827872e-014
0.0350	0.9993875625577780	0.9993875625578920	1.136868e-013
0.0400	0.9992001066999190	0.9992001067000950	1.757483e-013
0.0450	0.9989876708913390	0.9989876708916910	3.522738e-013
0.0500	0.9987502604429080	0.9987502604433770	4.684031e-013

Table 3. The y-exact, y-approximate and error of OSCHMM for problem 2Table 4: The y-exact, y-approximate and error of TSCHMM for problem 2

X	y_1-exact	y_2-exact	y_1- approximate	y_2- approximate	Error in y_1	Error in y_2
0.1	0.9950041652780	0.0998334166468	0.99500416527	0.99833416647	1.2891E-14	3.0923E-13
0.2	0.9800665778412	0.1986693307950	0.98006657784	0.198669330795	5.1308E-14	6.1279E-13
0.3	0.9553364891256	0.2955202066613	0.95533648912	0.295520206662	1.1448E-13	9.0512E-13
0.4	0.9210609940028	0.3894183423086	0.92106099400	0.389418342309	2.0114E-13	1.1807E-12
0.5	0.8775825618903	0.4794255386042	0.87758256189	0.479425538605	3.0954E-13	1.4346E-12
0.6	0.8253356149096	0.5646424733950	0.82533561490	0.564642473396	4.3747E-13	1.6617E-12
0.7	0.7648421872844	0.6442176872376	0.76484218728	0.644217687239	5.8231E-13	1.8576E-12
0.8	0.6967067093471	0.7173560908995	0.69670670934	0.717356090901	7.4105E-13	2.0184E-12
0.9	0.6216099682706	0.7833269096274	0.62160996826	0.783326909629	9.1035E-13	2.1406E-12
1.0	0.5403023058681	0.8414709848078	0.54030230586	0.841470984810	1.0865E-12	2.2211E-12

Table 4. The y-exact, y-approximate and error of TSCHMM for problem 2

X	y_1-exact	y_2-exact	y_1- approximate	y_2- approximate	Error in y_1	Error in y_2
0.1	0.9950041652780	0.0998334166468	0.99500416527	0.99833416666	1.6897E-12	1.9734E-11
0.2	0.9800665778412	0.1986693307950	0.98006657783	0.19866933084	3.2802E-12	3.9280E-11
0.3	0.9553364891256	0.2955202066613	0.95533648911	0.295520206719	8.1501E-12	5.7761E-11
0.4	0.9210609940028	0.3894183423086	0.92106099399	0.389418342843	1.2859E-11	7.5690E-11
0.5	0.8775825618903	0.4794255386042	0.87758256186	0.479425538695	2.0553E-11	9.1547E-11
0.6	0.8253356149096	0.5646424733950	0.82533561488	0.564642473501	2.7968E-11	1.0652E-10
0.7	0.7648421872844	0.6442176872376	0.76484218724	0.644217687356	3.7894E-11	1.1853E-10
0.8	0.6967067093471	0.7173560908995	0.69670670929	0.717356091028	4.7377E-11	1.2941E-10
0.9	0.6216099682706	0.7833269096274	0.62160996821	0.783326909764	5.8742E-11	1.3657E-10
1.0	0.5403023058681	0.8414709848078	0.54030230579	0.841470984950	6.9468E-11	1.4243E-10

Table 5. The y-exact, y-approximate and error of OSCHMM for problem 3

X	y-exact	y-approximate	Error in OSCHMM (of problem 3)
0.1	0.90483741803595957316	0.90483741803591096220	4.861096E-14
0.2	0.81873075307798185867	0.81873075307788970825	9.215042E-14
0.3	0.74081822068171786607	0.74081822068158785625	1.300098E-13
0.4	0.67032004603563930074	0.67032004603547725100	1.620497E-13
0.5	0.60653065971263342360	0.60653065971244499082	1.884327E-13
0.6	0.54881163609402643263	0.54881163609381692607	2.095065E-13
0.7	0.49658530379140951470	0.49658530379118379194	2.257227E-13
0.8	0.44932896411722159143	0.44932896411698400947	2.375819E-13
0.9	0.40656965974059911188	0.40656965974035351527	2.455966E-13
1.0	0.36787944117144232160	0.36787944117119205438	2.502672E-13

