Fourth Order Fitted Scheme for Second Order Singular Perturbation Boundary Value Problems

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Abstract   Fitted fourth order central difference scheme is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end point. A fitting factor is introduced in a tri-diagonal finite difference scheme and is obtained from the theory of singular perturbations. Thomas Algorithm is used to solve the system and its stability is investigated. To demonstrate the applicability of the method, we have solved linear and nonlinear problems. From the results, it is observed that the present method approximates the exact solution very well.

Keywords   Singular Perturbation Problems, Finite Differences, Fitted Scheme

1. Introduction

There are varieties of physical processes in which a boundary layer may arise in the solutions for certain parameter ranges. These types of problems may be generally characterized as singular perturbation problems, and the parameter is termed as the perturbation parameter. Detailed theory and analytical discussion on singular perturbation problems, can be referred in Bender and Orsazag[1], Kevorkian and Cole[3], Nayfeh[5-6], O'Mally[7] and Van Dyke[12] and for exponential fitted methods Miller,J.J.H., O'Riordan,E. and Shishkin, G.I.[4], Y.N.Reddy, P.Pramod. Chakravarthy[9] and Reddy Y.N., Awoke A.[10-11].

Fitted stable fourth order scheme is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end point. A fitting factor is introduced in a tri-diagonal finite difference scheme and is obtained from the theory of singular perturbations. Thomas Algorithm is used to solve the system and its stability is investigated. To demonstrate the applicability of the method, we have solved several linear and nonlinear problems. From the results, it is observed that the present method approximates the exact solution very well.

2. Fourth Order Finite Difference Method

Consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \text{ , x} \in [0,1] \qquad (1)$$

$$\text{with y }(0) = \alpha \qquad (2a)$$

$$\text{and y }(1) = \beta; \qquad (2b)$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in $[0,1]$. Further more, we assume that $b(x) \leq 0$, $a(x) \geq M > 0$ throughout the interval $[0,1]$, where M is some positive constant.

A finite difference scheme is often a convenient choice for the numerical solution of two point boundary value problems, because of their simplicity. Let us divide the interval $[0,1]$ in to N equal parts , each of length h where $h = 1/N$ and then we have $x_i = ih$, i = 0,1,...,N . For simplicity, let us denote $a(x_i) = a_i, b(x_i) = b_i, f(x_i) = f_i, y(x_i) = y_i, y'(x_i) = y'_i \text{ and } y''(x_i) = y''_i$.

Assuming that y has continuous fourth order derivatives in the interval [0,1], by using Taylor series expansion we obtain the central difference formulas for y'_i, y''_i given by.

$$y'_i \cong \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6}y'''_i + R_1 \qquad (3)$$

and

$$y''_i \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12}y_i^{(4)} + R_2 \qquad (4)$$

where, $R_1 = -\frac{h^4 y^{(5)}(\eta)}{5!}$ and $R_2 = -\frac{2h^4 y^{(6)}(\xi)}{6!}$ for $\xi, \eta \in [x_{i-1}, x_{i+1}]$.

From (1), we have

$$\varepsilon y'' = f(x) - a(x)y' - b(x)y \qquad (5)$$

Differentiating both sides of equation (5) with respect to x, we get

$$y''' = \frac{1}{\varepsilon}(f' - ay'' - (a' + b)y' - b'y) \qquad (6)$$

Differentiating twice both sides of equation (5) with respect to x, we get

$$y^{(4)} = \frac{1}{\varepsilon}(f'' - ay''' - (2a' + b)y'' - (a'' + 2b')y' - b''y) \quad (7)$$

