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On Linear Superposition Principle Applying to Hirota Bilinear Equations

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Abstract The linear superposition principle of exponential travelling waves is analysed for equations of Hirota bilinear type, with an aim to construct a specific subclass of *N*-soliton solutions formed by linear combination of exponential travelling waves. Applications are made for Sawada-Kotera and a (2+1)-dimensional equations, thereby presenting their *N*-wave solutions. An opposite question is also put forward and discussed about generating Hirota bilinear equations possesses the indicated *N*-wave solutions and a few illustrative examples are presented, together with an algorithm using weights.

Keywords Hirota Bilinear Form, Soliton Equations, N-Wave Solution

1. Introduction

Searching for exact solution to nonlinear differential equations is significantly important in mathematical physics. Exact solution plays a vital role in understanding various qualitative and quantitative feature of nonlinear phenomenon. There are various classes of interesting exact solutions such as soliton and travelling wave solution, but it often needs for a specific mathematical technique to construct exact solution due to nonlinearity property in dynamics see[1]

Among the various method for getting the exact solution of the nonlinear differential equations,[2,3,4], Hirota bilinear method provide a direct and powerful approach to nonlinear integrable equations and it is widely used in constructing soliton solution[5] . The existence of *N*-soliton solution often implies integrability of the considered differential equation[6, 7]. Interactions between solitons are elastic and nonlinear, but unfortunately, the linear superposition principle does not hold for soliton any more. However, bilinear equations are the nearest neighbours to linear equations, and expected to have some features similar to those of linear equations.

In this paper, we would like extend the work[8, 9] to a (2+1)-dimensional and Sawada-Kotera equations in order to explore a key feature of the linear superposition principle that linear equations possess for Hirota bilinear equations while aiming to construct a specific sub-class of N-soliton solutions formed by linear combination of exponential travelling waves. Moreover as for integrable equations, there are detailed solution procedures by using determinant theories (see, e. g transaction of the nonlinear mathematical society 357(2005), 1753-1778 and applied mathematics and computation 217(2010), 10006-10013).

To be specific, we will, in section two of the paper, prove that a linear superposition principle can be applied to exponential travelling waves of Hirota bilinear equations, while in section three an application procedure for generating Hirota bilinear equations possessing *N*-waves solutions of linear combinations of exponential waves will be treated and in section four we propose an opposite procedure for generating Hirota bilinear equations possess *N*-wave solutions of linear combinations of exponential waves, along with algorithm using weights as in[10, 11]. A few new and general such Hirota bilinear equations are computed. And finally the paper will be concluded in section 5.

2. Linear Superposition Principle

We begin with a Hirota bilinear equation

$$P(D_{x_1}, D_{x_2}, \dots D_{x_m}) f.f = 0 \tag{1}$$

Where P is a polynomial in the indicated variables satisfying

$$P(0,0,...,0) = 0 \tag{2}$$

and $D_{x_i}, 1 \le i \le M$ are Hirota's differential operator defined by

$$\begin{aligned} D_y^p f(y).g(y) &= (\partial_y - \partial_{y'})^p f(y) g(y)|_{y'=y} \\ &= \partial_{y'}^p f(y+y') g(y-y')|_{y'=0}, p \ge 1 \end{aligned} \tag{3}$$

Various nonlinear equations of mathematical physics are written in Hirota form through a dependent variable transformation

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Let us introduce N wave variables

$$\eta_i = k_{1,i}x_1 + k_{2,i}x_2 + ... + k_{M,i}x_M, \quad 1 \le i \le N \quad (4)$$

and N exponential wave function

$$f_i = e^{\eta_i} = e^{k_{1,i}x_1 + k_{2,i}x_2 + ... + k_{M,i}x_M}, \quad 1 \le i \le N \quad (5)$$

where $k_{i,j's}$ are constants. Observing that we have a bilinear identity

$$P(D_{x_1}, D_{x_2}, ..., D_{x_M})e^{\eta_i} = p(k_{1,i} - k_{1,j}, k_{2,i} - k_{2,j}, ..., k_{M,i} - k_{M,j})e^{\eta_i + \eta_j} \quad (6)$$

It follows immediately from (2) that exponential wave functions $f_i, 1 \le i \le N,$ solves the Hirota bilinear equation (1).

