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The Existence of at most Twenty Seven Nonnegative Equilibrium Points in a Class of 3-D Competitive Cubic Systems

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Abstract This paper presents the stability analysis of equilibrium points of a model involving competition between three species subject to a strong Allee effect which occurs at low population density. By using the software of MAPLE 10, we prove that, under certain conditions, the model has at most twenty seven nonnegative equilibrium points and, via Lyapunov function, we derive criteria for the asymptotical stability of the unique positive equilibrium point.

Keywords Allee Effect, Competition Model Of Three Species, Lyapunov Theorem

1. Introduction

The Allee model of growth has been widely and successfully used as a simple, yet adequate descriptor of the dynamics of small populations or *critical depensation model*[1], and many theoretical studies (e.g.[2],[3]) have been achieved. The Allee effect refers to reduced fitness or decline in population growth at low population densities. In population models, the Allee effect is often modeled as a threshold, below which there is population extinction.

In the present paper, we consider the Allee effect within the context of the symmetric model of three competing species. We wish to point out that an model of two competing species with Allee effect was proposed and studied in[2-4], and some papers dealing with experiments, simulations, or combinations of these competitive systems among others are described in[5-6]. In Section 2, we introduce the symmetric model of three competing species subject to the Allee effects. The main analytical results on stability analysis of the equilibrium points, are presented in Section 3. Section 4 is devoted to a discussion, in the context of numerical simulation, of the analytical results obtained in this paper. Concluding remarks on the paper are made at the end.

2. A three-species Competitive System Subject to the *Allee Effects*

In the three-species Lotka-Volterra competition models (e.g.[7-11]), it is possible for one- or two species extinction, or global stability of a positive three-species equilibrium, periodic solutions or a stable heteroclinic orbit (e.g.[12-14]). Here, we shall propose a new three-competitive model that specifically predicts Allee growth of species x, y, and z, respectively. Keeping this in mind, the model is described as follows:

$$
\begin{aligned}
x' &= x\,[(x-a)(1-x)-\alpha y-\beta z] = xf(x,y,z),\\
y' &= y\,[(y-b)(1-y)-\beta x-\alpha z] = yg(x,y,z), \quad (2.1)\\
z' &= z\,[(z-c)(1-z)-\alpha x-\beta y] = zh(x,y,z),\\
& x(0)\geq 0,\ y(0)\geq 0,\ z(0)\geq 0,
\end{aligned}
$$

where x, y and z are the population densities, $(x-a1-x$, $y-b1-y$ and $z-c1-z$ are the quadratic intrinsic growth rates at intermediate densities, $0<a<1, 0<b<1$, and $0<c<1$ are the lower threshold of the population densities $x, y,$ and z, respectively, α and β are the coefficients of interspecies competition, and ($'= d/dt$). Throughout this paper we assume that $0<\alpha<1/4$ and $0<\beta<1/4$.

In our model we consider that the intrinsic growth rates are quadratic and we prove that, under certain conditions, system (2.1) has at most twenty seven nonnegative equilibria. By using the software of MAPLE 10, a numerical example is provided to illustrate the behavior of the system (2.1) for a biologically reasonable range of parameters with only one asymptotically stable equilibrium point and seven unstable equilibrium points in R_+^3. We believe that this is the first time that the three-species competition system (2.1) has been formulated and analyzed in the literature.

2.1. Boundedness of the Solutions

Consider the system (2.1). Obviously the functions $f, g,$ and h are continuous and Lipschitzian with respect to all independent variables on $R_+^3 = \{(x,y,z)/x\geq 0, y\geq 0, z\geq 0\}$. Therefore, a solution of the system (2.1) with

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nonnegative initial conditions exists and is unique. The basic existence and uniqueness theorem for differential equations ensures that

Lemma 2.1. The positive cone Int R_+^3 is invariant for system (2.1).

Lemma 2.2. The solutions $x(t), y(t)$, and $z(t)$ of system (2.1) with positive initial conditions are bounded for all $t \geq 0$.

Proof. Since $x' \leq x\,[(x-a)(1-x)]$, then $limsup_{t\to+\infty} x(t) \leq 1$. Here, we consider the case of strong Allee effect :

$$x' = f(x) = x(x-a)(1-x),$$

where $0 < a < 1$ is the survival threshold. There are three equilibrium points $x_o = 0$, and $0 < x_+^1 = a < x_+^2 = 1$. The relative extrema of the function $f(x)$ are

$$x_*^{2,1} = \frac{\left[(a+1) \pm \sqrt{a^2 - a + 1}\right]}{3},$$

which give the points of inflection of the graph of x versus t. The solutions are increasing and concave down when $x_*^2 < x < 1$; increasing and concave up when $K_o < x < x_*^2$; decreasing and concave down when $x_*^1 < x < K_o$; decreasing and concave up when $0 = x_o < x < x_*^1$ or $x > 1$. We conclude that $x_o = 0$ and $x_+^2 = 1$ are sinks; and $x_+^1 = a$ is a source. Then if the initial population size is below a, the population $x = x(t)$ will die out.

Similarly to y and z, respectively.