By substituting y_i', y_i'' from equations (3) and (4) in (1) at $x = x_i$, we get the central difference approximation in a form that includes all the $o(h^2)$ error terms:

$$\varepsilon(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12}y^{(4)}{}_i) + a_i(\frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6}y_i''') + b_i y_i = f_i \, (8)$$

where, y_i''' and $y_i^{(4)}$ are defined accordingly in equations (6) and (7) respectively. Using equations (3, 4, 6 & 7) in equation (8), we get the fourth order scheme

$$[\frac{\varepsilon}{h^2} + \frac{a_i^2}{6\varepsilon} - \frac{(2a_i' + b_i)}{12}](y_{i+1} - 2y_i + y_{i-1})$$

$$+[\frac{ha_i(a_i' + b_i)}{12\varepsilon} + \frac{h(a_i'' + 2b_i')}{24} + \frac{a_i}{2h}](y_{i+1} - y_{i-1})$$

$$+[\frac{h^2 a_i b_i'}{6\varepsilon} + \frac{h^2 b_i''}{12} + b_i]y_i + R = f_i + \frac{h^2 f_i''}{12} + \frac{h^2 a_i f_i'}{6\varepsilon}, \quad i = 1,2,..,N\text{-}1$$

Where, $R = R_1 + R_2 = O(h^4)$, from equation (4). For detail discussions of the method refer Joshua Y.Choo and David H.Schultz[2].

3. Fitted Fourth Order Scheme

A difference scheme with a fitting factor containing exponential functions is known as exponentially fitted difference scheme. From the theory of singular perturbations it is known that the solution of (1)-(2) is of the form [cf. O' Malley[7]; pp 22-26]

$$y(x) = y_0(x) + \frac{a(0)}{a(x)}(\alpha - y_0(0))e^{-\int_0^x (\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)})dx} + O(\varepsilon) \quad (9)$$

where $y_0(x)$ is the solution of

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), \quad y_0(1) = \beta \quad (10)$$

By taking first terms of the Taylor's series expansion for a(x) and b(x) about the point '0', (10) becomes,

$$y(x) = y_0(x) + (\alpha - y_0(0))e^{-(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)})x} + O(\varepsilon) \quad (11)$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \ldots x_N = 1$ be the mesh points. Then we have $x_i = ih$; $i = 0, 1, 2, \ldots, N$.

From (11) we have

$$y(ih) = y_0(ih) + (\alpha - y_0(0))e^{-(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)})ih} + O(\varepsilon).$$

Therefore

$$\lim_{h \to 0} y(ih) = y_0(0) + (\alpha - y_0(0))e^{-(\frac{a^2(0) - \varepsilon b(0)}{a(0)})i\rho},$$

where $\rho = \frac{h}{\varepsilon}$ \quad (12)

Now, we consider the stable fourth order central difference scheme (8) and introduce the fitting factor $\sigma(\rho)$:

$$\sigma(\rho)[\frac{\varepsilon}{h^2} + \frac{a_i^2}{6\varepsilon} - \frac{(2a_i' + b_i)}{12}](y_{i+1} - 2y_i + y_{i-1})$$

$$+[\frac{ha_i(a_i' + b_i)}{12\varepsilon} + \frac{h(a_i'' + 2b_i')}{24} + \frac{a_i}{2h}](y_{i+1} - y_{i-1})$$

$$+[\frac{h^2 a_i b_i'}{6\varepsilon} + \frac{h^2 b_i''}{12} + b_i]y_i = f_i + \frac{h^2 f_i''}{12} + \frac{h^2 a_i f_i'}{6\varepsilon}, \quad i = 1,2,..,N\text{-}1 \quad (13)$$

Rewriting (13), we have

$$[\frac{\sigma\varepsilon}{h^2} + \frac{a_i^2}{6\varepsilon} - \frac{\sigma(2a_i' + b_i)}{12}](y_{i+1} - 2y_i + y_{i-1})$$

$$+[\frac{ha_i(a_i' + b_i)}{12\varepsilon} + \frac{h(a_i'' + 2b_i')}{24} + \frac{a_i}{2h}](y_{i+1} - y_{i-1})$$

$$+[\frac{h^2 a_i b_i'}{6\varepsilon} + \frac{h^2 b_i''}{12} + b_i]y_i = f_i + \frac{h^2 f_i''}{12} + \frac{h^2 a_i f_i'}{6\varepsilon}, \quad i = 1,2,..,N\text{-}1 \quad (14)$$

$y_0 = \alpha$; $y_N = \beta$; where $\sigma(\rho)$ is a fitting factor which is to be determined in such a way that the solution of (14) converges uniformly to the solution of (1)-(2).

