Piecewise Constant Control Set Systems

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Abstract   In this article we prove that for any measurable admissible control $w(\cdot)$ and for any $\varepsilon > 0$ there exists piecewise constant admissible control $\overline{w}(\cdot)$ such that for set solutions of control set system are ε-neighbouring.

Keywords   Set Differential Equation, Control System, Piecewise Constant Control

1. Introduction

In recent years the development of the calculus in metric spaces has attracted some attention[1-7]. Earlier, F.S. de Blasi, F. Iervolino[8] started the investigation of set differential equations (SDEs) in semilinear metric spaces. This has now evolved into the theory of SDEs as an independent discipline: properties of solutions[1-3,5,36], the impulse equations[1,2,37], control systems[38-41] and asymptotic methods[1-3,42-46]. On the other hand, SDEs are useful in other areas of mathematics. For example, SDEs are used, as an auxiliary tool, to prove existence results for differential inclusions[1,26,31,35]. Also, one can employ SDEs in the investigation of fuzzy differential equations[2,6,21-23, 25,26]. Moreover, SDEs are a natural generalization of the usual ordinary differential equations in finite (or infinite) dimensional Banach spaces.

In many engineering control systems piecewise constant controls, instead of measurable controls are applied. In this article we prove that for any measurable admissible control $w(\cdot)$ and for any $\varepsilon > 0$ there exists piecewise constant admissible control $\overline{w}(\cdot)$ such that for set solutions of control set system are ε-neighbouring.

2. Preliminaries

Let $comp(R^n)\left(conv(R^n)\right)$ be a set of all nonempty (convex) compact subsets from the space R^n,

$$h(A,B) = \min_{r \geq 0}\left\{S_r(A) \supset B, S_r(B) \supset A\right\}$$

be Hausdorff distance between sets A and B, $S_r(A)$ is r-neighborhood of set A.

Let A, B, C be in $conv(R^n)$. The set C is the Hukuhara difference of A and B, if $B + C = A$, i.e.

$$C = A \overset{H}{\longrightarrow} B$$

From Radstrom's Cancellation Lemma[47], it follows that if this difference exists, then it is unique.

Definition 1[48]. A mapping $X:[0,T] \to conv(R^n)$ is differentiable in the sense of Hukuhara at $t \in [0,T]$ if for some $\delta > 0$ the Hukuhara differences

$$X(t+\Delta) \overset{H}{\longrightarrow} X(t), \quad X(t) \overset{H}{\longrightarrow} X(t-\Delta)$$

exists in $conv(R^n)$ for all $0 < \Delta < \delta$ and there exists an $DX(t) \in conv(R^n)$ such that

$$\lim_{\Delta \to 0_+} h\left(\Delta^{-1}\left(X(t+\Delta) \overset{H}{\longrightarrow} X(t)\right), DX(t)\right) = 0 \quad \text{and}$$

$$\lim_{\Delta \to 0_+} h\left(\Delta^{-1}\left(X(t) \overset{H}{\longrightarrow} X(t-\Delta)\right), DX(t)\right) = 0$$

Here $DX(t)$ is called the Hukuhara derivative of $X(t)$ at t.

Consider the Cauchy problem with small parameter

$$DX = A(t)X + G(t), \quad X(0) = X_0, \tag{1}$$

where $A(t)$ is $(n \times n)$-dimensional matrix-valued function; $G: R_+ \to conv(R^n)$ is the set-valued map, $X_0 \in conv(R^n)$.

Definition 2. A mapping $X:[0,T] \to conv(R^n)$ is a solution to the problem (1) if and only if it is continuous and satisfies the integral equation

$$X(t) = X_0 + \int_0^t \left[A(s)X(s) + G(s)\right]ds$$

for all $t \in [0,T]$. Here the integral is understood in the sense of[48] (the integral exists for example if $X(\cdot)$ is measurable and the real mapping $t \to h(X(t),\{0\})$ is integrable on $I \subset R_+$).

Theorem 1[2]. Let the following conditions are true:

1) $A(\cdot)$ is measurable on $[0,T]$;

2)   There exists $a > 0$ such that $\|A(t)\| \leq a$ for almost every $t \in [0,T]$;

3)   The set-valued map $G:[0,T] \to conv(R^n)$ is measurable on $[0,T]$;

4)   There exists $g(\cdot) \in L_2[0,T]$ such that

$h(G(t),\{0\}) \le g(t)$ almost everywhere on $[0,T]$.