3. Existence and Stability of Equilibrium Points

Computations of the boundary equilibria and the analysis of the existence of positive equilibrium points and their stability for system (2.1), provide the information needed to determine the coexistence or extinction of species. To do so, we compute the Jacobian matrix $J(E)$ of (2.1). The signs of the real parts of the eigenvalues of $J(E)$ evaluated at a given equilibrium point $E = (x, y, z)$ determine its stability. Here

$$J(E) = \begin{bmatrix} (a+1-2x)x + f & -\alpha x & -\beta x \\ -\beta y & (b+1-2y)y + g & -\alpha y \\ -\alpha z & -\beta z & (c+1-2z)z + h \end{bmatrix} \quad (3.1)$$

where $f, g,$ and h are as in (2.1), $a < 1,\ b < 1,\ c < 1$, $\alpha < \frac{1}{4}$ and $\beta < \frac{1}{4}$, and all the parameters are positive. System (2.1) has at most twenty seven non-negative equilibria:

$E_{000} = (0,0,0)$, with eigenvalues $J(E_{000}) = \{-a < 0, -b < 0, -c < 0\}$. Thus, E_{000} is locally asymptotically stable;

$E_{a00} = (a, 0,0)$, with eigenvalues $J(E_{a00}) = \{-b - \beta a < 0, -c - \alpha a < 0, a(1-a) > 0\}$. This implies that E_{a00} is unstable;

$E_{100} = (1,0,0)$, where eigenvalues $J(E_{100}) = \{a - 1 < 0, -b - \beta < 0, -c - \alpha < 0\}$. Thus, E_{100} is locally asymptotically stable;

$E_{0b0} = (0, b, 0)$, where eigenvalues $J(E_{0b0}) = \{b(1-b) > 0, -a - \alpha b < 0, -c - \beta b < 0\}$. This implies that E_{0b0} is unstable;

$E_{010} = (0,1,0)$, with eigenvalues $J(E_{010}) = \{-a - \alpha < 0, b - 1 < 0, -c - \beta < 0\}$. Thus, E_{010} is locally asymptotically stable;

$E_{00c} = (0,0,c)$, where eigenvalues $J(E_{00c}) = \{c(1-c) > 0, -a - \beta c < 0, -b - \alpha c < 0\}$. This implies that E_{00c} is unstable;

$E_{001} = (0,0,1)$, with eigenvalues $J(E_{001}) = \{-a - \beta < 0, -b - \alpha < 0, c - 1 < 0\}$. Thus , E_{001} is locally asymptotically stable;

Now we establish criteria for the existence and stability of the equilibrium $E_{++0} = (x^*, y^*, 0)$. For this case, one of the three competitors goes to extinction depending on the initial values and the coexistence of three competing species described by (2.1) is not possible. Thus the *exclusion principle* holds[10].

When system (2.1) is restricted to R_{xy}^2, we obtain the following subsystem:

$$x' = x\,[(x-a)(1-x) - \alpha y] = xf(x, y, 0),$$
$$y' = y\,[(y-b)(1-y) - \beta x] = yg(x, y, 0). \quad (3.2)$$

An interior planar equilibrium $E_{++0} = (x^*, y^*, 0)$ occurring in the $x - y$ plane exists if and only if the algebraic system $f(x^*, y^*, 0) = 0,\ g(x^*, y^*, 0) = 0$ has a positive solution. A routine computation yields

$$y^* = \frac{(x^* - a)(1 - x^*)}{\alpha}$$

and $p(x^*) = 0$, where

$$p(x) = \lambda_{14}x^4 + \lambda_{13}x^3 + \lambda_{12}x^2 + \lambda_{11}x + \lambda_{10}, \quad (3.3)$$

$$\lambda_{14} = \frac{1}{\alpha^2} > 0, \lambda_{13} = -\frac{2(a+1)}{\alpha^2} < 0,$$

$$\lambda_{12} = \frac{\left[(a+1)^2 + 2a + \alpha(b+1)\right]}{\alpha^2} > 0,$$

$$\lambda_{11} = \frac{\left[\alpha^2\beta - \alpha(a+1)(b+1) - 2a(a+1)\right]}{\alpha^2} < 0,$$

$$\lambda_{10} = \frac{\left[\alpha^2 b + a^2 + \alpha a(b+1)\right]}{\alpha^2} > 0$$

Using the rule of signs of Descartes it follows: (a) There are four sign changes in $p(x)$, so there are 4, 2 or 0 positive roots; (b) There are no sign changes at $p(-x)$, so there are no negative roots. Hence, at most four positive equilibrium points are possible in the $x - y$ plane[15].

Under certain conditions on the parameters we have the following geometric interpretation (see Fig.3.1):

Proposition 3.1. Let $E_{++0} = (x^*, y^*, 0)$ denote an interior equilibrium in the $x - y$ plane. Then E_{++0} is the intersection of ellipses E_1 and E_2 defined by

$$E_1 : x = A + B\cos(t), y = C\sin(t), \quad (3.4)$$
$$E_2 : x = E\cos(t), y = D + F\sin(t), -\pi \leq t \leq \pi.$$

where

$$\Delta = (a+1)(b+1) - \alpha\beta > 0, A = \frac{\Delta}{2(b+1)}, D = \frac{\Delta}{2(a+1)},$$

$$B^2 = \frac{\Delta^2 - 4(b+1)\left[a(b+1) + \alpha b\right]}{4(b+1)^2}, C^2 = \left(\frac{b+1}{\alpha}\right)B^2,$$

$$F^2 = \frac{\Delta^2 - 4(a+1)\left[b(a+1) + \beta a\right]}{4(a+1)^2}, E^2 = \left(\frac{a+1}{\beta}\right)F^2.$$

Proposition 3.2. Consider the system (2.1). Suppose that there are four interior equilibria $E^i_{++0} = (x^*, y^*, 0), i = 1,2,3,4,$ in the $x - y$ plane; that is, A, B, C and D, respectively. Then only one equilibrium C of coexisting populations is locally stable and its basin of attraction is bounded by the stable separatrices of the saddles B and D, both coming from the unstable node A.

