Numerical Solution of Parabolic Volterra Integro-Differential Equations via Backward-Euler Scheme

Ali Filiz

Department of Mathematics, Adnan Menderes University, Aydin, 09010, Turkey

Abstract   In this study, we investigate convergence properties of time and space discretization of Parabolic Volterra integro-differential equations (PVIDE) is outlined. The rectangle and the trapezoidal rules applied for integral term of these equations and finite difference method (Backward-Euler method) used for partial differential part. The integral is approximated in each case by the quadrature rule with relatively high-order truncation error. We consider time step methods based on the backward-Euler method and combined with the appropriate quadrature rules.

Keywords   Finite difference method, Backward-Euler method, Parabolic Volterra integro-differential equations, Integral equations, Partial differential equations, Time and space discretization, Mixed rule, Quadrature rule, Error

1. Introduction

In[5] and[1], Linz and Baker considered the numerical solution of Volterra integral equations of the second kind using the rectangle, the trapezium and Simpson's rules for finding $u(t)$ with the quadrature rule. Whereas, in this paper we introduce the numerical treatment of parabolic Volterra integro-differential equations using the backward-Euler scheme for finding $u(x,t)$ with the finite difference method. In[2], Douglas introduced the numerical treatment of parabolic Volterra equations using the backward-Euler and Crank-Nicolson methods for finding $u(x,t)$ with the finite difference method of the form

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + \int_0^t g(t,s)u(x,s)ds. \quad (1)$$

Douglas's has used quite simple equation, but it is important that he used firstly time discretization using the backward-Euler and Crank-Nicolson methods. However, In[2] and[7] the numerical examples are not given. In this paper the time and space discretization are studied with examples. In this paper we will study the numerical solution by the time-continuous finite difference method of parabolic Volterra equations of the form

$$\frac{\partial u}{\partial t}(x,t) = Au(x,t) + \lambda \int_0^t K(t,s)\,Bu(x,s)ds + S(x,t), \quad (2)$$

in $D = B \times L$. In the (x,t) plane let $D = \{(x,t): 0 < x < \pi,\ 0 < t < T\}$, and let $u(x,t)$ have the initial and boundary conditions $u(x,0) = f(x)$, on $0 \le x \le \pi$, $u(0,t) = u(\pi,t) = 0$.

Here we denote by $\overline{D}$ the closure of D. A is an elliptic second order partial differential operator and B is a second order partial differential operator respectively.

We assume that both operators are smooth. The equation (2) is to be solved subject to the above initial and boundary conditions. As they defined in[5] and[1], the kernel $K(s,t)$ is a smooth, real-valued function of both variables for $0 \le s \le t$, and $S(x,t)$ is supposed to be a smooth function of x and t. The numerical methods will be a combination of finite difference and quadrature schemes for the time stepping, with time step k. Introducing a lattice in $\overline{D}$, we define for positive integers m and n, $h = \pi/m, \quad k = T/n$,

$$\overline{D_{i,j}} = \{(ih, jk) : i = 0,1,2,..,n; j = 0,1,2,..,m\}.$$

For any function U defined on $\overline{D_{i,j}}$, we set $U_{i,j} = U(ih, jk)$. We replace the time derivative in (2) by a difference quotient, and use a quadrature rule for the integral term of the form

$$\int_0^{nk} K(t,s)u(x,s)ds \approx \sum_{j=0}^{n-1} \alpha_{n,j}K(t_n, jk)U(ih, jk), (3)$$

where $\alpha_{n,j}$ a quadrature is weight and $U(ih, jk)$ is the approximation to $u(x,s)$.

2. Numerical Methods for Parabolic Volterra Integro-differential Equations

In this section we will study the initial boundary value problem first considered in the introduction, i.e.,

$$\frac{\partial u}{\partial t}(x,t) = Au(x,t) + \lambda \int_0^t K(t,s)Bu(x,s)ds + S(x,t), \quad (4)$$

in $D = B \times L$. In the (x,t) plane let $D = \{(x,t): 0 < x < \pi,\ 0 < t < T\}$, and let $u(x,t)$ have the initial and boundary conditions $u(x,0) = f(x)$, on $0 \le x \le \pi$, $u(0,t) = u(\pi,t) = 0$.