Now consider the N waves testing function

$$\begin{aligned} f &= \varepsilon_1 f_1 + \varepsilon_2 f_2 + ... + \varepsilon_N f_N \\ &= \varepsilon_1 e^{\eta_1} + \varepsilon_2 e^{\eta_2} + ... + \varepsilon_N e^{\eta_N}, \end{aligned} \quad (7)$$

where $\varepsilon_i = 1 \le i \le N,$ are arbitrary constants. This is a general linear combination of N exponential travelling wave solutions. We supposed to have asked if it will present a solution to the Hirota bilinear equation (1) as each f_i does. The answer is positive. We are going to show that a linear superposition principle of those exponential waves will apply to Hirota bilinear equations, under some additional conditions on the exponential waves and possibly on the polynomial p as well.

Following (6) we can compute

$$\begin{aligned} &P(D_{x_1}, D_{x_2}, ..., D_{x_M}) f.f \\ &= \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j P(D_{x_1}, D_{x_2}, ..., D_{x_M})e^{\eta_i}.e^{\eta_j} \\ &= \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j P(k_{1,i} - k_{1,j}, ..., k_{M,i} - k_{M,j})e^{\eta_i + \eta_j} \end{aligned} \quad (8)$$

The bilinear property will play a vital role in establishing the linear superposition principle for the exponential wave $e^{\eta_i}, 1 \le i \le N,$ we can directly see from (8) that any linear combination of exponential wave solution $e^{\eta_i}, 1 \le i \le N,$ solves the Hirota bilinear equation (1) if the following condition

$$\begin{aligned} P(k_{1,i} - k_{1,j}, k_{2,i} - k_{2,j}, ..., k_{M,i} - k_{M,j}) = 0, \\ 1 \le i \ne j \le N, ... \end{aligned} \quad (9)$$

is satisfied. Here $i = j$ is excluded, since it is a consequences of (3), the condition (9) gives us a system of nonlinear algebraic equations on the wave related numbers $k_{i,j}'s$ when the polynomial P is fixed. We will show that there is also a possibility of existence of solution for the variables $k_{i,j}'s$ in lower dimension cases.

Theorem 2 (linear superposition principle), let $P(x_1, x_2, ..., x_M)$ be a multivariate polynomial satisfying (3) and the wave, variables $\eta_i, 1 \le i \le N$ be defined by (3). then any linear combination of the exponential waves $e^{\eta_i}, 1 \le i \le N,$ solves the Hirota bilinear equation (2) if condition (9) is satisfied.

This shows a linear superposition principle of exponential wave solution that applies to hirota bilinear equations and paves a way of constructing N wave solutions from linear combination of exponential waves within the Hirota bilinear formalism. The system (9) is a key condition we need to handle. Once we get a solution of the wave related numbers $k_{i,j}'s$ by solving (9), we can present an N wave solution formed by (7), to the considered nonlinear equation. Details of the theorem will be found in[8]

Taking one of the variables $\eta_i, 1 \le i \le N$, to be constant, for example, taking

$$\eta_{i0} = \varepsilon_{i0}, \qquad i.e\ k_{1,i0} = 0,\ 1 \le i \le M \quad (10)$$

Where $1 \le i_0 \le N$ is fixed, the N wave solution conditions (9) subsequently requires all other wave related numbers to satisfy the dispersion relation

$$P(k_{1,i}, k_{2,i}, ..., k_{M,i}) = 0, \quad 1 \le i \le N,\ i \ne i_o \quad (11)$$

This corresponds to a specific case of N soliton solutions by the Hirota perturbation technique[6] truncated at the second order perturbation term. However, it is not generally necessary to satisfy the dispersion relation.

3. Application to Soliton Equations

We would like to consider two application examples of constructing the N-wave solutions by using the linear superposition principle in theorem (2).