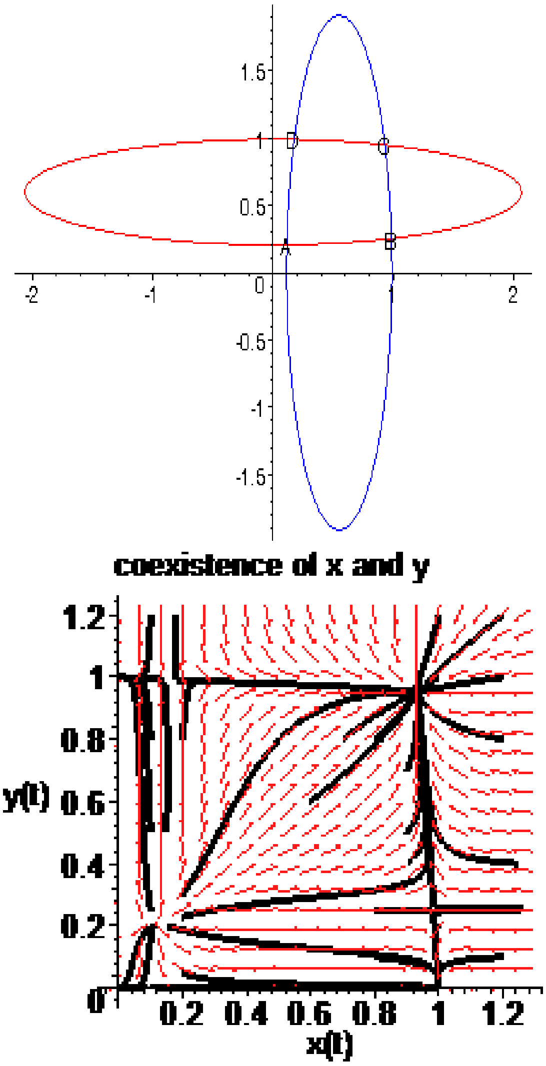

Figure 3.1. Intersection between two ellipses E_1, E_2. The equilibrium points are: $A = E^1_{++0}$ is an unstable node; $B = E^2_{++0}$ is a saddle point; $C = E^3_{++0}$ is locally asymptotically stable; $D = E^4_{++0}$is a saddle point. Here $a = 0.1, b = 0.2, c = 0.15, \alpha = \frac{1}{16}, \beta = \frac{1}{25}$ (see Section 4)

Similar results for the existence and stability of the equilibrium points $E_{+0+} = (x^*, 0, z^*)$ and $E_{0++} = (0, y^*, z^*)$ are obtained from (2.1).

3.1. Existence, Stability and Linearization of Positive Equilibrium Points

Let $E = (x^*, y^*, z^*)$ denote an interior equilibrium point of R^3_+, if it exists. It follows from direct substitution and algebraic manipulation:

Proposition 3.3. System (2.1) has at most eight equilibrium points in the interior of R^3_+. Their equilibrium values x^*, y^* and z^* are given by

$$x^* = A_1 + A_2\sqrt{\Delta_1},$$

$$z^* = B_1 + B_2\sqrt{\Delta_2},$$

and y^* is a positive root of

$$F(y) = D_8 y^8 + D_7 y^7 + D_6 y^6 + D_5 y^5 + D_4 y^4 + D_3 y^3 + D_2 y^2 + D_1 y + D_0 \quad (3.5)$$

where

$$D_8 = -P_1^4 < 0, D_7 = -4P_2 P_1^3 > 0,$$

$$D_6 = 2M_1 C_1^2 P_1^2 + 2N_1 C_2^2 P_1^2 - \left(4P_3 P_1^3 + 6P_1^2 P_2^2\right),$$

$$D_5 = 2C_1^2\left(2M_1 P_1 P_2 + M_2 P_1^2\right) + 2C_2^2\left(2N_1 P_1 P_2 + N_2 P_1^2\right) - (4P_1 P_2^3 + 12P_2 P_3 P_1^2)$$

$$D_4 = 2M_1 N_1 C_1^2 C_2^2 - M_1^2 C_1^4 - N_1^2 C_2^4 +$$
$$2C_1^2\left(M_1 P_2^2 + 2M_1 P_1 P_3 + 2M_2 P_1 P_2 + M_3 P_1^2\right) +$$
$$2C_2^2\left(N_1 P_2^2 + 2N_1 P_1 P_3 + 2N_2 P_1 P_2 + N_3 P_1^2\right) -$$
$$(6P_1^2 P_3^2 + 12P_1 P_3 P_2^2 + P_2^4),$$

$$D_3 = 2C_1^2 C_2^2\left(M_1 N_2 + M_2 N_1\right)$$
$$-2M_1 M_2 C_1^4 - 2N_1 N_2 C_2^4 +$$
$$2C_1^2\left(2M_1 P_2 P_3 + M_2 P_2^2 + 2M_2 P_1 P_3 + 2M_3 P_1 P_2\right) +$$
$$2C_2^2\left(2N_1 P_2 P_3 + N_2 P_2^2 + 2N_2 P_1 P_3 + 2N_3 P_1 P_2\right) -$$
$$\left(4P_3 P_2^3 + 12P_1 P_2 P_3^2\right),$$