Multiplying (14) by h and taking the limit as $h \to 0$; we get

$$\lim_{h \to 0} \left[\begin{array}{c} (\frac{\sigma}{\rho} + \frac{\sigma a(0)^2 \rho}{6})(y((i+1)h) - 2y(ih) + y((i-1)h)) \\ + \frac{a(0)}{2}(y((i+1)h) - y((i-1)h)) \end{array} \right] = 0 \quad (15)$$

By using (12) in (15) and simplifying, we get

$$\lim_{h \to 0} S\sigma = \frac{a(0)}{2} \coth((\frac{a^2(0) - \varepsilon b(0)}{a(0)})\frac{\rho}{2}) \quad (16)$$

Where $S = \frac{6 + a^2(0)\rho^2}{6\rho}$ \quad (17)

We have; $\sigma = \frac{a(0)}{2S} \coth((\frac{a^2(0) - \varepsilon b(0)}{a(0)})\frac{\rho}{2})$ \quad (18)

σ given by (18) is the constant fitting factor.
From (14) we have

$$[\frac{\sigma\varepsilon}{h^2} + \frac{\sigma a_i^2}{6\varepsilon} - \frac{\sigma(2a_i' + b_i)}{12} - \frac{ha_i(a_i' + b_i)}{12\varepsilon} - \frac{h(a_i'' + 2b_i')}{24} - \frac{a_i}{2h}]y_{i-1}$$

$$-[2[\frac{\sigma\varepsilon}{h^2} + \frac{\sigma a_i^2}{6\varepsilon} - \frac{\sigma(2a_i' + b_i)}{12}] - \frac{h^2 a_i b_i'}{6\varepsilon} - \frac{h^2 b_i''}{12} - b_i]y_i$$

$$+[\frac{\sigma\varepsilon}{h^2} + \frac{\sigma a_i^2}{6\varepsilon} - \frac{\sigma(2a_i' + b_i)}{12} - \frac{ha_i(a_i' + b_i)}{12\varepsilon} - \frac{h(a_i'' + 2b_i')}{24} - \frac{a_i}{2h}]y_{i+1}$$

$$= f_i + \frac{h^2 f_i''}{12} + \frac{h^2 a_i f_i'}{6\varepsilon}, \quad i = 1,2,..,N\text{-}1 \quad (19)$$

where the fitting factor σ is given by (18).

The equivalent three term recurrence relation of equation (19) is given by:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \; ; \; i=1,2,3, \ldots,N\text{-}1 \quad (20)$$

where

$$E_i = \frac{\sigma\varepsilon}{h^2} + \frac{\sigma a_i^2}{6\varepsilon} - \frac{\sigma(2a_i' + b_i)}{12} + \frac{ha_i(a_i' + b_i)}{12\varepsilon} + \frac{h(a_i'' + 2b_i')}{24} + \frac{a_i}{2h}$$

$$F_i = 2[\frac{\sigma\varepsilon}{h^2} + \frac{\sigma a_i^2}{6\varepsilon} - \frac{\sigma(2a_i' + b_i)}{12}] - \frac{h^2 a_i b_i'}{6\varepsilon} - \frac{h^2 b_i''}{12} - b_i$$

$$G_i = \frac{\sigma\varepsilon}{h^2} + \frac{\sigma a_i^2}{6\varepsilon} - \frac{\sigma(2a_i' + b_i)}{12} - \frac{ha_i(a_i' + b_i)}{12} - \frac{h(a_i'' + 2b_i')}{24} - \frac{a_i}{2h}$$

$$H_i = f_i + \frac{h^2 f_i''}{12} + \frac{h^2 a_i f_i'}{6\varepsilon}$$

This gives us the tri-diagonal system which can be solved easily by Thomas Algorithm.