One of the difficulties involved in such a method is that if $\alpha_{n,j} \neq 0$ for $j \le n$, then all values of $U(ih, jk)$ have to be retained, causing great demands for data storage. In Sloan and Thomée[7] a quadrature rule based on fewer points is proposed, thus reducing the number of time levels at which the data need be saved.

The quadrature rule will be of the form

$$\sigma^n(g) = \sum_{j=0}^{n-1} \alpha_{n,j}g(t_j) \approx \int_0^{nh} g(s)\,ds, \quad t_j = jk. \quad (5)$$

We define the backward difference method by

$$\frac{U_n - U_{n-1}}{k} = AU_n + \lambda \sum_{j=0}^{n-1} \alpha_{n,j}K(t_n, t_j)U(ih, jk). \quad (6)$$

The simplest quadrature rule of type (3) which is consistent with the order of accuracy of the backward-Euler scheme is the rectangle rule

$$\sigma^n(g) = \sum_{j=0}^{n-1} g(t_j),\ t_j = jk, \quad (7)$$

which corresponds to choosing $\alpha_{n,j} = k$ for $0 \le j \le n - 1$, so that

$$\sum_{j=0}^{n-1} \alpha_{n,j} = nk = t_n = T. \quad (8)$$

Let $m = [1/\sqrt{k}]$ and $m \in Z^+$. Set $k_1 = mk$, and let l be the largest integer such that $lk_1 \le t_n \le nk$. In

approximating the integral term over $[0, nk]$ we shall apply the trapezoidal rule with step size k_1 on $[0, lk_1]$ and the rectangle rule with step-size k on the remaining part $[lk_1, nk]$. More precisely, with $\overline{t_j} = jk_1$,

$$\sigma^n(g) = \sum_{j=0}^{n-1} \alpha_{n,j}g(t_j),$$

$$= \frac{k_1}{2}\sum_{j=1}^{l}[g(\overline{t_j}) + (\overline{t_{j-1}})] + k\sum_{j=ml}^{n-1} g(t_j). \quad (9)$$

Note that this rule has a storage requirement of $O(k^{-1/2})$ as opposed to $O(k^{-1})$ for the simple rectangle rule. Sloan and Thomée[7] deal with stability and convergence results for two different time discretization of (6), based on the backward-Euler and Crank-Nicolson methods respectively. In their paper, the time discretization is studied in the treatment of the integral term. Let k be the time step in the backward-Euler scheme for the equation

$$\frac{\partial u}{\partial t}(x,t) = \kappa \frac{\partial^2 u}{\partial x^2}(x,t) + \lambda \int_0^t K(t,s)u(x,s)ds + S(x,t). \quad (10)$$

or for the following form with elliptic operator under the integral term,

$$\frac{\partial u}{\partial t}(x,t) = \kappa \frac{\partial^2 u}{\partial x^2}(x,t) + \lambda \int_0^t K(t,s)\frac{\partial^2 u}{\partial x^2}(x,s)ds + S(x,t). \quad (11)$$

If we substitute the backward difference for u_t, centred difference for u_{xx} in (10), and a quadrature rule for the integral term, we obtain the following form

$$\frac{U_{i,j} - U_{i,j-1}}{k} = \kappa \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2}$$

$$+ \lambda \sum_{v=0}^{j-1} \alpha_{j,v}K(jk, vk)U(ih, vk) + S_{i,j}, \quad j \ge 1, \quad (12)$$

and if we substitute the backward difference for u_t, centred difference for u_{xx} in (11), and a quadrature rule for the integral term, we obtain the following form

$$\frac{U_{i,j} - U_{i,j-1}}{k} = \kappa \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2}$$

$$+ \lambda \sum_{v=0}^{j-1} \alpha_{j,v}K(jk, vk)\left(\frac{U_{i-1,v} - 2U_{i,v} + U_{i+1,v}}{h^2}\right) \quad (13)$$

$$+ S_{i,j}, \quad j \ge 1,$$

where h is the mesh step for x. The sum on the RHS is the quadrature approximation to the integral term of (10) or (11) at the point $t_j = jk$. The weights in the quadrature rule

correspond to (9) above. For sufficiently regular functions g the truncation error is of order $O(k_1^2 + k_1 k)$, assuming that t is bounded. If we take $k_1 = O(k^{1/2})$ the accuracy is similar to that of the backward-Euler scheme. If the weights α_v^1 are trapezoidal weights for step-size k_1 and α_v^2 are rectangle rule weights for step-size k, we can write

$$\alpha_{j,v} = \begin{cases} \alpha_v^1 = k_1, & when & j \equiv 0 \ (\mathrm{mod}\, m) \\ \alpha_v^1 = 0, & when & j \neq (\mathrm{mod}\, m), j - v > m \\ \alpha_v^2 = k, & otherwise. \end{cases} \quad (14)$$