Table 6. Accuracy Comparison of TSCHMM for Problem 4

X	y-exact	y-approx (of problem4)	Error in TSCHMM p = 5	Error in Jator & Li (2009), p = 5
0.1	2.3941125769963956181	2.3941125055703059230	7.14260896951E-08	5.10704 E-06
0.2	2.7481413324264235256	2.7481411575175150467	1.74908908478 E-07	1.49586 E-05
0.3	3.0078669405110678859	3.0078665760231918731	3.64487876012 E-07	2.78532 E-05
0.4	3.1017624057742078185	3.1017617867959027691	6.18978305049 E-07	4.28908 E-05
0.5	2.9395431007452620774	2.9395421018471828849	9.98898079192 E-07	6.70307 E-05
0.6	2.4118365344157147255	2.4118350550184086061	1.47939730611 E-06	1.02637 E-04
0.7	1.3915548304898433104	1.3915527282654101737	2.10222443313E-06	1.44907 E-04
0.8	-0.262326758334357631	-0.2623295992140394222	2.84087968179 E-06	1.90905 E-04
0.9	-2.697771160773070925	-2.6977748297643383106	3.66899126738 E-06	2.39733 E-04
1.0	-6.058560720845666951	-6.0585652825136523363	4.56166798538 E-06	2.94670 E-04

Table 7. Accuracy Comparison of OSCHMM and TSCHMM with Implicit 3-point-1 block (13PIB)

H	METHOD	MAX ERR
0.01	I3P1B	2.14918(-8)
	OSCHMM	9.115552(-10)
	TSCHMM	7.294766(-9)
0.005	I3P1B	1.34949(-9)
	OSCHMM	1.39353(-10)
	TSCHMM	9.11552(-10)
0.001	I3P1B	8.64701(-11)
	OSCHMM	9.114593(-13)
	TSCHMM	7.291698(-12)
0.0005	I3P1B	3.95277(-10)
	OSCHMM	1.139323(-13)
	TSCHMM	9.114593(-13)

Table 8. y-exact, y-approximate and error in TSCHMM for problem 5

X	y_1-exact	y_2-exact	y_1-approx (prob5)	y_2-approx (TSCHMM)	Error y_1	Error y_2
0.1	0.9999951220742781	0.003123432421368851	0.9999951220742781	0.003123432421368851	1.64E-18	7.20E-21
0.2	0.9999804883447010	0.006246834371010263	0.9999804883447010	0.006246834371010263	2.87E-18	2.40E-21
0.3	0.9999560989540329	0.009370175377494076	0.9999560989540329	0.009370175377494076	1.26E-18	4.33E-20
0.4	0.9999219541402128	0.012493424969984680	0.9999219541402128	0.012493424969984680	5.73E-18	6.30E-20
0.5	0.9998780542363521	0.015616552678538286	0.9998780542363521	0.015616552678538286	4.10E-18	1.09E-19
0.6	0.9998243996707314	0.018739528034400182	0.9998243996707314	0.018739528034400182	8.60E-18	1.15E-19
0.7	0.9997609909667961	0.021862320570301988	0.9997609909667961	0.021862320570301988	6.97E-18	1.85E-19
0.8	0.9996878287431515	0.024984899820758884	0.9996878287431515	0.024984899820758884	1.14E-17	1.81E-19
0.9	0.9996049137135566	0.028107235322366826	0.9996049137135566	0.028107235322366826	9.83E-18	2.79E-19
1.0	0.9995122466869173	0.031229296614099748	0.9995122466869173	0.031229296614099748	1.43E-17	2.61E-19

On tables 3 and 4, y-exact, y-computed and error of OSCHMM and TSCHMM for problem 2 are shown while the y-exact, y-computed and error of TSCHMM for problem 3 is shown on table 5. On table 6, it is observed that the maximum absolute error of the TSCHMM is 7.14260896951E-08 is (smaller) more accurate than 5.10704 E-06 of Jator & Li (2009) for problem 4.

The accuracy comparison of the new methods and I31PB are shown on table 7. The new One-step method (OSCHMM) and Two-step method (TSCHMM) are substantially more accurate than the numerical solution of initial-value problems (IVPs) using I3P1B, as the maximum absolute error is smaller with variable h is 9.115552(-10) while that the maximum absolute error of TSCHMM is 7.294766(-9) which is smaller than 2.14918(-8) of I3P1B for h = 0.01.

5. Conclusions

We have presented continuous hybrid multistep methods with Legendre polynomial as basis function for the direct solution of system of second order ODEs. The derived methods were implemented in block mode which have the advantages of being self-starting, uniformly of the same order of accuracy and do not need predictors, having good accuracy as shown on table 7. It should be noted that accuracy and efficiency rate of a method is dependent on the implementation strategies. If economical computation is required, then the new methods are the better choice. The new methods are therefore recommended for general purposed use. Finally, the region of absolute stability of the block methods of One-step and Two-step methods were presented in figures 1 and 2. Maple and Matlab software package were employed to generate the schemes and results.

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