$$D_2 = 2C_1^2 C_2^2\left(M_1 N_3 + M_2 N_2 + M_3 N_1\right) -$$
$$C_1^4\left(M_2^2 + 2M_1 M_3\right) - C_2^4\left(N_2^2 + 2N_1 N_3\right) +$$
$$2C_1^2\left(M_1 P_3^2 + 2M_2 P_2 P_3 + M_3 P_2^2 + 2M_3 P_1 P_3\right) +$$
$$2C_2^2\left(N_1 P_3^2 + 2N_2 P_2 P_3 + N_3 P_2^2 + 2N_3 P_1 P_3\right) -$$
$$\left(6P_2^2 P_3^2 + 4P_1 P_3^3\right)$$

$$D_1 = 2C_1^2 C_2^2\left(M_2 N_3 + M_3 N_2\right) -$$
$$2M_2 M_3 C_1^4 - 2N_2 N_3 C_2^4 +$$
$$2C_1^2\left(M_2 P_3^2 + 2M_3 P_2 P_3\right) +$$
$$2C_2^2\left(N_2 P_3^2 + 2N_3 P_2 P_3\right) - 4P_2 P_3^3,$$

$$D_0 = P_3^4 + 2M_3 N_3 C_1^2 C_2^2 - (P_3^2 - M_3 C_1^2)^2 - (P_3^2 - N_3 C_2^2)^2,$$

$$A_1 = \frac{a+1}{2} + \frac{\beta^2}{2\alpha}, A_2 = \frac{1}{2\alpha}, B_1 = \frac{c+1}{2} + \frac{\alpha^2}{2\beta}, B_2 = \frac{1}{2\beta},$$

$$\Delta_1 = M_1 y^2 + M_2 y + M_3,$$

$$\Delta_2 = N_1 y^2 + N_2 y + N_3,$$

$$M_1 = N_1 = 4\alpha\beta > 0, M_2 = -4\alpha[(b+1)\beta + \alpha^2] < 0,$$

$$M_3 = \left[(a+1)\alpha + \beta^2\right]^2 + 4\alpha(b\beta - a\alpha)$$
$$= \beta^4 + 2(a+1)\alpha\beta^2 + 4b\alpha\beta + (a-1)^2\alpha^2 > 0,$$

$$N_2 = -4\beta[(b+1)\alpha + \beta^2] < 0,$$

$$N_3 = [(c+1)\beta + \alpha^2]^2 + 4\beta(b\alpha - c\beta)$$
$$= \alpha^4 + 2(c+1)\beta\alpha^2 + 4b\alpha\beta + (c-1)^2\beta^2 > 0,$$

$$C_1 = (a+1)A_2 - 2A_1A_2 = -\frac{\beta^2}{\alpha^2} < 0, C_2 = -\beta B_2 = -\frac{1}{2},$$

$$P_1 = M_1A_2^2 = \frac{\beta}{\alpha} > 0, P_2 = \alpha + M_2A_2^2 = -\frac{(b+1)\beta}{\alpha} < 0,$$

$$P_3 = A_1^2 - (a+1)A_1 + M_3A_2^2 + \beta B_1 + a$$

$$= \frac{\beta^4}{2\alpha^2} + \frac{(a+1)\beta^2}{2\alpha} + \left(\frac{b}{\alpha} + \frac{c+1}{2}\right)\beta + \frac{\alpha^2}{2} > 0.$$

Corollary 3.1. Suppose that $D_6 < 0, D_5 > 0, D_4 < 0, D_3 > 0, D_2 < 0, D_1 > 0$, and $D_0 < 0$. Then $F(y)$ has 8, 6, 4, 2 or 0 positive roots.

Corollary 3.2. Suppose $D_0 > 0$. Then $F(y)$ has at least one positive root.

Proof. Clearly, $F(0) = D_0 > 0$, and $\lim_{y\to\infty} F(y) = -\infty$. Hence, there exists a $y^*\epsilon(0,\infty)$ so that $F(y^*) = 0$. This completes the proof.

Remark 3.1 Using the software MAPLE, we obtain the following numerical examples:(i) $a = 0.5, b = 0.25, c = 0.2, \alpha = \frac{1}{16}, \beta = \frac{1}{9}$ => none positive equilibrium; (ii) $a = 0.4, b = 0.25, c = 0.3, \alpha = \frac{1}{16}, \beta = \frac{1}{9}$ => two positive equilibriums with eigenvalues (-,+,+),(+,+,+); (iii) $a = 0.4, b = 0.25, c = 0.2, \alpha = \frac{1}{16}, \beta = \frac{1}{9}$ => two positive equilibriums with eigenvalues (-,+,+),(+,α±iβ), α> 0; (iv) $a = 0.4, b = 0.25, c = 0.1, \alpha = \frac{1}{16}, \beta = \frac{1}{9}$ => four positive equilibriums with eigenvalues (+,+,+),(+,-,+),(-,-,+),(-,+,+).