Thomas Algorithm

A brief discussion on solving the tri-diagonal system using Thomas algorithm is presented as follows:

Consider the scheme given in (20):

$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i$; i=1,2,3, …….,N-1

subject to the boundary conditions

$$y_0 = y(0) = \alpha \qquad \qquad (21a)$$
$$y_N = y(1) = \beta \qquad \qquad (21b)$$

We set $y_i = W_i y_{i+1} + T_i$ for i = N-1, N-2, ….2, 1. (22)

where $W_i = W(x_i)$ and $T_i = T(x_i)$ which are to be determined. From (22), we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \qquad \qquad (23)$$

By substituting (23) in (20), we get

$$E_i \left(W_{i-1} y_i + T_{i-1} \right) - F_i y_i + G_i y_{i+1} = H_i .$$

$$y_i = \left(\frac{G_i}{F_i - E_i W_{i-1}} \right) y_{i+1} + \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \right) \qquad (24)$$

By comparing (24) and (22), we get the recurrence relations

$$W_i = \left(\frac{G_i}{F_i - E_i W_{i-1}} \right) \qquad \qquad (25a)$$

$$T_i = \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \right) \qquad \qquad (25b)$$

To solve these recurrence relations for i=0,1,2,3, ,N-1, we need the initial conditions for W_0 and T_0. For this we take $y_0 = \alpha = W_0 y_1 + T_0$. We choose $W_0 = 0$ so that the value of $T_0 = \alpha$. With these initial values, we compute W_i and T_i for i=1,2,3,….,N-1 from (25) in forward process, and then obtain y_i in the backward process from (22)and (21b).

Stability Analysis

We will now show that the algorithm is computationally stable. By stability, we mean that the effect of an error made in one stage of the calculation is not propagated into larger errors at later stages of the calculations. Let us now examine the recurrence relation given by (25a). Suppose that a small error e_{i-1} has been made in the calculation of W_{i-1}; then, we have

$\overline{W}_{i-1} = W_{i-1} + e_{i-1}$ and we are actually calculating

$$\overline{W}_i = \left(\frac{G_i}{F_i - E_i \overline{W}_{i-1}} \right) \qquad \qquad (26)$$

From (26) and (25a), we have

$$e_i = \left(\frac{G_i}{F_i - E_i(W_{i-1} + e_{i-1})} \right) - \left(\frac{G_i}{F_i - E_i W_{i-1}} \right)$$

$$= \left(\frac{G_i E_i e_{i-1}}{\left(F_i - E_i(W_{i-1} + e_{i-1}) \right) \left(F_i - E_i W_{i-1} \right)} \right)$$

$$= \left(\frac{W_i^2 E_i}{G_i} \right) e_{i-1} \qquad \qquad (27)$$

under the assumption that the error is small initially. From the assumptions made earlier that a(x)>0 , b(x)≤0 and its derivatives also non-positive, we have

$$F_i \geq E_i + G_i \ ; i=1, 2, 3, …, N-1$$

Form (25a) we have $W_1 = \frac{G_1}{F_1} < 1$, since $F_1 > G_1$

$$W_2 = \frac{G_2}{F_2 - E_2 W_1} < \frac{G_2}{F_2 - E_2} \ ; \text{ since } W_1 < 1,$$

$$< \frac{G_2}{E_2 + G_2 - E_2} = 1 \ ; \text{ since } F_2 \geq E_2 + G_2$$

Successively, it follows that $|e_i| = |W_i|^2 \left| \frac{E_i}{G_i} \right| |e_{i-1}| <$ $|e_{i-1}|$, since $|E_i| \leq |G_i|$.

Therefore the recurrence relation (25a) is stable. Similarly we can prove that the recurrence relation (25b) is also stable. Finally the convergence of the Thomas Algorithm is ensured by the condition $|W_i| < 1$, i=1, 2, 3,…., N-1.

