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Solitary Wave Solutions of the High-order Nonlinear Schrödinger Equation in Dispersive Single Mode Optical Fibers

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Abstract The dynamics of propagation of waves in the transmission supports as high-nonlinear and dispersive optical fibers is governed by Schrödinger equations that integrate supplementary nonlinear terms. This particularity complicates the resolution of this equation. So the resolution of such an equation requires a suitable method. In this article, we construct solutions of the type solitary wave by means of a method which supposes from the onset that the equation admits a solution that is a combination of the bright soliton and dark soliton and thereafter proceed by elimination of the constants until the exact solutions or those that are nearer to the exact solutions are obtained.

Keywords Solitary wave, Schrödinger equation, Optical fibers, Bright soliton, Dark soliton

1. Introduction

The nonlinear partial differential equations that describe the propagation of waves in the atomic chains, electric lines and optical fibers are for most of schrödinger's type. They present nonlinear terms that are most often of cubic order, quintic and so on. The research of the analytical solutions for these equations is never an easy task. It is justified besides by the multitude of mathematical methods that many authors use every day in the resolution of mathematical problems [1-13]. The nonlinear partial differential equations that describe the propagation of waves in the atomic chains, electric lines and optical fibers are for most of schrödinger's type. If we come back to the optical fiber transmission support that is currently in the center of modern telecommunications, one realizes that in the case of high-nonlinear dispersive single mode optical fiber, the dynamics of propagation of waves is governed by the following Schrödinger equation [14].

$$U_z = -i\frac{\beta_2}{2}U_{tt} + \frac{\beta_3}{6}U_{ttt} + i\frac{\beta_4}{24}U_{tttt} + \frac{i\gamma|U|^2 U}{1+\Gamma|U|^2} - \frac{\alpha}{2}U \,, \quad (1)$$

$$U_z = -i\frac{\beta_2}{2}U_{tt} + \frac{\beta_3}{6}U_{ttt} + i\frac{\beta_4}{24}U_{tttt} + \frac{i\gamma|U|^2 U}{1+\Gamma|U|^2} - \frac{\alpha}{2}U$$

where U is the slowly varying amplitude of the electrical field envelope, β_n is the n^{th} order of the dispersion parameter, α is the linear loss parameter, Γ is the parameter of saturation of the nonlinearity and γ designates the magnitude of the kerr parameter and nonlinear absorption and $i^2 = -1$. This nonlinear partial differential equation is not easy to solve. This difficulty to solve is translated by very few works that propose analytical solutions of this equation as presented above. Our aim in this work is to propose some analytical solutions to this equation, or merely to find the solutions very close to the exact solutions. The equation (1) is high-nonlinear and dispersive; then susceptible to have some solitary solutions. Thus, the method of research of the solutions, consist in supposing initially that the equation (1) admits a solution that is a combination of solitary waves of nature bright and kink of the shape

$$U(z,t) = \sum_{l=1}^{n}\left(a_{jl}\,\frac{\sinh^j \alpha t}{\cosh^l \alpha t}\right)\exp(-ikz), \quad (2)$$

where $i^2 = -1$; $j = 0,1$; $l = 1,2,....,n$; n designates the number of terms of equation (2); a_{jl} , α and k the constants to determine as a function of the parameter of the equation (1). When the equation (2) and its different derivatives are introduced in the equation (1), we get the complicated coefficient equations and of which the elimination of the constants case by case, permits to obtain the solutions progressively. In this work, we will limit in the case where we have four complex coefficients a_{jl} . So the

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manuscript is presented as follows: In section 2, we obtain the equations of the coefficients a_{jl}. In the section 3, we analyze the different possibilities of obtaining the solutions. Finally section 4 concludes our work.

