Exponential Membership Functions in Fuzzy Goal Programming: A Computational Application to a Production Problem in the Textile Industry

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Abstract   The linear membership function is considered the most common type that is used in fuzzy goal programs. In this paper, the exponential membership function, whether with increasing or with decreasing rate of change, is used. Each of the two types is utilized within a fuzzy goal program. Two main forms of fuzzy goal program are implemented. The first is based on the lexicographical minimization, while the second is based on a preemptive goal hierarchy. A computational comparison between the two forms is carried out on a production planning problem in the textile industry. This problem was in the form of fuzzy linear programming, and it is amended to be in the form of fuzzy goal programming.

Keywords   Fuzzy Goal Programming, Exponential Membership Function, Lexicographical Minimization, Preemptive Goal Hierarchy

1. Introduction

Over the last decades, numerous attempts have been made in the field of Goal Programming (GP) [1-3]. The preemptive priority GP is considered one of the main approaches in modelling GP programs. Hence, each goal is set to a certain predefined priority level, and different solution methods have been introduced to solve the preemptive priority GP problems. Two main criticisms have been made about practicality of the preemptive GP. Firstly, preemptive priorities imply an infinite trade-off between goals placed in different levels, which may lead to high goal achievement of higher goal priority levels and by far too low goal achievement for lower goal priority levels, which leads to unsatisfactory results. Secondly, in a real GP problem it is not easy to define priority levels. This is due to the uncertainty of the relative importance of goals. The first criticism may be avoided by using the weighted GP, where a single objective function of the weighted sum of the deviation variables is to be minimized [4]. However, the decision-maker may find determining goal priority levels in some situations is better than assigning precise weights for the goals. An extensive survey of different GP models was given by Jones and Tamiz [5]. Also, Hannan [6] stated major criticisms and limitations of GP.

In many real world problems, the decision-maker may not be certain about the used data. Therefore, fuzzy GP has been introduced to deal with different goal programs under uncertainty and vagueness. An early attempt in fuzzy GP was given by Narasimhan [7] which was followed by contributions from several other researchers [8-13]. Later, Chanas and Kuchta [14] provided a survey of various fuzzy GP models to represent a satisfaction degree of the decision-maker with respect to his/her preference structure. Aköz and Petrovic [15] expressed the preferences of the decision-maker in an imprecise way to overcome the difficulty of setting a crisp preemptive priority structure. In their proposed fuzzy GP model, the goal importance levels are defined and represented by fuzzy relations. Moreover, different types of membership functions can be considered in fuzzy programming. For instance, the logistic membership function has been utilized in fuzzy linear programming to be applied to a production planning problem in the textile industry [16, 17]. In addition, Iskander [18] utilized the exponential membership functions in stochastic fuzzy goal programming.

In this paper, two well known forms of fuzzy goal programs are used. The first is based on lexicographical minimization, while the second depends on a preemptive goal hierarchy. A computational comparison between the two forms in the case of exponential membership function, whether with increasing rate of change or with decreasing rate of change, is applied to the production planning problem of Elamvazuthi et al. [17]. In the next section, the formulation of the two fuzzy goal programming models, with the exponential membership functions are given. The

computational comparison, which is based on a production planning problem is stated in Section 3. Finally, conclusions are provided in the last section.

2. Fuzzy Goal Programming Models

Assume the following fuzzy goal constraints:

$$\sum_{j=1}^{n} a_{ij} x_j \underset{\sim}{>} b_i, i = 1, 2,\ldots, m_1, \qquad (1)$$

$$\sum_{j=1}^{n} a_{ij} x_j \underset{\sim}{<} b_i, i = m_1+1, m_1+2,\ldots, m, \qquad (2)$$

where $x_j, j = 1, 2,\ldots, n$, are non-negative decision variables, b_i, $i = 1, 2,\ldots, m$, are the required aspiration levels, while a_{ij} represents the coefficient of the jth decision variable in the ith fuzzy goal constraint. The symbols $\underset{\sim}{>}$ and $\underset{\sim}{<}$ indicate approximately greater than or equal to and approximately less than or equal to, respectively. Let l_i and u_i be the lower and the upper tolerance limits for the fuzzy goal constraints (1) and (2) respectively. Accordingly, for the set of fuzzy goal constraints (1), the ith exponential membership function with increasing rate of change is presented as

$$\zeta_i = (\exp\{1\} - 1)^{-1} (\exp\{(\sum_{j=1}^{n} a_{ij} x_j - l_i) / (b_i - l_i)\} - 1), (3)$$

while with decreasing rate of change, it is stated as

$$\zeta_i = (1 - \exp\{-1\})^{-1} (1 - \exp\{(l_i - \sum_{j=1}^{n} a_{ij} x_j) / (b_i - l_i)\}). (4)$$