4. Numerical Examples

4.1. Example

Consider the following homogeneous singular perturbation problem from Bender and Orszag[[1]], page 480; problem 9.17 with α=0]
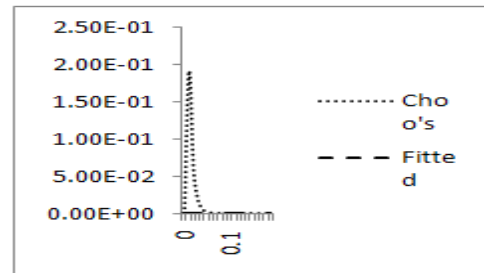
$$\varepsilon y''(x) + y'(x) - y(x) = 0 \ ; \ x \in [0,1], \text{with } y(0)=1 \text{ and } y(1)=1.$$

The exact solution is given by :
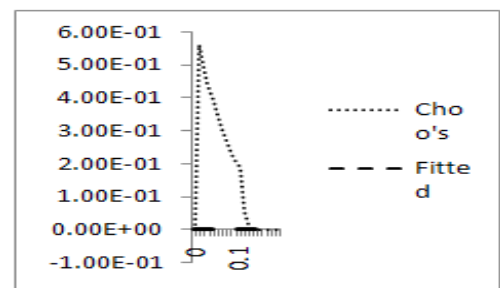
$$y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]}$$

Where $m_1 = (-1 + \sqrt{1+4\varepsilon})/(2\varepsilon)$ and $m_2 = (-1 - \sqrt{1+4\varepsilon})/(2\varepsilon)$

Error plot for Example 4.1, $\varepsilon = 10^{-3}$, $h = 10^{-2}$

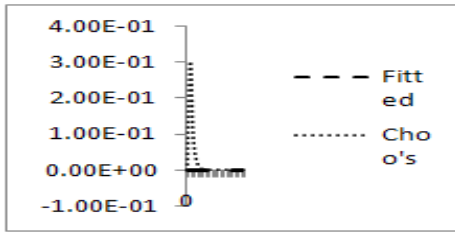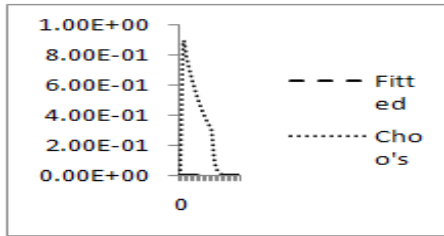Error plot for example 4.1, $\varepsilon = 10^{-4}$, $h = 10^{-2}$.

4.2. Example

Let us consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity, Reinhardt[[8]], example 2]

$$\varepsilon y''(x) + y'(x) = 1 + 2x \ ; \ x \in [0,1], \text{with } y(0)=0 \text{ and } y(1)=1.$$

The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - e^{-x/\varepsilon})}{(1 - e^{-1/\varepsilon})}$$

Error plot for Example 4.2, $\varepsilon=10^{-3}$, $h=10^{-2}$

Errorplot for Example 4.2, $\varepsilon=10^{-4}$, $h=10^{-2}$

5. Non-linear Problems

Nonlinear singular perturbation problems were converted as a sequence of linear singular perturbation problems by using quasilinearization (Replacing the non-linear problem by a sequence of linear problems) method. The outer solution (the solution of the given problem by putting $\varepsilon=0$) is taken to be the initial approximation.