2. Coefficient Equations

After reducing in the same denominator, equation (1) becomes

$$\begin{aligned}&U_z+\Gamma|U|^2U_z+i\frac{\beta_2}{2}U_{tt}+i\frac{\beta_2}{2}\Gamma|U|^2U_{tt}-\frac{\beta_3}{6}U_{ttt}-\frac{\beta_3}{6}\Gamma|U|^2U_{ttt}\\&-i\frac{\beta_4}{24}U_{tttt}-i\frac{\beta_4\Gamma}{24}|U|^2U_{tttt}-i\gamma|U|^2U+\frac{\alpha}{2}U+\frac{\alpha}{2}\Gamma|U|^2U=0.\end{aligned} \tag{3}$$

The aim is to construct explicitly the solutions of equation (2) in the form

$$U(z,t)=\left(\frac{a}{\cosh\Omega_0t}+b\frac{\sinh\Omega_0t}{\cosh\Omega_0t}+\frac{c}{\cosh^2\Omega_0t}+d\frac{\sinh\Omega_0t}{\cosh^2\Omega_0t}\right)\exp-ikz, \tag{4}$$

where the coefficients a_{jl} of equation (2) are substituted by $a=a_r+ia_i$, $b=b_r+ib_i$, $c=c_r+ic_i$ and $d=d_r+id_i$ are complex numbers such that a_r, a_i, b_r, b_i, c_r, c_i, d_r and d_i are real constants to be determined as a function of the parameters of the system. So equation (3) inserted into equation (1) yields

$$\begin{aligned}&\sum_{n=1}^{10}F_n(a_r,a_i,b_r,b_i,c_r,c_i,d_r,d_i)\frac{1}{\cosh^n\Omega_0t}\\&+\sum_{k=1}^{10}F'_k(a_r,a_i,b_r,b_i,c_r,c_i,d_r,d_i)\frac{\sinh\Omega_0t}{\cosh^k\Omega_0t}\\&+i\left[\begin{aligned}&\sum_{l=1}^{10}G_l(a_r,a_i,b_r,b_i,c_r,c_i,d_r,d_i)\frac{1}{\cosh^l\Omega_0t}\\&+\sum_{m=1}^{10}F'_m(a_r,a_i,b_r,b_i,c_r,c_i,d_r,d_i)\frac{\sinh\Omega_0t}{\cosh^m\Omega_0t}\end{aligned}\right]=0,\end{aligned} \tag{5}$$

where $i^2=-1$, $F_n(a_r,a_i,b_r,b_i,c_r,c_i,d_r,d_i)$, $F'_k(a_r,a_i,b_r,b_i,c_r,c_i,d_r,d_i)$, $G_l(a_r,a_i,b_r,b_i,c_r,c_i,d_r,d_i)$ and $F'_m(a_r,a_i,b_r,b_i,c_r,c_i,d_r,d_i)$ are function of the constants a_r, a_i, b_r, b_i, c_r, c_i, d_r and d_i. From the real and imaginary part of equation (5), we obtain according to the terms in $1/\cosh^n\Omega_0t$ and $1/\cosh^l\Omega_0t$ the series of equations

$$F_n(a_r,a_i,b_r,b_i,c_r,c_i,d_r,d_i)=0, \tag{6}$$

and

$$G_l(a_r,a_i,b_r,b_i,c_r,c_i,d_r,d_i)=0. \tag{7}$$

Similarly the terms in $\sinh\Omega_0t/\cosh^k\Omega_0t$ and $\sinh\Omega_0t/\cosh^m\Omega_0t$ lead to the following equations

$$F'_k(a_r,a_i,b_r,b_i,c_r,c_i,d_r,d_i)=0, \tag{8}$$

and

$$F'_m(a_r,a_i,b_r,b_i,c_r,c_i,d_r,d_i)=0. \tag{9}$$

Equations (4), …, (7) are the main equations which permits to investigate the form of solutions as in equation(4). Since these equations have eight unknowns, and not easy to solve, we proceed by different analyses.