From (14) we have

$$\alpha_{j,v} \leq \max(\alpha_v^1, \alpha_v^2) = \alpha_v, \quad (15)$$

where, as easily verified (assuming $k_1 \leq 1$),

$$\sum_{v=0}^{j-1} \alpha_v \leq 2jk + 1. \quad (16)$$

Note that in the numerical solution of Example 2.1, Example 2.2 and Example 2.4, we did not include the function $S(x,t)$. Examples with the function $S(x,t)$ can be done in a similar manner.

Example 2.1: Consider the initial-boundary value problem

$$\frac{\partial u}{\partial t}(x,t) = \kappa \frac{\partial^2 u}{\partial x^2}(x,t) + \lambda \int_0^t e^{-(t-s)} u(x,s)ds, \quad (17)$$

for $0 \leq t \leq T$, $0 \leq x \leq \pi$, with $u(x,0) = \sin(x)$, $u(0,t) = 0$, and $u(\pi,t) = 0$, which has the exact solution

$$u(x,t) = e^{-(\alpha t)}\left(\cos(\beta t) + \left(\frac{(1-\alpha)}{\beta}\right)\sin(\beta t)\right)\sin(x), \,(18)$$

where $\alpha = (\kappa + 1)/2$ and $\beta = (\sqrt{-\kappa^2 + 2\kappa - 1 - 4\lambda})/2$.
Taking $\kappa = 1$ in the equation (18) we have the solution

$$u(x,t) = e^{-t}\cosh(\sqrt{\lambda}t)\sin(x), \quad for \ \lambda > 0. \quad (19)$$

If we put $\kappa = 1$ and $\lambda = -1$ in (18) we obtain the solution

$$u(x,t) = e^{-t}\cos(t)\sin(x). \quad (20)$$

The last result corresponds to the simplest form of our main problem (17). We will solve the equation (17) with the backward-Euler method. Firstly, we use the rectangle rule throughout for the integral term in equation (17). Table 1 shows numerical results for this case. Unless otherwise indicated, throughout this section and the next section we take $\kappa = 1$ and $\lambda = -1$, and T = 0.5. Table 1 shows that the combined backward-Euler and rectangle rule has error $O(k)$ for this Example 2.1

Table 1. Rectangle rule showing effect of time-step k, for Example 2.1, has an error of order $O(k)$ with fixed $h = \pi/100$

$t = 0.5$		$k=0.005$	$k=0.0025$	$k=0.00125$
x	$Exact$	$Error_1$	$Error_2$	$Error_3$
0.31	0.16448	0.109841E-03	0.584606E-04	0.274675E-04
0.63	0.31287	0.208929E-03	0.111199E-03	0.522463E-04
0.94	0.43062	0.287567E-03	0.153052E-03	0.719108E-04
1.26	0.50623	0.338055E-03	0.179923E-03	0.845362E-04
1.57	0.53228	0.355452E-03	0.189182E-03	0.888866E-04
‖error‖		2.513425E-03	1.337721E-03	7.498422E-04

In this test we see that the space error has the expected order in our numerical calculations.