It is always informative to draw the set of positive equilibrium points of the system (2.1) in R_+^3. Here the set is defined by the intersection of the surfaces:

$$f(x,y,z) = 0, g(x,y,z) = 0, h(x,y,z) = 0 \quad (3.6)$$

Under certain conditions on the parameters of the system (2.1), we obtain (see Fig 3.2):

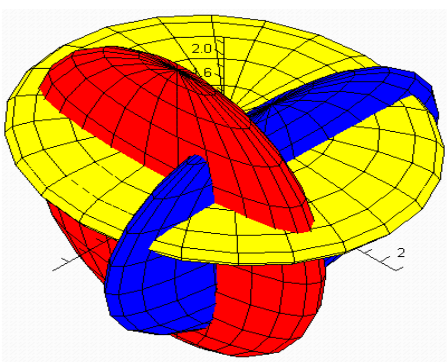

Figure 3.2. Intersection between ellipsoids $S1, S2$, and $S3$. There are 8 equilibrium points in R_+^3. Here $a = 0.1, b = 0.2, c = 0.15, \alpha = \frac{1}{16}, \beta = \frac{1}{25}$.

Proposition 3.4. Let $E = (x^*, y^*, z^*)$ denote the interior equilibrium of the system (2.1), if it exists. Then E is the intersection of three ellipsoids S_1, S_2 and S_3 :

$$S1: x = \Delta_{11} + \nabla_{11} \sin\sin(s)\cos\cos(t), y = \nabla_{12} \sin\sin(s)\sin\sin(t), z = \nabla_{13} \cos\cos(s),$$

$$S2: x = \nabla_{21} \sin\sin(s)\cos\cos(t), y = \Delta_{22} + \nabla_{22} \sin\sin(s)\sin\sin(t), z = \nabla_{23} \cos\cos(s),$$

$$S3: x = \nabla_{31} \sin\sin(s)\cos\cos(t), y = \nabla_{32} \sin\sin(s)\sin\sin(t), z = \Delta_{33} + \nabla_{33} \cos\cos(s),$$

$$0 \le s \le \pi, 0 \le t \le 2\pi;$$

$$\Delta^2 - 4A_{ii}\nabla_i > 0; i = 1,2,3. \quad (3.7)$$

where

$$\Delta_{11} = \frac{\Delta}{2A_{11}}, \nabla_{11}^2 = \frac{\Delta^2 - 4A_{11}\nabla_1}{4A_{11}^2},$$

$$\nabla_{12}^2 = \frac{\Delta^2 - 4A_{11}\nabla_1}{4A_{11}A_{12}}, \nabla_{13}^2 = \frac{\Delta^2 - 4A_{11}\nabla_1}{4A_{11}A_{13}},$$

$$\Delta_{22} = \frac{\Delta}{2A_{22}}, \nabla_{21}^2 = \frac{\Delta^2 - 4A_{22}\nabla_2}{4A_{22}A_{21}},$$

$$\nabla_{22}^2 = \frac{\Delta^2 - 4A_{22}\nabla_2}{4A_{22}^2}, \nabla_{23}^2 = \frac{\Delta^2 - 4A_{22}\nabla_2}{4A_{22}A_{23}},$$

$$\Delta_{33} = \frac{\Delta}{2A_{33}}, \nabla_{31}^2 = \frac{\Delta^2 - 4A_{33}\nabla_3}{4A_{33}A_{31}},$$

$$\nabla_{32}^2 = \frac{\Delta^2 - 4A_{33}\nabla_3}{4A_{33}A_{32}} \nabla_{33}^2 = \frac{\Delta^2 - 4A_{33}\nabla_3}{4A_{33}^2},$$

$$A_{11} = \det\begin{bmatrix} b+1 & -\alpha \\ -\beta & c+1 \end{bmatrix} = (b+1)(c+1) - \alpha\beta > 0,$$

$$A_{12} = -\det\begin{bmatrix} -\alpha & -\beta \\ -\beta & c+1 \end{bmatrix} = (c+1)\alpha + \beta^2 > 0,$$

$$A_{13} = \det\begin{bmatrix} -\alpha & -\beta \\ b+1 & -\alpha \end{bmatrix} = \alpha^2 + (b+1)\beta > 0,$$

$$A_{21} = -\det\begin{bmatrix} -\beta & -\alpha \\ -\alpha & c+1 \end{bmatrix} = (c+1)\beta + \alpha^2 > 0,$$

$$A_{22} = \det\begin{bmatrix} a+1 & -\beta \\ -\alpha & c+1 \end{bmatrix} = (a+1)(c+1) - \alpha\beta > 0,$$

$$A_{23} = -\det\begin{bmatrix} a+1 & -\beta \\ -\beta & -\alpha \end{bmatrix} = (a+1)\alpha + \beta^2 > 0,$$

$$A_{31} = \det\begin{bmatrix} -\beta & b+1 \\ -\alpha & -\beta \end{bmatrix} = (b+1)\alpha + \beta^2 > 0,$$

$$A_{32} = -\det\begin{bmatrix} a+1 & -\alpha \\ -\alpha & -\beta \end{bmatrix} = (a+1)\beta + \alpha^2 > 0,$$

$$A_{33} = \det\begin{bmatrix} a+1 & -\alpha \\ -\beta & b+1 \end{bmatrix} = (a+1)(b+1) - \alpha\beta > 0,$$

$$\Delta = \det\begin{bmatrix} a+1 & -\alpha & -\beta \\ -\beta & b+1 & -\alpha \\ -\alpha & -\beta & c+1 \end{bmatrix}, \nabla_1 = \det\begin{bmatrix} a & -\alpha & -\beta \\ b & b+1 & -\alpha \\ c & -\beta & c+1 \end{bmatrix},$$

$$\nabla_2 = \det\begin{bmatrix} a+1 & a & -\beta \\ -\beta & b & -\alpha \\ -\alpha & c & c+1 \end{bmatrix},$$

$$\nabla_3 = \det\begin{bmatrix} a+1 & -\alpha & a \\ -\beta & b+1 & b \\ -\alpha & -\beta & c \end{bmatrix}.$$