3. Analysis of the Solutions

a) First case: $a_i = 0$, $b_r = 0$, $c_i = 0$, $d_r = 0$

From equations (6) and (7) we obtain

Term in $1/\cosh^{10}\Omega_0 t$,

$$c_r^3 - c_r d_i^2 = 0, \tag{10}$$

Term in $1/\cosh^9\Omega_0 t$,

$$-5\beta_4\Omega_0\left(a_r c_r^2 - b_i d_i c_r\right) + 2\beta_3\left(d_i c_r^2 - d_i^3\right) = 0, \tag{11}$$

Term in $1/\cosh^8\Omega_0 t$,

$$\beta_4\Omega_0^2\left(5c_r - 10d_i^2 - 5a_r^2 + 5b_i^2\right) + \beta_3\Omega_0\left(b_i c_r + 8a_r d_i\right) - 3\beta_2\left(c_r - d_i^2\right) = 0, \tag{12}$$

Term in $1/\cosh^7\Omega_0 t$,

$$\begin{aligned}&-\beta_4\Omega_0^2\left(-226a_r c_r^2 - 14a_r d_i^2 + 280b_i d_i c_r\right)\\ &-4\beta_3\Omega_0\left(20d_i c_r^2 - 44d_i^3 - 12a_r c_r b_i - 12b_i^2 d_i - 24d_i a_r^2\right)\\ &+24\beta_2\left(-7a_r c_r^2 + a_r d_i^2\right) = 0.\end{aligned} \tag{13}$$

The combination of equation (10) and (11) gives

$$c_r = d_i; \quad a_r = b_i. \tag{14}$$

Insertion of (14) in (12) and (13) respectively yields

$$a_1 d_i + a_2 b_i = 0, \tag{15}$$

and

$$b_1 b_i d_i + b_2 b_i^2 + b_3 d_i^2 = 0, \tag{16}$$

where $a_1 = 3\beta_2 - 10\beta_4\Omega_0^2$; $a_2 = 9\beta_2$, $a_3 = 3\beta_2$, $b_1 = -\left(10\beta_4\Omega_0^2 + 36\beta_2\right)$; $b_2 = 48\beta_3\Omega_0$, $b_3 = 24\beta_3\Omega_0$.

Eliminating d_i between equation (13) and (14) gives the quadratic equation

$$Ab_i^2 + Bb_i + C = 0, \tag{17}$$

where the coefficients A, B and C are given by $A = a_2^2 b_3 - a_1 a_2 b_1 + a_1^2 b_2$; $B = a_1 a_3 b_1 - 2a_3 a_2 b_3$ and $c = b_3 a_3^2$.

- For $A = 0$, i.e. $a_2^2 b_3 = a_1 a_2 b_1 + a_1^2 b_2$ we obtain

$$d_i = \frac{a_3\left(a_2 b_3 - a_1 b_1\right)}{a_1\left(2a_3 b_3 - a_1 b_1\right)}, \tag{18}$$

$$c_r = \frac{\pm a_3\left(a_2 b_3 - a_1 b_1\right)}{a_1\left(2a_3 b_3 - a_1 b_1\right)}, \tag{19}$$

$$b_i = a_r = \frac{b_3 a_3}{2a_2 b_3 - a_1 b_1}. \tag{20}$$

The resulting solution of equation (3) in this case is given by

$$U_1(z,t) = \left[\begin{array}{l} \dfrac{b_3 a_3}{2a_2 b_3 - a_1 b_1}\left(\dfrac{1}{\cosh \Omega_0 t} + i\dfrac{\sinh \Omega_0 t}{\cosh \Omega_0 t} \right) \\ + \dfrac{a_3 (a_2 b_3 - a_1 b_1)}{a_1 (2a_2 b_3 - a_1 b_1)}\left(\dfrac{\pm 1}{\cosh^2 \Omega_0 t} + i\dfrac{\sinh \Omega_0 t}{\cosh^2 \Omega_0 t} \right) \end{array} \right] \exp - ikz. \tag{21}$$