On the other hand, for the set of fuzzy goal constraints (2), the ith exponential membership function with increasing rate of change is presented as

$$\zeta_i = (\exp\{1\} - 1)^{-1} (\exp\{(u_i - \sum_{j=1}^{n} a_{ij} x_j) / (u_i - b_i)\} - 1), (5)$$

while with decreasing rate of change, it is stated as

$$\zeta_i = (1 - \exp\{-1\})^{-1} (1 - \exp\{(\sum_{j=1}^{n} a_{ij} x_j - u_i) / (u_i - b_i)\}). (6)$$

These basic forms of exponential membership functions (3)-(6) have been implemented in the case of stochastic fuzzy goal programming [18]. They have been denoted by either exponential membership functions with increasing or with decreasing rate of change, according to the corresponding basic form. However, in that paper, some of them may not be strictly with increasing or with decreasing rate of change due to the adjustments in the basic forms to fit the stochastic fuzzy problem. Moreover, in that paper and for some of the tested models, the values of the decision variables in the case of increasing rate of change are similar to those in the case of decreasing rate of change. This situation may be satisfactory for the decision-maker.

Accordingly, let I_p be the index set of fuzzy goal constraints having priority level $p, p = 1, 2,\ldots, P$, where P is

the number of priority levels, then the lexicographical minimization can be presented as follows:

Lexicographically Minimize $\{ \sum_{i \in I_p} d_i : p = 1, 2,\ldots, P \}$

subject to:

$$\mu_i = \min \{\zeta_i, 1\}, i = 1, 2,\ldots, m, \qquad (7)$$
$$\mu_i + d_i \geq 1, i = 1, 2,\ldots, m,$$
$$x_j, \mu_i, d_i \geq 0, j = 1, 2,\ldots, n; i = 1, 2,\ldots, m,$$

where μ_i is the exponential achievement degree, either with increasing or with decreasing rate of change, for the ith fuzzy goal constraint, while d_i denotes the corresponding underachievement value. It is obvious that regardless of the non-negativity constraint of d_i, it must be non-negative. However, incorporating this constraint into the model may help in efficiently reaching the optimal solution.

On the other hand, the model with a preemptive goal hierarchy is stated as

Maximize $\sum_{p=1}^{P} \sum_{i \in I_p} \dfrac{(\mu_i)_p}{k_p}$

subject to:

$$\mu_i = \min \{\zeta_i, 1\}, i = 1, 2,\ldots, m, \qquad (8)$$
$$(\mu_i)_p \geq (\mu_v)_{p+1}, i \neq v; p = 1, 2,\ldots, P-1,$$
$$x_j, \mu_i \geq 0, j = 1, 2,\ldots, n; i = 1, 2,\ldots, m,$$

where $(\mu_i)_p$ is the exponential achievement degree of the ith fuzzy goal constraint having priority level $p, p = 1, 2,\ldots, P$, while k_p is the number of fuzzy goal constraints having priority level p, i.e., the number of elements in set I_p. It is obvious that the objective function has been modified. Instead of maximizing the sum of all membership functions, the average of the membership functions for each priority level is evaluated while the sum of these averages is maximized. The main purpose of this modification is to remove the effect of the number of the membership functions in each priority level.

In the two models (7) and (8), a set of system constraints $\phi_s(x) \geq 0, s = 1, 2,\ldots, S$, can be incorporated, where $\phi_s(x)$ is the sth real-valued function of x (an n-vector of decision variables), and S is the number of system constraints.

In the next section, each of the two models is applied to a textile production problem, where a computational comparison between the two models is investigated in the case of increasing and the case of decreasing rate of change.