Secondly, we use the trapezoidal rule throughout for the integral term in equation (17). The numerical results are given by Table 2. Thirdly, we obtain the numerical solution of equation (11) using the trapezoidal rule and rectangle rule as in (9) which is called the mixed rule. Table 2 illustrates that the combined backward-Euler and trapezoidal rule also has error of order k for Example 2.1

Table 2. Trapezoidal rule showing effect of time-step k, for Example 2.1, has an error of order $O(k)$ with fixed $h = \pi/100$

$t = 0.5$		$k=0.005$	$k=0.0025$
x	$Exact$	$Error_1$	$Error_2$
0.31	0.16448	0.119405E-03	0.632306E-04
0.63	0.31287	0.227122E-03	0.120272E-03
0.94	0.43062	0.312607E-03	0.165540E-03
1.26	0.50623	0.367491E-03	0.194604E-03
1.57	0.53228	0.386403E-03	0.204619E-03
‖error‖		2.732282E-03	1.446872E-03

Table 3 shows numerical results for the mixed rule (backward-Euler scheme). In Table 3, we take m = integral part of $[1/\sqrt{k}]$, with $km = k_1$; for the cases in Table 3, we have $[1/\sqrt{k}]$ = integer. First, we will apply the simplest rule which is the rectangle rule for the integral term. Table 3 illustrates that the combined trapezoidal and rectangle rule has error of order k for Example 2.1.

Table 3. Mixed rule showing effect of time-step k, for Example 2.1, has an error of order $O(k)$ with fixed $h = \pi/100$

$t = 0.5$		$k = 0.01$ $k_1 = 0.1$	$k = 0.0025$ $k_1 = 0.05$	$k = 0.00111$ $k_1 = 0.0333$
x	$Exact$	$Error_1$	$Error_2$	$Error_3$
0.31	0.16448	0.240776E-03	0.667188E-04	0.337623E-04
0.63	0.31287	0.457984E-03	0.126907E-03	0.642197E-04
0.94	0.43062	0.603361E-03	0.174672E-03	0.883908E-04
1.26	0.50623	0.741033E-03	0.205339E-03	0.103910E-03
1.57	0.53228	0.779169E-03	0.215907E-03	0.109257E-03
‖error‖		0.550955E-02	0.152669E-02	0.772564E-03

Now consider the equation (11) where we have an elliptic operator under the integral sign. The numerical results for this equation are given in Table 4.

Example 2.2: Consider the initial-boundary value problem

$$\frac{\partial u}{\partial t}(x,t) = \kappa \frac{\partial^2 u}{\partial x^2}(x,t) + \lambda \int_0^t e^{-(t-s)} \frac{\partial^2 u}{\partial x^2}(x,s)ds, \quad (21)$$

for $0 \le t \le 1$, $0 \le x \le \pi$, with $u(x,0) = \sin(px)$, $u(0,t) = 0$, p is a positive integer, $p=1, 2, 3,…$, $u(\pi,t) = 0$. This has the exact solution

$$u(x,t) = e^{-(at/2)}\left(\cosh(ct/2) + \left(\frac{(2-a)}{c}\right)\sinh(ct/2)\right)\sin(px), (22)$$

where $a = (\kappa p^2 + 1)$, $b = p^2(\kappa + \lambda)$ and $c = (\sqrt{a^2 - 4b})$.

If we choose $\kappa = 1$, $\lambda = -1$ and $p = 1$ in (22) we obtain the solution

$$u(x,t) = \frac{1}{2}(1 + e^{-2t}\cos(t))\sin(x). \quad (23)$$

Table 4 illustrates that the combined backward-Euler and rectangle rule has error $O(k)$ for Example 2.2

Table 4. Rectangle rule showing effect of time-step k, for Example 2.2, has an error of order $O(k)$ with fixed $h = \pi/100$

$t = 0.5$		$k=0.005$	$k=0.0025$	$k=0.00125$
x	$Exact$	$Error_1$	$Error_2$	$Error_3$
0.31	0.16448	0.381092E-03	0.194095E-03	0.955590E-4
0.63	0.31287	0.724881E-03	0.369190E-03	0.181764E-3
0.94	0.43062	0.997714E-03	0.508147E-03	0.250178E-3
1.26	0.50623	0.117288E-02	0.597365E-03	0.294103E-3
1.57	0.53228	0.123325E-02	0.628109E-03	0.309241E-3
‖error‖		1.141703E-02	5.813977E-03	2.987913E-03

The last result corresponds to the simplest form of our main problem (21). In Table 4 we use the rectangle rule throughout for the integral term in equation (21). Equation (21) can be solved with the backward-Euler and Crank-Nicolson method (CN) in a similar manner. After these calculations we will extend our simple examples for more general initial conditions. First, we take the initial condition $u(x,0) = x(\pi - x)$ on $0 \le x \le \pi$. We can obtain the exact solution for general initial conditions by using Fourier expansions.