– For $A \neq 0$; i.e. $a_2^2 b_3 \neq a_1 a_2 b_1 + a_1^2 b_2$, we obtain when $B^2 \neq 4AC$ the following values

$$a_r = b_i = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \tag{22}$$

$$d_i = \frac{a_3}{a_1} - \frac{a_2}{a_1}\left(\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \right), \tag{23}$$

and

$$c_r = \pm d_i = \pm \left[\frac{a_3}{a_1} - \frac{a_2}{a_1}\left(\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \right) \right]. \tag{24}$$

The resulting solution of equation (3) in this case is given by

$$U_2(z,t) == \left[\begin{array}{l} \dfrac{-B + \sqrt{B^2 - 4AC}}{2A}\left(\dfrac{1}{\cosh \Omega_0 t} + i\dfrac{\sinh \Omega_0 t}{\cosh \Omega_0 t} \right) \\ \pm \left[\dfrac{a_3}{a_1} - \dfrac{a_2}{a_1}\left(\dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A} \right) \right]\left(\dfrac{\pm 1}{\cosh^2 \Omega_0 t} + i\dfrac{\sinh \Omega_0 t}{\cosh^2 \Omega_0 t} \right) \end{array} \right] \exp - ikz \cdot \tag{25}$$

– For $B^2 = 4AC$, equation (25) reads

$$U_3(z,t) = \left[\begin{array}{l} \dfrac{-B}{2A}\left(\dfrac{1}{\cosh \Omega_0 t} + i\dfrac{\sinh \Omega_0 t}{\cosh \Omega_0 t} \right) \\ \pm \left(\dfrac{a_3}{a_1} + \dfrac{a_2 B}{2a_1 A} \right)\left(\dfrac{\pm 1}{\cosh^2 \Omega_0 t} + i\dfrac{\sinh \Omega_0 t}{\cosh^2 \Omega_0 t} \right) \end{array} \right] \exp - ikz \cdot \tag{26}$$

b) Second case: only $a_r \neq 0$; $c_i \neq 0$

We obtain from order $1/\cosh^7 \Omega_0 t$ to $1/\cosh^6 \Omega_0 t$ of equation (6) and (7), the following equations

$$7\Omega_0^2 \beta_4 c_i + 12\beta_2 = 0, \tag{27}$$

and

$$6(\Gamma + \gamma)c_i^2 + \beta_4 \Gamma \Omega_0^2 \left(4c_i^2 - 30a_r^2\right) - \beta_2 \Gamma \Omega_0^2 \left(12c_i^2 - 18a_r^2\right) = 0 \cdot \tag{28}$$

From equations (27) and (28) one finds

$$c_i = -12\beta_2 / 7\Omega_0^2 \beta_4, \tag{29}$$

and

$$a_r = \pm\left(\frac{12\beta_2}{7\Omega_0^2\beta_4}\right)\sqrt{\frac{\Gamma+\gamma+\frac{2}{3}\beta_4\Gamma\Omega_0^4-2\beta_2\Gamma\Omega_0^2}{5\beta_4\Gamma\Omega_0^4-3\beta_2\Gamma\Omega_0^2}}\,. \tag{30}$$

Here, the solution of equation (3) is given by

$$U_4(z,t)=\pm\left(\frac{12\beta_2}{7\Omega_0^2\beta_4}\right)\sqrt{\frac{\Gamma+\gamma+\frac{2}{3}\beta_4\Gamma\Omega_0^4-2\beta_2\Gamma\Omega_0^2}{5\beta_4\Gamma\Omega_0^4-3\beta_2\Gamma\Omega_0^2}}\left(\frac{1}{\cosh\Omega_0 t}\right)-\frac{12i\beta_2}{7\Omega_0^2\beta_4}\left(\frac{1}{\cosh^2\Omega_0 t}\right). \tag{31}$$

c) third case: only $c_r \neq 0$, $d_i \neq 0$

From the terms in $1/\cosh^8\Omega_0 t$ and $1/\cosh^7\Omega_0 t$ of equations (6) and (7) the equations