3.1. Sawada-Kotera Equation

The first example to be considered is the Sawada-Kotera equation

$$u_{xxxxx} + 15uu_{xxx} + 15u_x u_{xx} + 45u^2 u_x + u_t = 0 \quad (12)$$

Through a dependent variable transformation $u = 2(\ln f)_{xx}$ equation (10) can be written as

$$(D_x^6 + D_x D_t) f.f = 0 \quad (13)$$

Which is equivalent to

$$P(k_i - k_j, \omega_i - \omega_j) = 0 \quad (14)$$

Assumed that the N-wave variables (4) are determined by $\eta_i = k_i x + \omega_i t$

now since

$$P(D_x, D_t) = (D_x^6 + D_x D_t) \quad (15)$$

Then (9) becomes

$$P(k_i - k_j, \omega_i - \omega_j) = 0 \quad (16)$$

Which is equivalent to

$$(k_i-k_j)^6+(k_i-k_j)(\omega_i-\omega_j)=0$$
$$k_i^6-6k_i^5k_j+15k_i^4k_j^2-20k_i^3k_j^3+15k_i^2k_j^4-6k_ik_j^5+$$
$$k_j^6+k_i\omega_i-k_i\omega_j-k_j\omega_i+k_j\omega_j=0 \tag{17}$$

applying the dispersion relation

$$P(k_i,\omega_i)=0,\ \ P(k_j,\omega_j)=0$$

Then (3.6) becomes

$$-6k_i^5k_j+15k_i^4k_j^2-20k_i^3k_j^3+15k_i^2k_j^4-$$
$$6k_ik_j^5-k_i\omega_j-k_j\omega_i=0 \tag{18}$$

For i = j we have

$$-6k_i^6+15k_i^6-20k_i^6+15k_i^6-6k_i^6-$$
$$k_i\omega_i-k_i\omega_i=0 \tag{19}$$
$$=-2k_i^6-2k_i\omega_i=0$$
$$\omega_i=-k_i^5$$

Hence

$$\eta_i=k_ix-k_i^5t$$

And therefor,e by the linear superposition principle in theorem (2.1) the Sawada-Kotera equation (3.1) has the following N-wave solutions

$$f=\sum_{i=1}^{N}\varepsilon_i f_i=\sum_{i=1}^{N}\varepsilon_i e^{k_ix-k_i^5t}$$

using $u=2(\ln f)_{xx}$

Where $k_i\text{'s and }\varepsilon_i\text{'s}$ are arbitrary constant.

3.2. (2+1)-Dimensional Equation

The (2+1)-dimensional equation

$$v_t=-v_{xxy}+4vv_y+2v_x\partial_x^{-1}v_y \tag{20}$$

can be transformed in to

$$(D_xD_t+D_yD_x^3)f.f=0 \tag{21}$$

Using a dependent variable transformation

$$v=-2(\ln f)_{xx}$$

Assumed the N-wave variables are determined by

$$\eta_i=k_ix+l_iy+\omega_it$$

Then

$$P(D_x,D_y,D_t)=D_xD_t+D_yD_x^3$$

Therefore, (9) becomes

$$P(k_i-k_j,l_i-l_j,\omega_i-\omega_j)=0 \tag{22}$$

Using (9) we have

$$(k_i-k_j)(\omega_i-\omega_j+(l_i-l_j)(k_i-k_j)^3=0 \tag{23}$$

Expanding and applying the dispersion relation $P(k_i,l_i,\omega_i)=0$

$$=-k_i\omega_j-k_j\omega_i-3l_ik_i^2k_j+3l_ik_ik_j^2-l_ik_j^3-l_jk_i^3+$$
$$3l_jk_i^2k_j-3l_jk_ik_j^2=0$$

For i = j we have

$$-2k_i\omega_i-2l_ik_i^3=0$$

Solving to get

$$l_i=\frac{1}{k_i},\qquad \omega_i=-k_i$$

Hence by linear superposition in theorem (2) the (2+1)-dimensional equation (18) has the following N-wave equation

$$v=-2(\ln f)_{xx},\ \ f=\sum_{i=1}^{N}\varepsilon_i f_i=\sum_{i=1}^{N}\varepsilon_i e^{k_ix+\frac{1}{k_i}y-k_it}$$