Example 2.3: Solve the problem

$$\frac{\partial u}{\partial t}(x,t) - \kappa \frac{\partial^2 u}{\partial x^2}(x,t) = 0 \quad for \;\; 0 < x < \pi, t > 0, \quad (24)$$

$$u(x,0) = x(\pi - x), \quad (25)$$

$$u(0,t) = u(\pi,t) = 0. \quad (26)$$

The Fourier coefficient for the initial values of $u(x,t)$ is

$$b_n = \frac{2}{\pi} \int_0^\pi x(\pi - x)\sin(nx)dx,$$

$$= \frac{4}{n^3\pi}(1 - \cos(n\pi)), \quad (27)$$

$$= \frac{4}{n^3\pi}(1 - (-1)^n), \quad n = 1, 2, 3,.....$$

Thus $b_n = \frac{8}{n^3\pi}$ if n is odd or zero is even, giving

$$u(x,t) = \frac{8}{\pi}\sum_{m=1}^{\infty} e^{-(2m-1)^2\kappa t}\frac{\sin((2m-1)x)}{(2m-1)^3}. \quad (28)$$

In the following example, we have solved the equation (17) with a different initial condition. In Example 2.1, we have chosen the initial condition $u(x,0) = \sin(x)$, however in Example 2.4, we have taken $u(x,0) = x(\pi - x)$.

Example 2.4: Solve the problem

$$\frac{\partial u}{\partial t}(x,t) = \kappa \frac{\partial^2 u}{\partial x^2}(x,t) + \lambda \int_0^t e^{-(t-s)}u(x,s)ds, \quad (29)$$

$$u(x,0) = x(\pi - x), \quad (30)$$

$$u(0,t) = u(\pi,t) = 0, \quad (31)$$

for $0 < x < \pi, t > 0$.

If we consider the Fourier expansion of $x(\pi - x)$, then we get the solution, for $\kappa = 1$ and $\lambda = -1$

$$u(x,t) = \frac{8}{\pi}\sum_{m=1}^{\infty} v(p,t)\frac{\sin(px)}{p^3}, \quad (32)$$

where p takes odd values, $v(p,t)$ is given by

$$v(p,t) = e^{-\alpha t}\left(\cosh(\beta t) - \frac{(\alpha - 1)}{\beta}\sinh(\beta t)\right), \quad (33)$$

Where $\alpha = (p^2 + 1)/2$ and $\beta = (\sqrt{\alpha^2 - 2\alpha})$.

The first term of (24), for $p = 1$, is

$$u_1(x,t) = \frac{8}{\pi}e^{-t}\sin(x)\cos(t), \quad (34)$$

which is $8/\pi$ times the solution of Example 2.1, because the first term of the Fourier series of the initial condition of Example 2.4 is equal to $8/\pi$ times initial condition of Example 2.1. It is enough to take the first twelve terms of the solution (32), to obtain 9 figures in the results. The errors for the rectangle rule are given in Table 5. Table 5 illustrates that the combined backward-Euler and rectangle rule has error of order k for Example 2.4.

Table 5. Rectangle rule showing effect of time-step k, for Example 2.4, has an error of order $O(k)$ with fixed $h = \pi /100$

t =0.5		k=0.005	k=0.0025
x	*Exact*	$Error_1$	$Error_2$
0.31	0.41904	0.341195E-03	0.180052E-03
0.63	0.79695	0.608422E-03	0.321954E-03
0.94	1.09668	0.759394E-03	0.403546E-03
1.26	1.28977	0.813800E-03	0.434301E-03
1.57	1.35517	0.823146E-03	0.440112E-03
‖error‖		6.425446E-03	3.418732E-03

In Table 6, we used the mixed rule for Example 2.4. In this calculation, we have obtained errors of $O(k)$. Table 6 illustrates that the combined backward-Euler (mixed rule) has error of order k.