$$\left(5\beta_4\Omega_0^2-3\beta_2\right)c_r^2+\left(3\beta_2-10\beta_4\Omega_0^2\right)d_i^2=0\,, \tag{32}$$

and

$$c_r^2 = 20d_i^2\,. \tag{33}$$

Substituting equation (33) into equation (32) with the constraint $\beta_2 = 30\beta_4\Omega_0^2/19$ gives the solution of the form

$$U_5(z,t)=d_i\left(\frac{\pm i\sqrt{20}}{\cosh^2\Omega_0 t}+\frac{i\sinh\Omega_0 t}{\cosh^2\Omega_0 t}\right)\exp -ikz,\quad d_i\neq 0. \tag{34}$$

d) Fourth case: only $a_r \neq 0$ and $b_i \neq 0$

We obtain for any parameter coefficient of the nonlinear partial differential equation (3) $b_i = \pm a_r$. So the solution of equation (3) in this case is given by

$$U_6(z,t)=a_r\left(\frac{1}{\cosh\Omega_0 t}\pm i\frac{\sinh\Omega_0 t}{\cosh\Omega_0 t}\right)\exp -ikz,\quad a_r\neq 0\,. \tag{35}$$

e) Fifth case: only $c_i \neq 0$, $d_r \neq 0$

From the order $1/\cosh^{10}\Omega_0 t$ to order $1/\cosh^9\Omega_0 t$ of equations (6) and (7), we obtain $c_i = \pm d_r$ and the solution is given by

$$U_7(z,t)=d_r\left(\frac{\pm i}{\cosh^2\Omega_0 t}+\frac{\sinh\Omega_0 t}{\cosh^2\Omega_0 t}\right)\exp -ikz,\quad d_r\neq 0. \tag{36}$$

f) sixth case: only $b_r \neq 0$, $c_i \neq 0$

Here both right-hand side and left-hand side of the obtained equations (6),…, (9) vanish for $\beta_3 = \beta_2 = 0$. Then any solution of the following form verify equation (3)

$$U(z,t)=\left(b_r\frac{\sinh\Omega_0 t}{\cosh\Omega_0 t}+i\frac{c_i}{\cosh^2\Omega_0 t}\right)\exp -ikz\,, \tag{37}$$

where b_r and c_i are real numbers.

g) Seventh case: only $b_r \neq 0$

The equation which derives from the term in $\sinh\Omega_0 t/\cosh^5\Omega_0 t$ gives $b_r = \sqrt{\beta_4\Omega_0^2/\Gamma\beta_2}$ with $\beta_3 = 0$. The solution in this case is given by

$$U(z,t)=\sqrt{\frac{\beta_4\Omega_0^2}{\Gamma\beta_2}}.\frac{1}{\cosh^2\Omega_0 t}\exp-ikz. \tag{38}$$

4. Conclusions

To start the research of solutions as proposed in this manuscript, we remark that in numerous studies of the modulational instability proposed by most authors on the equation (1), the perturbated solution is most often a plane wave and very rarely a solitary wave. This is how we took the option to determine some solitary waves of this equation. Knowing that there is no standard method of resolution of the nonlinear partial differential equations, we have use the effective method as used in this manuscript to attain our objective. Certainly it necessitates a lot of concentration, but also gives a lot of satisfaction. Apart from the above solutions obtained, the following cases:

$a_i=b_i=c_i=d_i=0$;

$a_r=b_r=c_r=d_r=a_i=c_i=0$;

$a_r=c_r=d_r=a_i=b_i=c_i=0$;

$a_r=b_r=d_r=a_i=b_i=c_i=0$;

$a_r=b_r=c_r=d_r=a_i=b_i=c_i=0$

lead to trivial solutions or impossibilities. This principle of looking for solution maybe extended to other types of nonlinear partial differential equations susceptible to have some solitary wave solutions. A numerical approach can also be considered.

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