4. An Opposite Question

Here we will like also to apply the algorithm proposed by[8] in order to construct an opposite procedure for conversely constructing Hirota bilinear equations possesses *N*-wave solutions formed by linear combinations of exponential waves. This is an opposite question on applying the linear superposition principle in theorem 2

We first find a multivariate polynomial $P(x_1,x_2,...,x_M)$) with no constant term such that

$$P(k_{1,1}-k_{2,1},k_{1,2}-k_{2,2},...,k_{1,M}-k_{2,M})=0, \tag{24}$$

for two parameters $k_{i,1},k_{i,2},...,k_{i,M}$, i – 1,2. Each of which would better contain at least one free parameter. Then formulate a Hirota bilinear equation through (2) using the polynomial P. Theorem 2 tells that the resulting Hirota bilinear equation posses multiple wave solution of linear combination of exponential travelling waves. Such as multivariate polynomial P can be normally found by balancing the involved free parameter in (24), and sometimes upon assuming that two sets of parameters satisfy the dispersion relation

$$P(k_{i,1},k_{i,2},...,k_{i,M})=0 \text{ i=1,2.} \tag{25}$$

Examples of equations expressed in f

An algorithm can be given to use the concept of weights, to compute examples of Hirota bilinear equations that possess the linear superposition principle of exponential waves. Let us first define the weights of independent variables $(w(x_1),w(x_2),...,w(x_M)))$ where each weight $w(x_i)$ is an integer, and then form homogenous polynomials $P(x_1,x_2,...,x_M)$ of some weights to check if it will still satisfy the condition (24). The simplest way to start our checking is to assume that the weights variables η_i 's involve arbitrary constants. This way we can compare power of those arbitrary constants in (24) to obtain algebraic equations on other constants and/or coefficients to solve. The following are some examples which apply the algorithm using weights.

Examples with N-waves satisfy the dispersion relation

Example 1 weights $(w(x), w(t)) = (1,2)$

Let us first introduce the weights of independent variables

$$(w(x), w(t)) = (1,2)$$

Then a general homogenous polynomials of weights 2 is

$$p = c_1 x^2 + c_2 t \tag{26}$$

Assume that the weights variables are

$$\eta_i = k_i x + b_1 k_i^2 t, \quad 1 \le \mathrm{i} \le \mathrm{N} \tag{27}$$

where k_i 's are arbitrary constants but b_1 is a constant to be determined this way, a direct computation tells that the corresponding Hirota bilinear equation $P(D_x, D_t) f.f = 0$ possesses an N-wave solution

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + b_1 k_i^2 t} \tag{28}$$

Where, ε_i 's and k_i 's are arbitrary but b_1 satisfies

$$c_1 + c_2 b_1 = 0 \tag{29}$$

and has the N-wave solution defined (28) with

$$b_1 = -\frac{c_1}{c_2}$$

Example 2. Weights $(w(x), w(y), w(t)) = (1,1,3)$

Let us secondly, introduce the weights of independent variables:

Weights

$$(w(x), w(y), w(t))) = (1,1,3)$$

Then, a general homogenous polynomial of weights 3 reads

$$P = c_1 x^3 + c_2 x^2 y + c_3 t = 0. \tag{30}$$

Assume that the wave variables are

$$\eta_i = k_i x + b_1 k_i^2 y + b_2 k_i^3, \quad 1 \le \mathrm{i} \tag{31}$$

where k_i 's $1 \le \mathrm{i} \le \mathrm{N}$, are arbitrary constants but b_2 and b_2 are constant to be determined. This way a direct computation tells us that the corresponding Hirota bilinear equation $P(D_x, D_t) f.f = 0$ possesses the N-wave solution.