Table 6. Mixed rule showing effect of time-step k, for Example 2.4, has an error of order $O(k)$ with fixed $h = \pi /100$

t =0.5		k =0.005 k_1 =0.0707	k =0.0025 k_1 =0.05	k =0.00125 k_1 =0.0353
x	*Exact*	$Error_1$	$Error_2$	$Error_3$
0.31	0.41904	0.373460E-03	0.196973E-03	0.108695E-03
0.63	0.79695	0.681001E-03	0.360494E-03	0.199164E-03
0.94	1.09668	0.863980E-03	0.459313E-03	0.254606E-03
1.26	1.28977	0.936862E-03	0.500141E-03	0.278308E-03
1.57	1.35517	0.952040E-03	0.509165E-03	0.283822E-03
‖error‖		0.732195E-02	0.389736E-02	0.216348E-02

3. Error Analysis in Time and Space for Backward-Euler

In this section, we introduce the discretization errors for our equation (12). In applications the operators *A* and *B* will not be differential operators such as

$$Au = \sum_{i,j}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j} \frac{\partial}{\partial x_i}\right), \qquad (35)$$

But rather approximate operators (arising from finite difference space discretization) which depend on some mesh parameter *h*. Our results will be useful if the error bounds are uniform in *h*. The error at a mesh point is defined by

$$e_{i,j} = u(x_i, t_j) - U(ih, jk). \qquad (36)$$

As we defined in Section 2, we abbreviate $u(x_i, t_j)$ and $U(ih, jk)$ by $u_{i,j}$ and $U_{i,j}$ respectively. Thus $U_{i,j} = u_{i,j} - e_{i,j}$. In equation (12) replace $U(ih, jk)$ by $u_{i,j} - e_{i,j}$ giving

$$\frac{u_{i,j} - e_{i,j} - (u_{i,j-1} - e_{i,j-1})}{k} =$$

$$\kappa \frac{u_{i-1,j} - e_{i-1,j} - 2(u_{i,j} - e_{i,j}) + u_{i+1,j} - e_{i+1,j}}{h^2} \qquad (37)$$

$$+ \lambda \sum_{v=0}^{j-1} \alpha_{j,v} K_{j,v} (u_{j,v} - e_{j,v}) + S_{i,j}, \; j \geq 1,$$

Then it follows from (10) and (12) that $e_{i,j}$ satisfy the equation

$$\frac{e_{i,j} - e_{i,j-1}}{k} = \kappa \frac{e_{i-1,j} - 2e_{i,j} + e_{i+1,j}}{h^2}$$

$$+ \lambda \sum_{v=0}^{j-1} \alpha_{j,v} K_{j,v} e_{j,v} + \tau_{i,j}, \; j \geq 1, \qquad (38)$$

where $\tau_{i,j}$ is the truncation error given by[3]. In Table 7, the space error is shown for the CN and trapezoidal rule (see[3] and[4]) for Example 2.1. The numerical results for the space error test give $O(h^2)$. The rectangle rule can be done similar manner. In Table 7, CN and trapezoidal rule showing effect of space-step for $k = 2.0/2000 = 0.001$.

Table 7. CN and trapezoidal rule has an error of order h^2

t =2.0		$h = \pi /10$	$h = \pi /20$	$h = \pi /40$
x	*Exact*	$Error_1$	$Error_2$	$Error_3$
0.31	-0.01740	0.145398E-04	0.338862E-05	0.829468E-06
0.63	-0.03310	0.276563E-04	0.644555E-05	0.157774E-05
0.94	-0.04556	0.380656E-04	0.887153E-05	0.217158E-05
1.26	-0.05356	0.447488E-04	0.104291E-04	0.255284E-05
1.57	-0.05632	0.470517E-04	0.109658E-04	0.268422E-05
‖error‖		7.345935E-02	1.835834E-02	4.588388E-03

In this test we see that the space error has the expected order in our numerical calculations.

4. Conclusions

In this paper, numerical method has been successfully developed for solving parabolic Volterra integro-differential equations. Numerical quadrature rules have been applied for integral term and backward-Euler method has been used for partial differential part. Errors values presented from Table 1 to Table 6 are calculated based upon backward-Euler and suitable quadrature methods with time step k. Values in these tables illustrate an error of order $O(k)$. Finally, in Table 7 the space error justifies $O(h^2)$ in our numerical solution. Numerical order of convergence is also calculated:

$$Ord = \frac{\ln(Error_1) - \ln(Error_2)}{\ln(2)}.$$

We expect that Ord = 1 in time-step k and Ord = 2 in space-step h. Obtained theoretical results are confirmed by numerical experiments.

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