Solution of the Modified Equal Width Equation, its Variant and Non-homogeneous Burgers' Equation by RDT Method

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Abstract   The modified equal width equation and its variant are investigated. We have presented a reduced differential transform method to solve the MEW equation, its variant and non-homogeneous Burgers' equation. This method is an alternative approach to overcome the demerit of complex calculation of differential transform method, capable of reducing the size of calculation and easily overcoming the difficulty of the perturbation technique or Adomain polynomials. The approximate analytical solutions of the equations are calculated in the form of series with easily computable components. Numerical results are derived and the obtained results are found in good agreement with the exact solutions.

Keywords   Reduced Differential Transform Method (RDTM), Modified Equal Width (MEW) Equation and Its Variant, Non-Homogeneous Burgers' Equation, Approximate Solution, Analytical Solution

1. Introduction

Nonlinear wave phenomena plays a major role in sciences such as fluid mechanics, plasma physics, solid state physics, optical fibers, chemical kinetics and geochemistry. The phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations[1]. Because of the complexity of the nonlinear wave equations, there is no unified method to find the solutions of these equations. Several methods, analytical and numerical, such as the homogeneous balance method[2], Hirota bilinear method[3], Variational iteration method[4], the decomposition method[5], Backlund transformation and inverse scattering method[6], Homotopy Perturbation method[7], the sine–cosine methods[8], the Tan hyperbolic function method[9] and the Reduced Differential transform method[10] are used to solve the partial differential equations governing the nonlinear waves. The MEW equation, which is related to the RLW equation, has solitary wave solutions with both positive and negative amplitudes, all of which have the same width[11,12]. The MEW equation is a nonlinear wave equation with cubic nonlinearity with a pulse like solitary wave solution. The present work is motivated by the intention to extend the use of earlier method in the literature to solve the MEW equation, its nonlinear variant and the non-homogeneous

Burgers' equation. In this paper, we solve the above problems by the Reduced Differential transform method[10]. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms. The structure of this paper is organized as follows:

In section 2, we begin with some basic definitions and explain the reduced differential transformation method. In section 3, we apply this method to solve the above three problems.

2. Analysis of the Method

The basic definitions in the Reduced Differential transform method[10] are as follows:

2.1. Definition

If function $u(x; t)$ is analytic and k-times continuously differentiable with
respect to time t and space x in the domain of interest, then let

$$U_k(x) = (\frac{1}{k!})(\frac{\partial^k u(x,t)}{\partial t^k})_{t=0} \tag{1}$$

where the function $U_k(x)$ is the transformed function of the original function $u(x,t)$. The differential inverse transform of $U_k(x)$ is defined as follows

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{2}$$

Then combining these two equations we write

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)_{t=0} t^k. \qquad (3)$$

From the above definitions, it can be found that the concept of the Reduced Differential transform method is derived from the Taylor's series expansion.

We write the gas dynamics equation in standard form

$$L(u) + R(u) + N(u) = 0, \qquad (4)$$

with initial condition

$$u(x,0) = f(x), \qquad (5)$$

where $(u) = u_t(x,t)$, $R \equiv \frac{\partial^3}{\partial x^2 \partial t}$ is a linear operator which has mixed partial derivatives and $N(u)$ is a nonlinear term. According to RDTM, the iteration formula is

$$(k+1)U_{k+1}(x) = -N(U_k(x)) - \frac{\partial^3}{\partial x^2 \partial t}\overline{u_k}(x,t), \qquad (6)$$

where $\frac{\partial^3}{\partial x^2 \partial t}\overline{u_k}(x,t)$ and $N(U_k(x))$ are the reduced differential transform transformations of the functions $R(u(x,t))$ and $N(u(x,t))$, respectively.

Definition 2.1 implies that the initial approximation $U_0(x)$ is given by the initial condition, that is

$$U_0(x) = u(x,0). \qquad (7)$$

Substituting $U_0(x)$ in the iteration formula (6), we obtain the values of $U_k(x)$. Then the differential inverse transformation of the set of values $[U_k(x)]_{k=0}^n$ gives approximation solution as

$$\overline{u_k}(x,t) = \sum_{k=0}^{n}\frac{1}{k!}\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)_{t=0} t^k. \qquad (8)$$

3. Applications

Now, we will solve three problems using the RDTM.

3.1. The Modified Equal Width (MEW) Equation

The modified equal width (MEW) equation

$$u_t + 3u^2 u_x - u_{xxt} = 0, -\infty < x < \infty, t > 0 \qquad (9)$$

has been discussed in[11,12]. Here $u = u(x,t)$ is its solution.

The MEW equation, which is related to the RLW equation, has solitary waves solutions with both positive and negative amplitudes, all of which have the same width. The MEW equation is a nonlinear wave equation with cubic nonlinearity with a pulse like solitary wave solution.