To determine the stability of a positive equilibrium point of (2.1), we will use the direct method of Lyapunov:

3.2. Direct Method of Lyapunov

Next let us consider the local stability of a positive equilibrium point $E = (x^*, y^*, z^*) \in \Omega \subset R_+^3$, where Ω is a neighborhood of E to be determined. Based on the "direct method" of Lyapunov, we construct a continuous function

$$V(x,y,z) = \delta_1\left[x - x^* - x^*\ln\ln\left(\frac{x}{x^*}\right)\right] + \delta_2\left[y - y^* - y^*\ln\ln\left(\frac{y}{y^*}\right)\right] + \delta_3\left[z - z^* - z^*\ln\ln\left(\frac{z}{z^*}\right)\right] \quad (3.8)$$

where δ_i (i=1,2,3) are positive constant numbers which are yet unspecified, satisfying the following properties:

(a) $V(E) = 0$,

(b) $V(x,y,z) > 0$ for $\Omega \setminus \{E\}$, that is, the equilibrium point E is an isolated minimum of V. In fact,

$$V_x(E) = V_y(E) = V_z(E) = 0,$$

$$V_{xx}(E) = \delta_1 \frac{x^*}{x^2} > 0; \det\begin{bmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{bmatrix} = \delta_1\delta_2 \frac{1}{x^* y^*} > 0;$$

$$\det\begin{bmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{bmatrix} = \delta_1\delta_2\delta_3 \frac{1}{x^* y^* z^*} > 0$$

where the partial derivatives are calculated at E.

(c) The function V is continuously differentiable on the neighborhood $\Omega \setminus \{E\}$, and, on this set, $V'(x,y,z) < 0$. Here,

$$V'(x(t),y(t),z(t)) = \delta_1(x - x^*)\left[(1-x)(x-a) - \alpha y - \beta z\right] + \delta_2(y - y^*)\left[(1-y)(y-b) - \beta x - \alpha z\right] + \delta_3(z - z^*)[(1-z)(z-c) - \alpha x - \beta y.$$

Since, $E = (x^*, y^*, z^*)$ is a positive equilibrium point of system (2.1), V' satisfies

$$\begin{aligned} &V'(x(t),y(t),z(t)) = \\ &\delta_1[-(x-x^*)^3 + (-2x^* + a + 1)(x - x^*)^2 - \\ &\alpha(x-x^*)(y-y^*) - \beta(x-x^*)(z-z^*)] + \\ &\delta_2[-(y-y^*)^3 + (-2y^* + b + 1)(y-y^*)^2 - \\ &\beta(x-x^*)(y-y^*) - \alpha(y-y^*)(z-z^*)] + \\ &\delta_3[-(z-z^*)^3 + (-2z^* + c + 1)(z-z^*)^2 - \\ &\alpha(x-x^*)(z-z^*) - \beta(y-y^*)(z-z^*)] - \\ &\alpha(x-x^*)(z-z^*) - \beta(y-y^*)(z-z^*)] \end{aligned} \quad (3.9)$$

If we prove that $E = (x^*, y^*, z^*)$ is an isolated maximum of $V'(x,y,z) = L(x,y,z)$, then (c) follows easily, that is:

(c1) We note that E is a critical point of the function $L(x,y,z)$, that is

$$\begin{aligned} &L_x(x,y,z) = -3\delta_1(x-x^*)^2 + \\ &2\delta_1\rho_1(x-x^*) - (\alpha\delta_1 + \beta\delta_2)(y-y^*) - \\ &(\beta\delta_1 + \alpha\delta_3)(z-z^*) L_y(x,y,z) \\ &= -3\delta_2(y-y^*)^2 + 2\delta_2\rho_2(y-y^*) - \\ &(\alpha\delta_1 + \beta\delta_2)(x-x^*) - (\alpha\delta_2 + \beta\delta_3)(z-z^*) \\ &L_z(x,y,z) = -3\delta_3(z-z)^2 + 2\delta_3\rho_3(z-z^*) - \\ &(\beta\delta_1 + \alpha\delta_3)(x-x^*) - (\alpha\delta_2 + \beta\delta_3)(y-y^*) \end{aligned}$$

implies $L_x(E) = L_y(E) = L_z(E) = 0$.

(c2) The equilibrium point E is a maximum point of $L(x,y,z)$ <=>

$$(i')L_{xx}(E) < 0; (ii')H_1(E) > 0; (iii')H(E) < 0 \quad (3.10)$$

Here

$$L_{xx}(E) = 2\delta_1\rho_1;$$

$$H_1(E) = \begin{pmatrix} L_{xx}(E) & L_{xy}(E) \\ L_{yx}(E) & L_{yy}(E) \end{pmatrix} = \begin{pmatrix} 2\delta_1\rho_1 & -\alpha\delta_1 - \beta\delta_2 \\ -\alpha\delta_1 - \beta\delta_2 & 2\delta_2\rho_2 \end{pmatrix};$$

$$H(E) = \begin{bmatrix} L_{xx}(E) & L_{xy}(E) & L_{xz}(E) \\ L_{xy}(E) & L_{yy}(E) & L_{yz}(E) \\ L_{xz}(E) & L_{yz}(E) & L_{zz}(E) \end{bmatrix} = \begin{bmatrix} 2\delta_1\rho_1 & -\alpha\delta_1 - \beta\delta_2 & -\beta\delta_1 - \alpha\delta_3 \\ -\alpha\delta_1 - \beta\delta_2 & 2\delta_2\rho_2 & -\alpha\delta_2 - \beta\delta_3 \\ -\beta\delta_1 - \alpha\delta_3 & -\alpha\delta_2 - \beta\delta_3 & 2\delta_3\rho_3 \end{bmatrix}$$