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + b_1 k_i^2 t + b2 k_i^3 t} \tag{32}$$

Where, ε_i 's and k_i 's are arbitrary but b_1 and b_2 satisfy

$$\left.\begin{aligned} c_3 b_2 + 2c_1 = 0 \\ c_2 b_1 - c_1 = 0 \end{aligned}\right\} \tag{33}$$

And the N-wave solution defined by (32) with

$$b_1 = \frac{c_1}{c_2}, \quad b_2 = \frac{-2c_1}{c_3},$$

Examples with N-wave not satisfying the dispersion relation

Weights $(w(x), w(y), w(t)) = (1,-1,2)$

Let us finally introduce the weights of independent variables

Weights $(w(x), w(y), w(t)) = (1,-1,2)$

Then, the homogenous polynomial of weights 2 reads

$$P = c_1 x^2 + c_2 xyt + c_3 t^2 y^2 + c_4 t \tag{34}$$

Assume that the wave variables are

$$\eta_i = k_i x + b_1 k_i^{-1} y + b_2 k_i^2 t, \quad 1 \le \mathrm{i} \le \mathrm{N}, \tag{35}$$

Where k_i are arbitrary but b_1 and b_2 are constants to be determined.

Now a similar direct computation tells that the corresponding Hirota bilinear equation

$$(c_1 D_x + c_2 D_x D_y D_t + c_3 D_t^2 D_y^2 + c_4 D_t) f.f = 0 \tag{36}$$

Possesses an N-wave solution

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + b_1 k_i^{-1} y + b_2 k_i^2 t} \tag{37}$$

Where, ε_i 's and k_i 's are arbitrary but b_1 and b_2 satisfy

$$\left.\begin{aligned} c_4 b_2 - c_2 b_1 b_2 = 0 \\ c_1 + 2c_2 b_1 b_2 = 0 \\ c_3 b_1^2 b_2^2 = 0 \end{aligned}\right\} \tag{38}$$

The taking $c_3 = 0$ tells that the Hirota bilinear equation

$$(c_1 D_x + c_2 D_x D_y D_t + c_4 D_t) f.f = 0 \tag{39}$$

Has N- wave solution defined by (34)

With $b_1 = \frac{c_4}{c_2}$ and $b_2 = \frac{-c_1}{2c_4}$

Example 2.

weights $(w(x), w(t)) = (1,2)$

Let us first introduce the weights of independent variables

$$(w(x), w(t)) = (1,-2)$$

Then general homogenous polynomials of weights 1 is

$$p = c_1 x + c_2 x^3 t \tag{40}$$

assume that the weights variables are

$$\eta_i = k_i x + b_1 k_i^{-2} t, \quad 1 \le \mathrm{i} \le \mathrm{N} \tag{41}$$

where k_i 's are arbitrary constants but b_1 is a constant to be determined.

Now, a similar direct computation tells that the corresponding Hirota bilinear equation

$$(c_1 D_x + c_2 D_x^3 D_t) f.f = 0 \tag{42}$$

Possesses N-wave solution

$$f=\sum_{i=1}^{N}\varepsilon_i f_i=\sum_{i=1}^{N}\varepsilon_i e k_i x+b_1 k_i^{-2}t \qquad (43)$$

Where, ε_i 's and k_i 's are arbitrary but b_1 satisfy $c_1+c_2b_1=0$ With $b_1=\frac{-c_1}{c_2}$,

5. Conclusions

Following the procedure proposed by[8], we have analyzed a specific sub-class of *N*-soliton solutions, formed by a linear combination of exponential travelling waves, for sawada-Kotera and a (2+1)-dimensional equations. The starting point is to solve a system of nonlinear algebraic equations for the wave related numbers called the N-wave solution condition. The resulting system tells what Hirota bilinear equation the linear superposition principle of exponential wave will apply to. Though it was shown that, high dimensional Hirota bilinear equations have a better opportunity to satisfy the *N*-wave solution condition because of their large parameters to choose from[10], we were able to also show for the lower dimensional Hirota bilinear equations. We also showed that linear superposition principle can hold for some special kind of wave solutions to Hirota bilinear equations, for example the exponential wave as we have explored. This also tells us why Hirota bilinear methods have advantages over some of the other methods

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