We consider above equation with the initial condition

$$u(x,0) = \text{sech}\, x. \qquad (10)$$

We know that the exact solution for above problem is

$$u(x,t) = \text{sech}(x - 0.5t) \qquad (11)$$

By RDTM, the transformed form of equation (9) is

$$(k+1)U_{k+1}(x) = -N(U_k(x)) + \frac{\partial^3}{\partial x^2 \partial t}\overline{u_k}(x,t) \qquad (12)$$

where $N(U_k(x))$ is the Reduced Differential transformation of $3u^2 u_x$.

Using the initial condition (11), we obtain

$$U_0(x) = \text{sech}\, x,$$
$$U_1(x) = 3\,(\text{sech}\, x)^3 \tanh x,$$
$$U_2(x) = -27\,(\text{sech}\, x)^7 + 22.5\,(\text{sech}\, x)^5$$
$$-30\,(\text{sech}\, x)^5 \tanh x + 13.5\,(\text{sech}\, x)^3 \tanh x,$$

and so on.

Then, the differential inverse transformation of the set of values $[U_k(x)]_{k=0}^2$ gives the second order approximation

solution as

$$\overline{u_2}(x,t) = \sum_{k=0}^{2} U_k(x)t^k$$
$$= U_0(x) + U_1(x)t + U_2(x)t^2$$
$$= \text{sech}\, x + 3\,\{(\text{sech}\, x)^3 \tanh x\}t$$
$$+ \left\{\begin{array}{l} -27\,(\text{sech}\, x)^7 + 22.5\,(\text{sech}\, x)^5 \\ -30\,(\text{sech}\, x)^5 \tanh x + 13.5\,(\text{sech}\, x)^3 \tanh x \end{array}\right\} t^2.$$

Therefore, the exact solution of (9) with the initial condition (10) is given by

$$u(x,t) = \lim_{n\to\infty}\overline{u_n}(x,t).$$

The comparison of the present approximation solutions with the exact solutions of the MEW equation is made in the following table:

Table 1. Comparison of the approximate solutions with the exact solutions of MEW equation (9) with initial condition (10)

t	x	RDTM Solution	Exact Solution	Absolute error
0.00015	0.5	0.886674	0.886788	0.00114
	0	1	1	0
	0.5	0.886964	0.88685	0.000114
0.00005	0.5	0.886771	0.886809	0.000038
	0	1	1	0
	0.5	0.886867	0.886829	0.000038
0.0005	0.5	0.886336	0.886716	0.00038
	0	0.999999	1	0.000001
	0.5	0.887302	0.886921	0.000381

3.2. The Variant of Modified Equal Width (MEW) Equation

Consider the variant of modified equal width (MEW) equation

$$u_t + \left(\frac{12}{7}\right)(u^6)_x - \left(\frac{3}{7}\right)(u^6)_{xxt} = 0,$$
$$-\infty < x < \infty, \quad t > 0, \qquad (13)$$

with the initial condition

$$u(x,0) = [\cosh\left(\tfrac{5x}{6}\right)]^{2/5}. \qquad (14)$$

Here $u = u(x,t)$.

We know that the exact solution for above problem is

$$u(x,t) = \left[\cosh\left(\frac{5(x-t)}{6}\right)\right]^{\frac{2}{5}}. \qquad (15)$$

By RDTM, the transformed form of equation (13) is

$$(k+1)U_{k+1}(x) = -N(U_k(x)) - \left(\tfrac{3}{7}\right)\frac{\partial^3}{\partial x^2 \partial t}(\overline{u_k}(x,t))^6 \qquad (16)$$

where $N(U_k(x))$ is the Reduced Differential transformation of $\left(\frac{12}{7}\right)(u^6)_x$.

Using the initial condition (14), we obtain

$$U_0(x) = [\cosh\left(\tfrac{5x}{6}\right)]^{2/5},$$

$$U_1(x) = -\left(\tfrac{24}{7}\right)[\cosh\left(\tfrac{5x}{6}\right)]^{7/5}\sinh\left(\tfrac{5x}{6}\right),$$

$$U_2(x) = \left(\tfrac{144}{49}\right)\left\{22\left[\cosh\left(\tfrac{5x}{6}\right)\right]^{\frac{22}{5}} - 17\left[\cosh\left(\tfrac{5x}{6}\right)\right]^{\frac{12}{5}}\right\}$$

$$+ \left(\tfrac{6}{49}\right)\{484\left[\cosh\left(\tfrac{5x}{6}\right)\right]^{\frac{17}{5}}\sinh\left(\tfrac{5x}{6}\right) -$$

$$204\left[\cosh\left(\tfrac{5x}{6}\right)\right]^{\frac{7}{5}}\sinh\left(\tfrac{5x}{6}\right)\},$$

and so on.