Letting $\delta_1 = \delta_2 = 1/2$, we can rewrite (3.10) as

$$\rho_1 < 0; \rho_1\rho_2 - \tau^2 > 0;$$

$$\rho_1\rho_2\rho_3 - (\rho_1 + \rho_2 + \rho_3)\tau^2 + 2\tau^3 < 0,$$

$$\tau = -\frac{(\alpha + \beta)}{2}. \quad (3.11)$$

Thus, $E = (x^*, y^*, z^*)$ is a isolated maximum of $V'(x,y,z)$, i.e., there is a neighborhood Ω of E such that

$V'(x,y,z) < 0$, on this set . The pertinent result, which we prove, is the following (see Fig.3.3):.

Proposition 3.5. Consider the Lyapunov function (3.8) defined in the neighborhood $\Omega \subset R_+^3$ of a positive equilibrium point E of the competitive system (2.1). If (3.11) occurs, then $E = (x^*, y^*, z^*)$ is locally asymptotically stable .

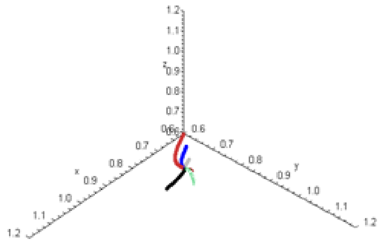

Figure 3.3. Each graph depicts a three-dimensional x, y, z population evolution in the state space for system (2.1). The initial conditions are (0,0,0),(0.9,0.9,0.9),(0.8,0.8,0.8),(0.7,0.7,0.7),(0.6,0.6,0.6),(1.0,0.9,0.9).The equilibrium point E_{+++}^5 is locally asymptotically stable. Here $a = 0.1, b = 0.2, c = 0.15, \alpha = \frac{1}{16}, \beta = \frac{1}{25}$

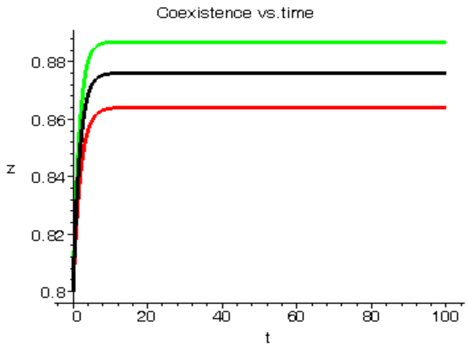

Figure 3.4. Each graph depicts one-dimensional x, y, z population changes with respect to time for system (2.1). Each trajectory starts at a point (0.8,0.8,0.8) near the equilibrium E_{+++}^5 locally asymptotically stable: green$-x$, red$-y$, black$-z$. Here $a = 0.1, b = 0.2, c = 0.15, \alpha = \frac{1}{16}, \beta = \frac{1}{25}$

4. Numerical example

By using the software of MAPLE 10, a numerical example has been provided to illustrate the behavior of the system (2.1) for a biologically reasonable range of parameters. Choosing the following set of values for the parameters in (2.1):

$$a = 0.1, b = 0.2, c = 0.15, \alpha = 1/16, \beta = 1/25,$$

we find that the inequalities given by (3.11) hold for a unique positive equilibrium point and we observe that there are 27 equilibrium points given by

$$E_{000} = (0,0,0), E_{a00} = (0.1,0,0), E_{100} = (1,0,0),$$
$$E_{0b0} = (0,0.2,0),$$
$$E_{010} = (0,1,0), E_{00c} = (0,0,0.15), E_{001} = (0,0,1),$$
$$E_{++0}^1 = (0.1145238077, 0.2057677744, 0),$$
$$E_{++0}^2 = (0.982105439, 0.2525582316, 0),$$
$$E_{++0}^3 = (0.9282751363, 0.9505267387, 0),$$
$$E_{++0}^4 = (0.1750956168, 0.9911472553, 0),$$
$$E_{+0+}^1 = (0.1145238077, 0, 0.1070755147),$$
$$E_{+0+}^2 = (0.9896422935, 0, 0.2303663443),$$
$$E_{+0+}^3 = (0.9569355368, 0, 0.9225867215),$$
$$E_{+0+}^4 = (0.1463466549, 0, 0.9890994253),$$
$$E_{0++}^1 = (0, 0.2127122137, 0.1601307297),$$
$$E_{0++}^2 = (0, 0.2863822446, 0.98863024559),$$
$$E_{0++}^3 = (0, 0.9167794511, 0.9544124694),$$
$$E_{0++}^4 = (0, 0.9841260905, 0.1991543450),$$
$$E_{+++}^1 = (0.1234346734, 0.2199416141, 0.1698917819),$$
$$E_{+++}^2 = (0.9689920329, 0.2745078943, 0.2447233248),$$
$$E_{+++}^3 = (0.1690910317, 0.2959630865, 0.9727666239),$$
$$E_{+++}^4 = (0.9306479263, 0.3426568704, 0.9047775459),$$
$$E_{+++}^5 = (0.8868214327, 0.8641473928, 0.8760527607),$$
$$E_{+++}^6 = (0.2205803755, 0.9043799491, 0.9364741050),$$
$$E_{+++}^7 = (0.9152759525, 0.9253052261, 0.2810475482),$$
$$E_{+++}^8 = (0.1851565751, 0.9730979666, 0.2142663096).$$