3. Computational Study

In this section, the textile production planning problem of Elamvazuthi et al. [17] is going to be utilized for a computational comparison between the two models. The structure of the problem takes the form of fuzzy linear programming. Therefore, this problem has been amended to be in the form of fuzzy goal programming. The problem

assumes that there are three types of textile products (sheets, pillow cases, and quilts) where the decision variables x_1, x_2, and x_3 denote the quantity produced from sheets, pillow cases, and quilts, respectively. The data of the problem are given in Tables 1 and 2.

Table 1. The lowest, most common, and highest profit per unit, as well as the minimum required quantity for each type

Types	Lowest profit	Most common profit	Highest profit	Minimum required quantity
Sheets	1.02	1.05	1.08	25000
Pillow cases	0.20	0.30	0.40	40000
Quilts	1.70	1.80	2.00	10000

Table 2. The process time per unit, for each type, and the monthly working hours in each department

Departments	Process time per unit (hour)			Working hours per month
	Sheet	Pillow case	Quilt	
Cutting	0.0033	0.001	0.0033	208
Sewing	0.0560	0.025	0.1000	4368
Pleating	0.0067	0.004	0.0170	520
Packaging	0.0100	0.010	0.0100	780

Three system constraints, for the minimum required quantity from each type, are introduced as follows:

$$x_1 \geq 25000,$$
$$x_2 \geq 40000,$$
$$x_3 \geq 10000.$$

Then, five fuzzy goals are proposed. The first four represent the process time for each of the four departments, while the fifth stands for the production profit. Therefore, the exponential membership functions of the five fuzzy goals can be stated as

(a) The Case of Increasing Rate of Change:

$$\zeta_1 = (\exp\{1\} - 1)^{-1} (\exp\{(208 - 0.0033x_1 - 0.001x_2 - 0.0033x_3) / (208 - 208(1 - \alpha))\} - 1),$$

$$\zeta_2 = (\exp\{1\} - 1)^{-1} (\exp\{(4368 - 0.056x_1 - 0.025x_2 - 0.1x_3) / (4368 - 4368(1 - \alpha))\} - 1),$$

$$\zeta_3 = (\exp\{1\} - 1)^{-1} (\exp\{(520 - 0.0067x_1 - 0.004x_2 - 0.017x_3) / (520 - 520(1 - \alpha))\} - 1),$$

$$\zeta_4 = (\exp\{1\} - 1)^{-1} (\exp\{(780 - 0.01x_1 - 0.01x_2 - 0.01x_3) / (780 - 780(1 - \alpha))\} - 1),$$

$$\zeta_5 = (\exp\{1\} - 1)^{-1} (\exp\{(1.05x_1 + 0.3x_2 + 1.8x_3 - 53718.447) / (66454.369 - 53718.447)\} - 1).$$

(b) The Case of Decreasing Rate of Change:

$$\zeta_1 = (1 - \exp\{-1\})^{-1} (1 - \exp\{(0.0033x_1 + 0.001x_2 + 0.0033x_3 - 208) / (208 - 208(1 - \alpha))\}),$$

$$\zeta_2 = (1 - \exp\{-1\})^{-1} (1 - \exp\{(0.056x_1 + 0.025x_2 + 0.1x_3 - 4368) / (4368 - 4368(1 - \alpha))\}),$$

$$\zeta_3 = (1 - \exp\{-1\})^{-1} (1 - \exp\{(0.0067x_1 + 0.004x_2 + 0.017x_3 - 520) / (520 - 520(1 - \alpha))\}),$$

$$\zeta_4 = (1 - \exp\{-1\})^{-1} (1 - \exp\{(0.01x_1 + 0.01x_2 + 0.01x_3 - 780) / (780 - 780(1 - \alpha))\}),$$

$$\zeta_5 = (1 - \exp\{-1\})^{-1} (1 - \exp\{(53718.447 - 1.05x_1 - 0.3x_2 - 1.8x_3) / (66454.369 - 53718.447)\}).$$

Table 3. The case of increasing rate of change

	The lexicographical minimization model			The model with a preemptive goal hierarchy		
α	0.05	0.10	0.15	0.05	0.10	0.15
x_1	25000.000	25000.000	25000.000	25000.000	25575.042	25016.545
x_2	40000.000	40000.000	40000.000	40000.000	40000.000	40000.000
x_3	10000.000	10000.000	10000.000	10000.000	10240.081	10241.882
μ_1	1.000	1.000	1.000	1.000	1.000	1.000
μ_2	1.000	1.000	1.000	1.000	1.000	1.000
μ_3	0.801	0.315	0.195	0.801	0.188	0.154
μ_4	0.674	0.273	0.170	0.674	0.188	0.154
μ_5	0.128	0.128	0.128	0.128	0.188	0.154