Then, the differential inverse transformation of the set of values $[U_k(x)]_{k=0}^2$ gives the second order approximation solution as

$$\overline{u_2}(x,t) = \sum_{k=0}^{2} U_k(x)t^k$$

$$= U_0(x) + U_1(x)t + U_2(x)t^2$$

$$= \left[\cosh\left(\frac{5x}{6}\right)\right]^{\frac{2}{5}} - \left(\frac{24}{7}\right)\left[\cosh\left(\frac{5x}{6}\right)\right]^{\frac{7}{5}}\sinh\left(\frac{5x}{6}\right)t + [\left(\frac{144}{49}\right)$$

$$\left\{22\left[\cosh\left(\frac{5x}{6}\right)\right]^{\frac{22}{5}} - 17\left[\cosh\left(\frac{5x}{6}\right)\right]^{\frac{12}{5}}\right\} + \left(\frac{6}{49}\right)$$

$$\left\{484\left[\cosh\left(\frac{5x}{6}\right)\right]^{\frac{17}{5}}\sinh\left(\frac{5x}{6}\right)\right.$$

$$\left. - 204\left[\cosh\left(\frac{5x}{6}\right)\right]^{\frac{7}{5}}\sinh\left(\frac{5x}{6}\right)\right\}]t^2.$$

Therefore, the exact solution of (13) with the initial condition (14) is given by
$$u(x,t) = \lim_{n\to\infty} \overline{u_n}(x,t).$$

The comparison of the present approximation solutions with the exact solutions of the variant of MEW equation is made in the following table:

Table 2. Comparison of the approximate solution with the exact solution of the equation (13) with initial condition (14)

t	x	RDTM Solution	Exact Solution	Absolute error
0.00015	-0.5	1.03459	1.03436	0.00023
	0	1	1	0
	0.5	1.03409	1.03432	0.00023
0.00005	-0.5	1.03442	1.03434	0.00006
	0	1	1	0
	0.5	1.03426	1.03433	0.00007
0.0005	-0.5	1.03517	1.03441	0.00076
	0	1	1	0
	0.5	1.03352	1.03427	0.00075

3.3. Burgers' Equation

Consider the non-homogeneous Burgers' equation
$$u_t + uu_x = x, \tag{17}$$
with the initial condition
$$u(x,0) = 2, \tag{18}$$
with $u = u(x,t)$ is its solution.

Taking Reduced Differential transform of equation (17), we obtain
$$(k+1)U_{k+1}(x) = -N_k\big(U_k(x)\big) + \delta(k)x \tag{19}$$
where $N_k\big(U_k(x)\big)$ is the Reduced Differential transformation of uu_x, and
$$\delta(k) = \begin{cases} 1, & k=0 \\ 0, & k\neq 0 \end{cases}.$$

Using the initial condition (18), we obtain the initial approximation as
$$U_0(x) = 2.$$

Using the definition of RDTM, first few terms N_k and U_k are as follows

$$N_0 = \left(\frac{1}{2}\right)\frac{\partial}{\partial x}\big(U_0(x)\big)^2 = 0, \qquad U_1 = x,$$

$$N_1 = \frac{\partial}{\partial x}(U_0 U_1) = 2, \qquad U_2(x) = -1,$$

$$N_2 = \left(\frac{1}{2}\right)\frac{\partial}{\partial x}(2U_0 U_2 + (U_1)^2) = x, \qquad U_3 = -\frac{x}{3},$$

$$N_3 = \frac{\partial}{\partial x}(U_0 U_3 + U_1 U_2) = -\frac{5}{3}, \qquad U_4 = 2\left(\frac{5}{4!}\right),$$

$$N_4 = \frac{1}{2}\frac{\partial}{\partial x}(2U_0 U_4 + 2U_1 U_3 + (U_2)^2) = -\frac{2x}{3}, U_5 = \frac{2x}{15},$$

and so on.

Then, the differential inverse transformation of the set of values $[U_k(x)]_{k=0}^\infty$ gives the solution as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k$$

$$= U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3 + U_4(x)t^4 + \ldots$$

$$= 2\left[1 - \frac{t^2}{2!} + \frac{5t^4}{4!} + \cdots\right] + x\left[t - \frac{t^3}{3} + \frac{2t^5}{15} + \cdots\right]$$

$$= 2\operatorname{sech} t + x\tanh t,$$

which is an exact solution of the equation (17) with initial condition (18).

5. Conclusions

The main goal of the present work is to implement the Reduced Differential transform method[10], and more importantly to overcome the computational difficulty of Differential Transform method. The aims of this work were achieved, and travelling wave solutions were formally obtained as a power series. The transformation technique was used to demonstrate that our analysis is applicable to a variety of nonlinear evolution equations.

The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms. Its small size of computation in comparison with the computational size required in other numerical methods, and its rapid convergence show that the method is reliable and introduces a significant improvement in solving the MEW equation and Burgers' equation over existing methods. As the method is usually tedious, we have used the software package "Mathematica" to calculate the series obtained from the RDTM.

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