The $y-$coordinate of the positive equilibrium points E_{+++}^i (i=1…8) are roots of (3.5), that is $F(y) = 0$, where

$$F(y) = -\left(\frac{65536}{390625}\right)y^8 + 0.8053063680y^7 - 1.622650388y^6 + 1.78271132y^5 - 1.162155201y^4 + 0.4589826936y^3 - 0.1073327879y^2 + 0.01363187734y - 0.0007228687716$$

In the equilibrium point E_{+++}^5, the characteristic equation (4.11) of $J(E_{+++}^5)$ reduces to $p(\lambda) = \lambda^3 + 1.581401696\,\lambda^2 + 0.8228981025\,\lambda + 0.1410366097$, with roots -0.6392646908, -0.5068784588, -0.4352585464. This implies that E_{+++}^5 is a locally asymptotically stable equilibrium point. Here, we observe that the equilibrium points E_{+++}^i $(i \neq 5)$ are unstable.

In the absence of a competitor, we have: (a) E_{++0}^3 is a locally asymptotically stable equilibrium point, with eigenvalues -0.7346114284, -0.6340454548 and -0.2460382656. The equilibrium points $E_{++0}^j (j \neq 3)$ are unstable. (b) E_{+0+}^3 is a locally asymptotically stable equilibrium point, with eigenvalues -0.7933441170, -0.6268358235, -0.2959390916. The equilibrium points $E_{+0+}^j (j \neq 3)$ are unstable. (c) E_{0++}^3 is a locally asymptotically stable equilibrium point, with eigenvalues -0.7381379376, -0.5669278285, -0.1954752145. The equilibrium points $E_{0++}^j (j \neq 3)$ are unstable.

In the absence of two competitors, we have: (d) E_{100} is a locally asymptotically stable equilibrium point, with eigenvalues $-0.9, -0.24, -0.2125$. The equilibrium $E_{a00} = (0.1,0,0)$ is unstable. (e) E_{010} is a locally asymptotically stable equilibrium point, with eigenvalues $-0.8, -0.14, -0.2625$. The equilibrium $E_{0b0} = (0,0.2,0)$ is unstable. (f) E_{001} is a locally asymptotically stable equilibrium point, with eigenvalues $-0.85, -0.241625, -0.19$. The equilibrium $E_{00c} = (0,0,0.15)$ is unstable.

Clearly $E_{000} = (0,0,0)$ is a locally asymptotically stable equilibrium point, with eigenvalues -0.1, -0.2, -0.15, respectively.

The $x-$coordinate of the equilibrium points E^{i}_{++0} (i=1…4) are roots of (3.3) $p(x) = 256x^4 - 563.2x^3 + 380.16x^2 - 77.4x + 4.68$. Intersection between two ellipses:

$$E_1 : (0.5489583333 + 0.436965199\cos\cos(t), 1.914685571\sin\sin(t))$$

$$E_2: (2.064590892\cos(t), 0.5988636364 + 0.3937020336\sin(t));\ t=0..2\pi.$$

The $x-$coordinate of the equilibrium points. E^{i}_{+0+} (i=1…4) are roots of $q(x) = 625x^4 - 1375x^3 + 910x^2 - 169.0625x + 9.275$. Intersection between two ellipses:

$$E_3 : (0.5489130435 + 0.4428184030\cos(t), 2.374349167\sin\sin(t));$$

$$E_4: (1.748148381\cos(t), 0.5738863664 + 0.4166985181\sin(t));\ t=0\ldots2\pi.$$

The equilibrium points are: E^{1}_{+0+} is an unstable node; $E^{2}_{+0+}$is a saddle point; $E^{3}_{+0+}$is locally asymptotically stable; $E^{4}_{+0+}$is a saddle point.

The $y-$coordinate of the equilibrium points E^{i}_{0++} (i=1…4) are roots of the polynomial: $r(y) = 256y^4 - 614.4y^3 + 489.44y^2 - 144.92y + 14.070$. Intersection between two ellipses:

$$E_5 : (0.5989130435 + 0.3880008501\cos\cos(t), 1.664338228\sin\sin(t))$$

$$E_6: (2.276586272\cos(t), 0.5739583333 + 0.415645885\sin(t));\ t=0\ldots2\pi.$$

The equilibrium points are: E^{1}_{0++} is an unstable node; $E^{2}_{0++}$is a saddle point; $E^{3}_{0++}$is locally asymptotically stable; $E^{4}_{0++}$is a saddle point.

6. Concluding Remarks

In this paper, a mathematical model of competition between three populations with lower threshold sizes has been proposed and investigated. The main focus was to analyze the question of existence and stability of nonnegative equilibria. Our results show that there exist at most twenty-seven equilibrium points for the system under consideration and, by using the software of MAPLE 10, a numerical example has been provided to illustrate the behavior of the system (2.1) for a biologically reasonable range of parameters with only one positive equilibrium asymptotically stable and 7 positive unstable equilibrium points.

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