Then problem (1) has on $[0,T]$ exactly one solution.

3. The Control Set Differential Equation

We consider following control set differential equation

$$DX = A(t)X + F(t,w), \quad X(0,w) = X_0, \qquad (2)$$

where $w \in R^m$ is the control, $F : R_+ \times R^m \to conv(R^n)$ is the set-valued map.

Let $W : R_+ \to conv(R^m)$ be the measurable set-valued map.

Definition 3. The set LW of all measurable single-valued branches of the set-valued map $W(\cdot)$ is the set of the admissible controls.

Obviously, the control set differential equation (2) turns into the ordinary set differential equation

$$DX = A(t)X + G(t), \quad X(0) = X_0, \qquad (3)$$

if the control $\widetilde{w}(\cdot) \in LW$ is fixed and $G(t) \equiv F(t,\widetilde{w}(t))$.

Let $X(t)$ denotes the set solution of the differential equation (3), then $X(t,w)$ denotes the set solution of the control differential equation (2) for the fixed $w(\cdot) \in LW$.

Definition 4. The set $Y(T) = \{X(T,w) : w(\cdot) \in LW\}$ be called the attainable set of the system (2).

Theorem 2[49]. Let the following conditions are true:

1) $A(\cdot)$ is measurable on $[0,T]$;

2)   There exists $a > 0$ such that $\|A(t)\| \le a$ for almost every $t \in [0,T]$;

3)   The set-valued map $W : [0,T] \to conv(R^m)$ is measurable on $[0,T]$;

4)   The set-valued map $F : [0,T] \times R^m \to conv(R^n)$ satisfies the conditions

      a)   measurable in t;
      b)   continuous in w;

5)   There exist $v(\cdot) \in L_2[0,T]$ and $l(\cdot) \in L_2[0,T]$ such that

$$h(W(t),\{0\}) \le v(t), \ h(F(t,w),\{0\}) \le l(t)$$

almost everywhere on $[0,T]$ and all $w \in W(t)$;

6)   The set $Q(t) = \{F(t,w(t)) : w(\cdot) \in LW\}$ is compact and convex for almost every $t \in [0,T]$.

Then for every $w(\cdot) \in LW$ there exists the set solution $X(\cdot,w)$ on $[0,T]$ and the attainable set $Y(T)$ is compact and convex.

Let $U = \prod\limits_{i=1}^{m}[u_{min}^i, u_{max}^i]$ and $W(t) \equiv U$ on $[0,T]$.

Now, we need to establish that for any measurable admissible control $w(\cdot)$ and for any $\varepsilon > 0$ there exists piecewise constant admissible control $\overline{w}(\cdot)$ such that for set solutions of system (2) holds for all $t \in [0,T]$

$$h(X(t,w), X(t,\overline{w})) < \varepsilon .$$

Theorem 3. Let the conditions of the theorem 2 are true, and

7) There exists constant $\gamma > 0$ such that

$$h\left(\int_0^t F(s,w_1(s))dt, \int_0^t F(s,w_2(s))ds\right) \le \gamma \left\|\int_0^t w_1(s)ds - \int_0^t w_2(s)ds\right\|$$

for all $w_1(\cdot), w_2(\cdot) \in LW$ and $t \in [0,T]$.

Then for every $w(\cdot) \in LW$ there exists $\overline{w}(\cdot) \in LW$ such that

1) $\overline{w}(t)$ is constant on every $\left[(i-1)\dfrac{T}{k}, i\dfrac{T}{k}\right]$, $i = \overline{1,k}$;

2) $\overline{w}_i(t) = \left\{\left(\overline{w}_i^1(t),...,\overline{w}_i^m(t)\right)^T \mid \overline{w}_i^j(t) \in \left\{u_{min}^j, u_{max}^j\right\}, i = \overline{1,k},\right.$

$\left. j = \overline{1,m}\right\}$ for every $t \in [0,T]$;

3) for all $t \in [0,T]$

$$h\left(X(t,w), X(t,\overline{w})\right) \le \gamma e^{aT}\frac{T}{2k}\|u_{max} - u_{min}\|,$$

where $u_{min} = \left(u_{min}^1,...,u_{min}^m\right)^T$, $u_{max} = \left(u_{max}^1,...,u_{max}^m\right)^T$.