5.1. Example

Consider the following singular perturbation problem from Bender and Orszag[[1], page 463; equations: 9.7.1]
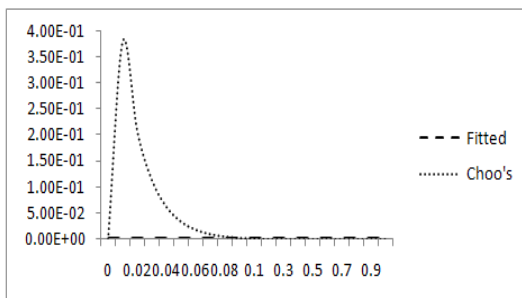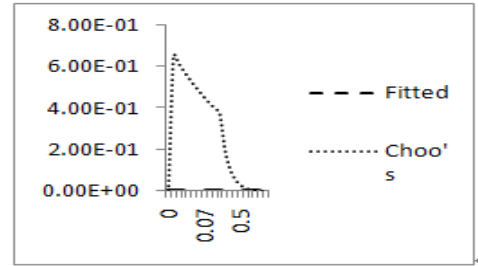$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0 \; ; x \in [0,1],$$
with y(0)=0 and y(1)=0.
The linear problem concerned to this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1}y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right)-1\right]$$

We have chosen to use Bender and Orszag's uniformly valid approximation[[1], page 463; equation: 9.7.6] for comparison, $y(x) = \log_e\left(\frac{2}{x+1}\right) - (\log_e 2)e^{-2x/\varepsilon}$

For this example, we have boundary layer of thickness $O(\varepsilon)$ at x=0. [cf. Bender and Orszag[1]].

Error plot for Example 5.1, $\varepsilon=10^{-3}$, $h=10^{-2}$

5.2. Example

Let us consider the following singular perturbation problem from Kevorkian and Cole[[3], page 56; equation 2.5.1]
$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0$; $x \in [0,1]$, with y(0)= -1 and y(1)=3.9995

The linear problem concerned to this example is
$$\varepsilon y''(x) + (x + 2.9995)y'(x) = x + 2.9995$$

We have chosen to use the Kevorkian and Cole's uniformly valid approximation[[3], pages 57 and 58; equations (2.5.5), (2.5.11) and (2.5.14)] for comparison,
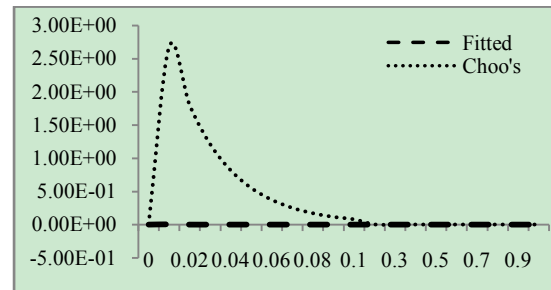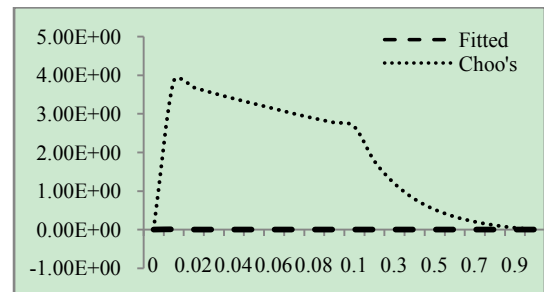
$$y(x) = x + c_1 \tanh\left(\left(\frac{c_1}{2}\right)\left(\frac{x}{\varepsilon}+c_2\right)\right)$$

Where $c_1=2.9995$ and $c_2=(1/c_1)\log_e[(c_1-1)/(c_1+1)]$

For this example also we have a boundary layer of width $O(\varepsilon)$ at x=0 [cf. Kevorkian and Cole[3], pages 56-66]. The numerical results are given in table 4(a), 4(b) for $\varepsilon=10^{-3}$ and 10^{-4} respectively.

Error plot for Example 5.2, $\varepsilon=10^{-3}$, $h=10^{-2}$

Error plot for Example 5.2, $\varepsilon=10^{-4}$, $h=10^{-2}$

6. Conclusions

We have presented fitted fourth order finite difference

method for solving singularly perturbed two-point boundary value problems The present fitted fourth order finite difference method for solving singularly perturbed two-point boundary value problems produce better approximation to the exact solution, specifically in the boundary layer region with step size $h > \varepsilon$, the perturbation parameter.

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