Table 4. The case of decreasing rate of change

	The lexicographical minimization model			The model with a preemptive goal hierarchy		
α	0.05	0.10	0.15	0.05	0.10	0.15
x_1	25000.000	25000.000	25000.000	25511.067	25575.042	25016.545
x_2	40000.000	40000.000	40000.000	40000.000	40000.000	40000.000
x_3	10000.000	10000.000	10000.000	10000.000	10240.081	10241.882
μ_1	1.000	1.000	1.000	1.000	1.000	1.000
μ_2	1.000	1.000	1.000	1.000	1.000	1.000
μ_3	0.916	0.556	0.396	0.822	0.386	0.330
μ_4	0.849	0.505	0.358	0.746	0.386	0.330
μ_5	0.285	0.285	0.285	0.339	0.386	0.330

The linear programming model of the given textile problem [16] is solved in two cases. In the first case, the lowest profit values for the three types are the used coefficients in the profit objective function that is required to be maximized. In the second case, the highest profit values are used. The value of the profit objective function in the two cases is 53718.447 and 66454.369, respectively. Hence, 66454.369 is considered the required aspiration level, while 53718.447 is the lower tolerance limit for the fifth fuzzy goal. For each of the first four fuzzy goals, the working hours per month represent the upper tolerance limit, while the required aspiration level is the upper tolerance limit weighted by $(1 - \alpha)$. Thus, α is defined to be the intolerance measure, $\alpha \in (0, 1)$. Two priority levels are presumed. The first is assigned to the process time in each department (i.e., the first four fuzzy goals) while the second is assigned to the production profit (i.e., the fifth fuzzy goal). The CONOPT solver embedded in the General Algebraic Modeling System (GAMS win32 23.8.2) package is utilized in this computational study. The results of the two models, in the case of increasing and the case of decreasing rate of change, as well as when $\alpha = 0.05$, 0.1, and 0.15, are given in Tables 3 and 4.

It is obvious that the values of the decision variables of the lexicographical minimization model have not been changed whether in the case of increasing or the case of decreasing rate of change, and for different values of α. This is mainly due to the system constraints which also prevent some of the goals having the first priority level to be achieved, since μ_3 and μ_4 are less than one, i.e., the third and the fourth goals are not fully achieved. This indicates that this model is less sensitive to the change in the results than the model with a preemptive goal hierarchy. On the other hand, it can be realized that whether in the case of increasing or the case of decreasing rate of change the total value of the membership functions of the five goals (the sum of the achievement degrees of all the goals) is higher in the case of the lexicographical minimization model than that in the case of the model with a preemptive goal hierarchy. This is for each value of α, except when $\alpha = 0.05$ in the case of increasing rate of change, since the results of the two models are exactly the

same. Moreover, the situation when $\alpha = 0.2$ was investigated. It has been found that, whether in the case of increasing rate of change or the case of decreasing rate of change, there is no feasible solution for the model with a preemptive goal hierarchy, while there is a solution for the lexicographical minimization model.

Afterward, the model with a preemptive goal hierarchy has been solved using the unmodified objective function, i.e., when the sum of all membership functions is required to be maximized. The results were similar to the results of the lexicographical minimization model that are given in Tables 3 and 4. Hence, the achievement value of the fifth fuzzy goal (which has the second priority level) is less than or equal to its corresponding value when the amended objective function is used, since the amended form removes the effect of the number of fuzzy goals having the first priority level.

Moreover, the two models have been solved with an opposite priority structure, i.e., the fifth fuzzy goal has the first priority while the others have the second priority. The results of the lexicographical minimization model are $x_1 = 27766.990$, $x_2 = 40000$, $x_3 = 10233.010$, $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 0$, and $\mu_4 = 0$ for the three values of α, and whether in the case of increasing or the case of decreasing rate of change. Only, $\mu_5 = 0.340$ in the case of increasing rate of change, while $\mu_5 = 0.583$ in the case of decreasing rate of change. Conversely, there is no solution for the model with a preemptive goal hierarchy in all the cases.