Proof. We have any $w(\cdot) \in LW$ and any $k \in N$. Let

$$W_i = \left(W_i^1,...,W_i^m\right)^T, \text{ where } W_i^j = \int_0^{i\frac{T}{k}} w^j(s)ds, \ i = \overline{1,k}, \ j = \overline{1,m}.$$

Obviously,

$$W_{i+1}^j - W_i^j = \int_0^{\frac{(i+1)T}{k}} w^j(s)ds - \int_0^{\frac{iT}{k}} w^j(s)ds = \int_{\frac{iT}{k}}^{\frac{(i+1)T}{k}} w^j(s)ds,$$

$$u_{min}^j \frac{T}{k} \le W_{i+1}^j - W_i^j \le u_{max}^j \frac{T}{k}, \quad j = \overline{1,m},$$

and

$$\|W_{i+1} - W_i\| \le \|u_{max} - u_{min}\|\frac{T}{k}.$$

Now we obtain

$$\overline{w}(t) = \begin{cases} \overline{w}_1, & t \in \left[0, \dfrac{T}{k}\right) \\ \quad \vdots \\ \overline{w}_{k-1}, & t \in \left[\dfrac{(k-2)T}{k}, \dfrac{(k-1)T}{k}\right), \\ \overline{w}_k, & t \in \left[\dfrac{(k-1)T}{k}, T\right] \end{cases}$$

such that

1) $\overline{w}_1 = \left(\overline{w}_1^1,...,\overline{w}_1^m\right)$, where

$$\overline{w}_1^j = \begin{cases} u_{max}^j, & if \ W_1^j \ge \dfrac{T}{2k}\left(u_{max}^j + u_{min}^j\right) \\ u_{min}^j, & if \ W_1^j < \dfrac{T}{2k}\left(u_{max}^j + u_{min}^j\right) \end{cases}, \ j = \overline{1,m};$$

2) $\overline{w}_i = \left(\overline{w}_i^1,...,\overline{w}_i^m\right)$, where

$$\overline{w}_i^j = \begin{cases} u_{max}^j, & if \ W_i^j - \sum\limits_{l=1}^{i-1}\overline{w}_l^j\dfrac{T}{k} \ge \dfrac{T}{2k}\left(u_{max}^j + u_{min}^j\right) \\ u_{min}^j, & if \ W_i^j - \sum\limits_{l=1}^{i-1}\overline{w}_l^j\dfrac{T}{k} < \dfrac{T}{2k}\left(u_{max}^j + u_{min}^j\right) \end{cases},$$

$j = \overline{1,m}$.

Obviously, for $i = 1$ and $j = \overline{1,m}$ we have

a) if $\overline{w}_1^j = u_{max}^j$, then

$$W_1^j - \overline{w}_1^j \frac{T}{k} \le 0 , \quad W_2^j - \overline{w}_1^j \frac{T}{k} \ge \frac{T}{2k}\left(u_{max}^j - u_{min}^j\right),$$

$$W_2^j \le W_1^j + u_{max}^j \frac{T}{k} , \quad 0 \ge W_1^j - \overline{w}_1^j \frac{T}{k} \ge -\frac{T}{2k}\left(u_{max}^j - u_{min}^j\right);$$

b) if $\overline{w}_1^j = u_{min}^j$, then

$$W_1^j - \overline{w}_1^j \frac{T}{k} \ge 0 , \quad W_2^j - \overline{w}_1^j \frac{T}{k} \le \frac{T}{2k}\left(u_{max}^j - u_{min}^j\right),$$

$$W_2^j \ge W_1^j + u_{max}^j \frac{T}{k} , \quad 0 \le W_1^j - \overline{w}_1^j \frac{T}{k} \le \frac{T}{2k}\left(u_{max}^j - u_{min}^j\right).$$

Hence we obtain

$$\left| W_1^j - \overline{w}_1^j \frac{T}{k} \right| \le \frac{T}{2k}\left(u_{max}^j - u_{min}^j\right),$$

and

$$\left\| W_1 - \overline{w}_1 \frac{T}{k} \right\| \le \frac{T}{2k}\left\| u_{max} - u_{min} \right\|.$$

Thus, by induction, we obtain that, for $i = \overline{2,k}$ and $j = \overline{1,m}$

$$\left| W_i^j - \sum_{l=1}^{i} \overline{w}_l^j \frac{T}{k} \right| \le \frac{T}{2k}\left(u_{max}^i - u_{min}^j\right), \quad j = \overline{1,m} \text{ ,and}$$

$$\left\| W_i - \sum_{l=1}^{i} \overline{w}_l \frac{T}{k} \right\| \le \frac{T}{2k}\left\| u_{max} - u_{min} \right\|. \tag{4}$$

Therefore, if $t_i = \frac{iT}{k}$, $i = \overline{1,k}$; then

$$\left\| \int_o^{t_i} w(s)ds - \int_0^{t_i} \overline{w}(s)ds \right\| \le \frac{T}{2k}\left\| u_{max} - u_{min} \right\|.$$

Now, we take $t \in \left(\frac{(i-1)T}{k}, \frac{iT}{k} \right)$. Then

$$\left\| \int_o^{t} w(s)ds - \int_0^{t} \overline{w}(s)ds \right\| \le \left\| W_{i-1} - \sum_{l=1}^{i-1} \overline{w}_l \frac{T}{k} + \int_{\frac{(i-1)T}{k}}^{t} \left(w(s) - \overline{w}_i\right)ds \right\|.$$

As for all $j = \overline{1,m}$

$$W_i^j - \sum_{l=1}^{i} \overline{w}_l^j \frac{T}{k} \ge W_{i-1}^j - \sum_{l=1}^{i-1} \overline{w}_l^j \frac{T}{k} + \int_{\frac{(i-1)T}{k}}^{t} \left(w(s) - \overline{w}_i^j\right)ds \ge$$

$$\ge W_{i-1}^j - \sum_{l=1}^{i-1} \overline{w}_l^j \frac{T}{k} ;$$

then

$$\left\| \int_o^{t} w(s)ds - \int_0^{t} \overline{w}(s)ds \right\| \le \max\left\{ \left\| W_i - \sum_{l=1}^{i} \overline{w}_l \frac{T}{k} \right\|, \left\| W_{i-1} - \sum_{l=1}^{i-1} \overline{w}_l \frac{T}{k} \right\| \right\}.$$

By (4), we get

$$\left\| \int_0^{t} w(s)ds - \int_0^{t} \overline{w}(s)ds \right\| \le \frac{T}{2k}\left\| u_{max} - u_{min} \right\| \tag{5}$$

for all $t \in [0,T]$.

Now, applying definition 2 and condition 7 of the theorem,

we obtain

$$h\left(X(t,w), X(t,\overline{w})\right) = h\left(\int_0^{t}\left[A(s)X(s,w) + F(s,w(s))\right]ds, \right.$$

$$\left. \int_0^{t}\left[A(s)X(s,\overline{w}) + F(s,\overline{w}(s))\right]ds \right) \le \int_0^{t} h\left(A(s)X(s,w), A(s)X(s,\overline{w})\right) +$$

$$+ h\left(\int_0^{t} F(s,w(s))ds, \int_0^{t} F(s,\overline{w}(s))ds \right) \le$$

$$\le a \int_0^{t} h\left(X(s,w), X(s,\overline{w})\right)ds + \gamma \left\| \int_0^{t} w(s)ds - \int_0^{t} \overline{w}(s)ds \right\|$$

Using Gronwall-Bellman's inequality, we obtain

$$h\left(X(t,w), X(t,\overline{w})\right) \le \gamma e^{aT} \left\| \int_0^{t} w(s)ds - \int_0^{t} \overline{w}(s)ds \right\|.$$

By (5), we have

$$h\left(X(t,w), X(t,\overline{w})\right) \le \gamma e^{aT} \frac{T}{2k}\left\| u_{max} - u_{min} \right\|.$$

Theorem is proved.

Remark. Obviously, if we take $k > \gamma e^{aT} \frac{T}{2\varepsilon}\left\| u_{max} - u_{min} \right\|$;

then $h\left(X(t,w), X(t,\overline{w})\right) < \varepsilon$ for all $t \in [0,T]$.

4. Conclusions

Here we used the approach of Hukuhara at definition of the derivative which has essential shortages. However the given approach is well investigated by many authors. Also in the literature exist other approaches to definition of the derivative[2,3,9,21,29,36], but also they have the shortages. It is easily possible to show that this outcome will be true for some other cases with little changes.

Also we remark that this result helps to build ε -optimal piecewise constant controls for optimal control set system (Mayer problem, time-optimal problem and other).

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