Since the unchanged values of the decision variables for the lexicographical minimization model are mainly due to the system constraints, another computational investigation has been conducted by relaxing the system constraints. Accordingly, it is assumed that the minimum required quantity from each type is reduced by 10%. Thus, the three system constraints become

$$x_1 \geq 22500,$$
$$x_2 \geq 36000,$$
$$x_3 \geq 9000.$$

The two original models were solved according to the amended system constraints. The results are presented in Tables 5 and 6.

Table 5. The case of increasing rate of change, according to the amended system constraints

α	The lexicographical minimization model			The model with a preemptive goal hierarchy		
	0.05	0.10	0.15	0.05	0.10	0.15
x_1	28902.913	25287.323	25446.140	28902.913	29795.064	29025.464
x_2	36000.000	36000.000	36000.000	36000.000	36000.000	36000.000
x_3	9197.087	9092.643	9000.000	9197.087	9194.209	9196.692
μ_1	1.000	1.000	1.000	1.000	1.000	1.000
μ_2	1.000	1.000	1.000	1.000	1.000	1.000
μ_3	1.000	1.000	0.559	1.000	0.274	0.222
μ_4	1.000	0.964	0.528	1.000	0.274	0.222
μ_5	0.214	0.000	0.000	0.214	0.274	0.222

Table 6. The case of decreasing rate of change, according to the amended system constraints

	The lexicographical minimization model			The model with a preemptive goal hierarchy		
α	0.05	0.10	0.15	0.05	0.10	0.15
x_1	28902.913	25287.323	25446.140	29963.475	29795.064	29025.464
x_2	36000.000	36000.000	36000.000	36000.000	36000.000	36000.000
x_3	9197.087	9092.643	9000.000	9000.000	9194.209	9196.692
μ_1	1.000	1.000	1.000	1.000	1.000	1.000
μ_2	1.000	1.000	1.000	1.000	1.000	1.000
μ_3	1.000	1.000	0.775	0.910	0.507	0.437
μ_4	1.000	0.986	0.752	0.856	0.507	0.437
μ_5	0.425	0.000	0.000	0.492	0.507	0.437

It is obvious that relaxing the system constraints leads to some interesting indications. First, it might be preferable to start with the lexicographical minimization model, in order to avoid having a solution that is not optimal, which may exist if the model with a preemptive goal hierarchy is used. For instance, in Table 5 and when $\alpha = 0.1$, according to the objective function of the model with a preemptive goal hierarchy, the solution of the lexicographical minimization model is better than that of the model with a preemptive goal hierarchy. Nevertheless, the GAMS/CONOPT solver could not find this feasible better solution for the model with a preemptive goal hierarchy. Second, the case of getting unchanged values of the decision variables for the lexicographical minimization model has been resolved. On the other hand, the model with a preemptive goal hierarchy may be recommended to avoid getting zero for the membership function of the fifth goal.

It should be noted that in the two models, a more general case can be introduced by using different intolerance measures for the four fuzzy goals, instead of using the same one for all of them. In this situation, an interactive approach may be implemented. This case can be investigated in a further study.

4. Conclusions

In this paper, a computational comparison between the lexicographical minimization model and the model with a preemptive goal hierarchy has been applied to a fuzzy textile production planning problem, where the membership functions of the fuzzy goals are considered exponential with either increasing or decreasing rate of change. It can be concluded that it is preferable to use the lexicographical minimization model, if the decision-maker is not sure about the tolerance limits of the fuzzy data, in order to minimize the sensitivity of the change in results due to the change in data. Also, the lexicographical minimization model is recommended when the achievement degrees of the goals with low priority levels are most likely to be relatively high. This is to avoid having infeasible solution, which may exist when the model with a preemptive goal hierarchy is used.

This recommendation is applicable whether in the case of increasing or the case of decreasing rate of change. On the other hand, if the decision-maker is seeking to get different sets of results based on different tolerance limits that reflect alternative production scenarios, then the model with a preemptive goal hierarchy can be preferable.

Finally, whether in the case of increasing or the case of decreasing rate of change, if an optimal solution exists for each of the two models, the trade-off between not getting zero for any membership function and achieving large values for the membership functions with high priority levels is a main criterion for choosing between the two models.

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